Rend. Circ. Mat. Palermo (2013) 62:61–66
DOI 10.1007/s12215-013-0109-7

Vanishing thetanulls on curves with involutions

Arnaud Beauville

Received: 29 June 2012 / Accepted: 3 October 2012 / Published online: 12 February 2013

Abstract The configuration of theta characteristics and vanishing thetanulls on a hyperelliptic curve is completely understood. We observe in this note that analogous results hold for the σ-invariant theta characteristics on any curve C with an involution σ. As a consequence we get examples of non hyperelliptic curves with a high number of vanishing thetanulls.

Keywords Thetanullwerte · Theta characteristics · Vanishing thetanulls · Curves with involution

Mathematics Subject Classification Primary 14H42; Secondary 14H51 · 14H45

1 Introduction

Let C be a smooth projective curve over $\mathbb{C}$. A *theta characteristic* on C is a line bundle κ such that $\kappa^2 \cong K_C$; it is even or odd according to the parity of $h^0(\kappa)$. An even theta characteristic κ with $h^0(\kappa) > 0$ is called a *vanishing thetanull*.

The terminology comes from the classical theory of theta functions. A theta characteristic κ corresponds to a symmetric theta divisor Θ_κ on the Jacobian JC, defined by a theta function θ_κ; this function is even or odd according to the parity of κ. Thus the numbers $\theta_\kappa(0)$ are 0 for κ odd; for κ even they are classical invariants attached to the curve ("thetanullwerte" or "thetanulls"). The thetanull $\theta_\kappa(0)$ vanishes if and only if κ is a vanishing thetanull in the above sense.

When C is hyperelliptic, the configuration of its theta characteristics and vanishing thetanulls is completely understood (see e.g. [4]). We observe in this note that analogous results hold for the σ-invariant theta characteristics on any curve C with an involution σ. As a consequence we obtain examples of non hyperelliptic curves with a high number of

A. Beauville (✉)
Laboratoire J.-A. Dieudonné, UMR 7351 du CNRS,
Université de Nice, Parc Valrose, 06108 Nice Cedex 2, France
e-mail: arnaud.beauville@unice.fr

vanishing thetanulls: for instance approximately one fourth of the even thetanulls vanish for a bielliptic curve.

2 σ-Invariant line bundles

Throughout the paper we consider a curve C of genus g, with an involution σ. We denote by $\pi : C \to B$ the quotient map, and by $R \subset C$ the fixed locus of σ. For a subset $E = \{p_1, \dots, p_k\}$ of R we will still denote by E the divisor $p_1 + \cdots + p_k$.

The double covering π determines a line bundle ρ on B such that $\rho^2 = \mathcal{O}_B(\pi_* R)$; we have $\pi^*\rho = \mathcal{O}_C(R)$, $\pi_*\mathcal{O}_C \cong \mathcal{O}_B \oplus \rho^{-1}$ and $K_C = \pi^*(K_B \otimes \rho)$.

We consider the map $\varphi : \mathbb{Z}^R \to \operatorname{Pic}(C)$ which maps $r \in R$ to the class of $\mathcal{O}_C(r)$. Its image lies in the subgroup $\operatorname{Pic}(C)^\sigma$ of σ-invariant line bundles.

Lemma 1 *φ induces a surjective homomorphism $\bar\varphi : (\mathbb{Z}/2)^R \to \operatorname{Pic}(C)^\sigma/\pi^*\operatorname{Pic}(B)$, whose kernel is $\mathbb{Z}/2 \cdot (1, \dots, 1)$.*

Proof Let R_C and R_B be the fields of rational functions of C and B, respectively. Let $\langle\sigma\rangle$ ($\cong \mathbb{Z}/2$) be the Galois group of the covering π. Consider the exact sequence of $\langle\sigma\rangle$-modules

$$1 \to R_C^*/\mathbb{C}^* \to \operatorname{Div}(C) \to \operatorname{Pic}(C) \to 0\ .$$

Since $H^1(\langle\sigma\rangle, R_C^*) = 0$ by Hilbert Theorem 90 and $H^2(\langle\sigma\rangle, \mathbb{C}^*) = 0$, we have $H^1(\langle\sigma\rangle, R_C^*/\mathbb{C}^*) = 0$, hence a diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_B^*/\mathbb{C}^* & \longrightarrow & \operatorname{Div}(B) & \longrightarrow & \operatorname{Pic}(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 1 & \longrightarrow & (R_C^*/\mathbb{C}^*)^\sigma & \longrightarrow & \operatorname{Div}(C)^\sigma & \longrightarrow & \operatorname{Pic}(C)^\sigma & \longrightarrow & 0 \end{array}$$

where the vertical arrows are induced by pull back.

If $R = \varnothing$, this shows that γ is surjective, hence there is nothing to prove. Assume $R \neq \varnothing$. Then γ is injective. Since $H^1(\langle\sigma\rangle, \mathbb{C}^*) = \mathbb{Z}/2$ and $(R_C^*)^\sigma = R_B^*$, the cokernel of α is $\mathbb{Z}/2$. The cokernel of β can be identified with $(\mathbb{Z}/2)^R$, so we get an exact sequence

$$0 \to \mathbb{Z}/2 \longrightarrow (\mathbb{Z}/2)^R \xrightarrow{\bar\varphi} \operatorname{Pic}(C)^\sigma/\pi^*\operatorname{Pic}(B) \to 0;$$

since $\mathcal{O}_C(R) \cong \pi^*\rho$, the vector $(1, \dots, 1)$ belongs to $\operatorname{Ker}\bar\varphi$, and therefore generates this kernel. □

Proposition 1 *Let M be a σ-invariant line bundle on C.*

(a) We have $M \cong \pi^*L(E)$ for some $L \in \operatorname{Pic}(B)$ and $E \subset R$. Any pair (L', E') satisfying $M \cong \pi^*L'(E')$ is equal to (L, E) or $(L \otimes \rho^{-1}(\pi_*E), R - E)$.
(b) There is a natural isomorphism $H^0(C, M) \cong H^0(B, L) \oplus H^0(B, L \otimes \rho^{-1}(\pi_*E))$.

Proof Part (*a*) follows directly from the Lemma. Let us prove (*b*). We view $\mathcal{O}_C(E)$ as the sheaf of rational functions on C with at most simple poles along E. Then σ induces a homomorphism $\mathcal{O}_C(E) \to \sigma_*\mathcal{O}_C(E)$, hence an involution of the rank 2 vector bundle $F := \pi_*\mathcal{O}_C(E)$; thus F admits a decomposition $F = F^+ \oplus F^-$ into eigen-subbundles for this involution. The section 1 of $\mathcal{O}_C(E)$ provides a section of F^+, which generates F^+; therefore

$F^- \cong \det F \cong \rho^{-1}(\pi_* E)$. This gives a canonical decomposition $\pi_*\mathcal{O}_C(E) \cong \mathcal{O}_B \oplus \rho^{-1}(\pi_* E)$. Taking tensor product with L and global sections gives the required isomorphism. □

3 σ-Invariant theta characteristics: the ramified case

In this section we assume $R \neq \varnothing$. We denote by b the genus of B and we put $r := g-2b+1$. By the Riemann–Hurwitz formula we have $\deg \rho = r$ and $\# R = 2r$.

We now specialize Proposition 1 to the case of theta characteristics.

Proposition 2 *Let κ be a σ-invariant theta characteristic on C.*

(a) We have $\kappa \cong \pi^* L(E)$ for some $L \in \operatorname{Pic}(B)$ and $E \subset R$ with $L^2 \cong K_B \otimes \rho\,(-\pi_* E)$. If another pair (L', E') satisfies $\kappa \cong \pi^* L'(E')$, we have $(L', E') = (L, E)$ or $(L', E') = (K_B \otimes L^{-1}, R - E)$.

(b) We have $h^0(\kappa) = h^0(L) + h^1(L)$, and the parity of κ is equal to $\deg(L) - (b-1) \pmod{2}$.

Proof (a) By Proposition 1(a) κ can be written $\pi^* L(E)$, with $L \in \operatorname{Pic}(B)$ and $E \subset R$. The condition $\kappa^2 = K_C$ translates as $\pi^*(L^2(\pi_* E)) \cong \pi^*(K_B \otimes \rho)$. Since π^* is injective (because $R \neq \varnothing$), this implies $L^2 \cong K_B \otimes \rho\,(-\pi_* E)$. The last assertion then follows from Proposition 1(a).

(b) The value of $h^0(\kappa)$ follows from Proposition 1(b), and its parity from the Riemann–Roch theorem. □

Lemma 2 *The group $(\operatorname{Pic}(C)[2])^\sigma$ of σ-invariant line bundles α on C with $\alpha^2 = \mathcal{O}_C$ is a vector space of dimension $2(g-b)$ over $\mathbb{Z}/2$.*

Proof By Lemma 1 we have an exact sequence

$$0 \to \operatorname{Pic}(B) \to \operatorname{Pic}(C)^\sigma \to (\mathbb{Z}/2)^{2r-1} \to 0. \tag{1}$$

For a $\mathbb{Z}$-module M, let $M[2] = \operatorname{Hom}(\mathbb{Z}/2, M)$ be the kernel of the multiplication by 2 in M. Note that $\operatorname{Ext}^1(\mathbb{Z}/2, M)$ is naturally isomorphic to $M/2M$. Applying $\operatorname{Hom}(\mathbb{Z}/2, -)$ to (1) gives an exact sequence of $(\mathbb{Z}/2)$-vector spaces

$$0 \to \operatorname{Pic}(B)[2] \to (\operatorname{Pic}(C)[2])^\sigma \to (\mathbb{Z}/2)^{2r-1} \to \operatorname{Pic}(B)/2\operatorname{Pic}(B) \to \operatorname{Pic}(C)^\sigma/2\operatorname{Pic}(C)^\sigma.$$

Let $p \in R$. The group $\operatorname{Pic}(B)/2\operatorname{Pic}(B)$ is generated by the class of $\mathcal{O}_B(\pi(p))$; since $\pi^*(\pi(p)) = 2p$, this class goes to 0 in $\operatorname{Pic}(C)^\sigma/2\operatorname{Pic}(C)^\sigma$. Thus the dimension of $(\operatorname{Pic}(C)[2])^\sigma$ over $\mathbb{Z}/2$ is $2b + 2r - 2 = 2(g-b)$. □

Proposition 3 *(a) The σ-invariant theta characteristics form an affine space of dimension $2(g-b)$ over $\mathbb{Z}/2$; among these, there are $2^{g-1}(2^{g-2b}+1)$ even theta characteristics and $2^{g-1}(2^{g-2b}-1)$ odd ones.*

(b) C admits (at least) $2^{g-1}\left(2^{g-2b} + 1 - 2^{-r+1}\binom{2r}{r}\right)$ vanishing thetanulls.

Proof The σ-invariant theta characteristics form an affine space under $(\operatorname{Pic}(C)[2])^\sigma$, which has dimension $2(g-b)$ by Lemma 2.

According to Proposition 2, a theta characteristic κ is determined by a subset $E \subset R$ and a line bundle L on B such that $L^2 \cong K_B \otimes \rho\,(-\pi_* E)$. This condition implies $\# E \equiv r \pmod{2}$. Moreover the parity of κ is that of $\deg(L) - (b-1) = \frac{1}{2}(r - \# E)$.

Once E is fixed we have 2^{2b} choices for L. Since E and $R \smallsetminus E$ give the same theta characteristic, we consider only the subsets E with $\# E \leq r$, counting only half of those with $\# E = r$. Thus the number of even σ-invariant theta characteristics is

$$\begin{aligned}
&2^{2b}\left[\frac{1}{2}\binom{2r}{r}+\binom{2r}{r-4}+\cdots\right]\\
&\quad=2^{2b-3}\left[(1+1)^{2r}+(-1)^r(1-1)^{2r}+(-i)^r(1+i)^{2r}+i^r(1-i)^{2r}\right]\\
&\quad=2^{2b+2r-3}+2^{2b+r-2}=2^{g-1}(2^{g-2b}+1)\ ,
\end{aligned}$$

which gives (a).

By Proposition 2(b) such a theta characteristic will be a vanishing thetanull a soon as $\deg L > b-1$, or equivalently $\#E < r$. Thus subtracting the number of theta characteristics $\kappa = \pi^*L(E)$ with $\#E = r$ we obtain (b). □

Remark 1) Note that there may be more σ-invariant vanishing thetanulls, namely those of the form $\pi^*L(E)$ with $\deg L = b-1$ but $h^0(L) > 0$. These will not occur for a general (C, σ).

2) Let $g \to \infty$ with b fixed. By the Stirling formula $\binom{2r}{r}$ is equivalent to $2^{2r}/\sqrt{\pi r}$, so $2^{-r+1}\binom{2r}{r}$ is negligible compared to $2^{g-2b} = 2^{r-1}$. Thus asymptotically we obtain $2^{2g-1-2b}$ vanishing thetanulls.

3) When $b = 0$ we recover the usual numbers for hyperelliptic curves. For $b = 1$ we obtain approximately 2^{2g-3} vanishing thetanulls, that is one fourth of the number of even theta characteristics.

4 σ-Invariant theta characteristics: the étale case

In this section we assume that σ is fixed point free ($R = \varnothing$).

Lemma 3 $(\mathrm{Pic}(C)[2])^\sigma$ *is a vector space of dimension* $g+1$ *over* $\mathbb{Z}/2$.

Proof Apply $\mathrm{Hom}(\mathbb{Z}/2, -)$ to the exact sequence

$$0 \to \mathbb{Z}/2 \to JB \xrightarrow{\pi^*} JC^\sigma \to 0.$$

□

Proposition 4 *(a) The σ-invariant theta characteristics form an affine space of dimension $g+1$ over $\mathbb{Z}/2$; among these, there are 3.2^{g-1} even theta characteristics and 2^{g-1} odd ones.*

(b) C admits a set $\mathcal{T}$ of $2^{g-2}-2^{\frac{g-3}{2}}$ σ-invariant vanishing thetanulls; it is contained in an affine subspace of dimension $g-1$ consisting of even theta characteristics.

The last property implies that for $\kappa_1, \kappa_2, \kappa_3$ in $\mathcal{T}$, the theta characteristic $\kappa_1 \otimes \kappa_2 \otimes \kappa_3^{-1}$ is even: in classical terms, $\mathcal{T}$ is *syzygetic*. The existence of these vanishing thetanulls appears already in [2].

Proof The first assertion follows from the previous Lemma. Let κ be a σ-invariant theta characteristic; we have $\kappa = \pi^*L$ for some line bundle L on C with $\pi^*L^2 = K_C = \pi^*K_B$, which implies either $L^2 = K_B \otimes \rho$ or $L^2 = K_B$. In the first case we have

$$h^0(\kappa) = h^0(L) + h^0(L \otimes \rho) = h^0(L) + h^0(K_B \otimes L^{-1}) \equiv 0 \pmod{2}.$$

Since $\pi^*L \cong \pi^*(L \otimes \rho)$, we get 2^{2b-1} even theta characteristics of C.

In the second case L is a theta characteristic on B. We recall briefly the theory of theta characteristics on a curve, as explained for instance in [3]. The group $V = \mathrm{Pic}(B)[2]$ is a vector space over $\mathbb{Z}/2$, equipped with a symplectic form e, the *Weil pairing*. A quadratic form on V associated to e is a function $q : V \to \mathbb{Z}/2$ satisfying

$$q(\alpha+\beta) = q(\alpha) + q(\beta) + e(\alpha, \beta) \; .$$

The set $\mathcal{Q}$ of such forms is an affine space over V. Now the set of theta characteristics on B is also an affine space over V, which is in fact canonically isomorphic to $\mathcal{Q}$: the isomorphism associates to a theta characteristic L the form $q_L \in \mathcal{Q}$ defined by $q_L(\alpha) = h^0(L\otimes\alpha) + h^0(L) \pmod{2}$. Moreover the parity of L is given by the Arf invariant $\mathrm{Arf}(q_L)$.

Coming back to our situation, let L be a theta characteristic on B, and $\kappa = \pi^*L$; we have

$$h^0(\kappa) = h^0(L) + h^0(L\otimes\rho) \equiv q_L(\rho) \pmod{2}.$$

The function $q \mapsto q(\rho)$ is an affine function on $\mathcal{Q}$, hence it takes equally often the values 0 and 1. Taking into account the isomorphism $\pi^*L \cong \pi^*(L\otimes\rho)$, we get 2^{2b-2} even theta characteristics on C and 2^{2b-2} odd ones; summing up we obtain (a).

Suppose $\kappa = \pi^*L$ is even, that is, $h^0(L) \equiv h^0(L\otimes\rho) \pmod{2}$; if we want $h^0(\kappa) > 0$, a good way (actually the only one if B is generic) is to choose L odd, that is, $\mathrm{Arf}(q_L) = 1$. Equivalently, we look for forms $q \in \mathcal{Q}$ with $q(\rho) = 0$ and $\mathrm{Arf}(q) = 1$.

Let ρ' be an element of V with $e(\rho, \rho') = 1$. ρ and ρ' span a plane $P \subset V$, such that $V = P \oplus P^\perp$. A form $q \in \mathcal{Q}$ is determined by its restriction to P and $P^\perp$, and we have $\mathrm{Arf}(q) = \mathrm{Arf}(q_{|P}) + \mathrm{Arf}(q_{|P^\perp})$. The condition $q(\rho) = 0$ implies $\mathrm{Arf}(q_{|P}) = q(\rho)q(\rho') = 0$; so q is determined by $q(\rho') \in \mathbb{Z}/2$ and a form q' on $P^\perp$ with Arf invariant 1. Since $\dim P^\perp = 2(b-1)$, there are $2^{b-2}(2^{b-1}-1)$ such forms, hence $2^{b-1}(2^{b-1}-1)$ forms $q \in \mathcal{Q}$ with $q(\rho) = 0$ and $\mathrm{Arf}(q) = 1$. Taking again into account the isomorphism $\pi^*L \cong \pi^*(L\otimes\rho)$, we obtain $2^{b-2}(2^{b-1}-1) = 2^{g-2} - 2^{\frac{g-3}{2}}$ vanishing thetanulls on C.

They are contained in the affine space of theta characteristics $\kappa = \pi^*L$ with $q_L(\rho) = 0$, which has dimension $2b-2 = g-1$ and consists of even theta characteristics. □

5 Low genus

Let C be a non hyperelliptic curve of genus g. How many vanishing thetanulls can C have? The answer is well-known up to genus 5. There is no vanishing thetanull in genus 3, and at most one in genus 4 (which occurs if and only if the unique quadric containing the canonical curve is singular).

Suppose $g = 5$. If C is trigonal it admits at most one vanishing thetanull. Otherwise the canonical curve $C \subset \mathbb{P}^4$ is the base locus of a net Π of quadrics. The discriminant curve (locus of the quadrics in Π of rank ≤ 4) is a plane quintic with only ordinary nodes; these nodes correspond to the rank 3 quadrics of Π, that is to the vanishing thetanulls of C. Therefore C can have any number ≤ 10 of vanishing thetanulls; they are syzygetic [1]. The maximum 10 is attained by the so-called Humbert curves, for which all the quadrics in Π can be simultaneously diagonalized. They have an action of the group $(\mathbb{Z}/2)^4$, generated by 5 involutions with elliptic quotient.

Starting with $g = 6$ very little seems to be known. By Proposition 3(b), if C is bielliptic (that is, C admits an involution with elliptic quotient), it has 40 vanishing thetanulls. This can be slightly improved as follows. We take an elliptic curve B, a line bundle α of degree

2 on B, a point $p \in B$, and disjoint divisors A in $|\alpha(p)|$, A_1, A_2, A_3 in $|\alpha|$ which do not contain p. We put $\rho = \alpha^2(p)$ and $\bar{R} = A_1 + A_2 + A_3 + A + p$, and construct the double covering $\pi : C \to B$ associated to $(\rho, \bar{R})$. The curve C has three extra vanishing thetanulls, namely $\mathcal{O}_C(\tilde{A}_i + \tilde{A}_j + \tilde{p})$ for $i < j$, where $\tilde{A}_i$ and $\tilde{p}$ are the lifts of A_i and p to C. Thus we get a genus 6 curve with 43 vanishing thetanulls; it is likely that one can do better.

References

1. Accola, R.: Some loci of Teichmüller space for genus five defined by vanishing theta nulls. Contributions to analysis (a collection of papers dedicated to Lipman Bers), pp. 11–18. Academic Press, New York (1974)
2. Farkas, H.: Automorphisms of compact Riemann surfaces and the vanishing of theta constants. Bull. Amer. Math. Soc. **73**, 231–232 (1967)
3. Mumford, D.: Theta characteristics of an algebraic curve. Ann. Sci. École Norm. Sup. **4**(4), 181–192 (1971)
4. Mumford, D.: Tata lectures on theta, II. Progress in Mathematics, vol. 43, Birkhäuser Boston, Inc., Boston (1984)