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Antisymplectic involutions of holomorphic symplectic manifolds

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Abstract

Let X be a holomorphic symplectic manifold, of dimension divisible by four, and σ be an antisymplectic involution of X. The fixed locus F of σ is a Lagrangian submanifold of X; we show that its $\hat{A}$-genus is one. As an application, we determine all possibilities for the Chern numbers of F when X is a deformation of the Hilbert square of a K3 surface.

Introduction

Let X be an irreducible holomorphic symplectic manifold admitting an antisymplectic involution σ (that is, σ changes the sign of the symplectic form). The fixed locus F of σ is a Lagrangian submanifold of X. The main observation of this note is that *when* $\dim(X)$ *is divisible by four, the $\hat{A}$-genus of F is equal to one*. Our proof, given in § 1, rests on a simple computation based on the holomorphic Lefschetz theorem.

In § 2, we apply this result when X is a symplectic four-fold with $b_2 = 23$ (this holds when X is the Hilbert square $S^{[2]}$ of a K3 surface). We show that there are exactly eleven possibilities for the pair of invariants $(K_F^2, \chi(\mathcal{O}_F))$ of the surface F, depending on the number of moduli of (X, σ). In § 3, we illustrate our results on a few examples, in particular, the *double Eisenbud–Popescu–Walter* (*EPW*) *sextics* studied by O'Grady [**9**], which form the only known family of pairs (X, σ) as above of maximal dimension twenty.

1. *The $\hat{A}$-genus of the fixed manifold*

1.1. Throughout this note, we consider an irreducible holomorphic symplectic manifold X (see [**2**]). This means that X is compact Kähler, simply connected, and admits a symplectic 2-form $\varphi \in H^0(X, \Omega_X^2)$, which generates the $\mathbb{C}$-algebra $H^0(X, \Omega_X^*)$. We denote by σ an antisymplectic involution of X (so that $\sigma^*\varphi = -\varphi$).

Lemma 1. *The fixed locus F of σ is a smooth Lagrangian submanifold of X.*

Proof. Let $x \in F$. We have a decomposition $T_x(X) = T^+ \oplus T^-$ into eigenspaces of $\sigma'(x)$. Because of the relation $\varphi_x(\sigma'(x).u, \sigma'(x).v) = -\varphi_x(u, v)$ for $u, v \in T_x(X)$, the two eigenspaces are isotropic, and therefore Lagrangian. As $T^+ = T_x(F)$, the lemma follows. □

1.2. Observe that the existence of the antisymplectic involution σ forces X to be *projective*: indeed, let $H^2(X, \mathbb{Q})^+ \subset H^2(X, \mathbb{R})^+$ be the $(+1)$-eigenspaces of σ^* in $H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R})$. The space $H^2(X, \mathbb{R})^+$ is contained in $H^{1,1}$, and contains a Kähler class; as $H^2(X, \mathbb{Q})^+$ is dense in $H^2(X, \mathbb{R})^+$, it also contains a Kähler class, which is ample.

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1.3. The *$\hat{A}$-genus* $\hat{A}(M)$ of a compact manifold M is a rational number that can be expressed as a polynomial in the Pontrjagin classes of M (see [7, § 26]). When M is a complex manifold of dimension n, we have

$$\hat{A}(M) = \int_M \operatorname{Todd}(M)\, e^{-c_1(M)/2},$$

where $\int_M : H^*(M,\mathbb{Q}) \to \mathbb{Q}$ is the evaluation on the fundamental class of M (see [7, Formula (12), p. 13]). If we extend the Euler–Poincaré characteristic χ as a $\mathbb{Q}$-linear homomorphism $K(M)\otimes\mathbb{Q} \to \mathbb{Q}$, then we have $\hat{A}(M) = \chi(\frac{1}{2}K_M)$, where K_M is the canonical bundle of M.

Theorem 1. *Let X be an irreducible symplectic manifold with $4 \mid \dim(X)$, σ be an antisymplectic involution of X and F be its fixed manifold. Then $\hat{A}(F) = 1$.*

Proof. As F is Lagrangian (Lemma 1), the symplectic form of X induces an isomorphism $T_F \xrightarrow{\sim} N^*_{F/X}$. We apply the holomorphic Lefschetz formula [1, 4.6]:

$$\sum_i (-1)^i \operatorname{Tr} \sigma^*_{|H^i(X,\mathcal{O}_X)} = \int_F \operatorname{Todd}(F)(\operatorname{ch} \wedge N^*_{F/X})^{-1} = \int_F \operatorname{Todd}(F)(\operatorname{ch}\wedge T_F)^{-1}.$$

Because X is irreducible symplectic, σ^* acts as $(-1)^i$ on $H^{2i}(X,\mathcal{O}_X)$; as $\dim(X)$ is divisible by four, this implies that the above expression is equal to one.

As usual, we write the Chern polynomial $c_t(T_F) = \prod_i (1+t\gamma_i)$, where the γ_i live in some overring of $H^*(F)$. We have

$$\operatorname{Todd}(F) = \prod_i \frac{\gamma_i}{1-e^{-\gamma_i}} \quad \text{and} \quad \operatorname{ch}(\wedge T_F) = \sum_{i_1<\ldots<i_k} e^{\gamma_{i_1}+\ldots+\gamma_{i_k}} = \prod_i (1+e^{\gamma_i}),$$

hence

$$\operatorname{Todd}(F)(\operatorname{ch}\wedge T_F)^{-1} = 2^{-n}\, e^{-c_1} \prod_i \frac{2\gamma_i}{1-e^{-2\gamma_i}}, \quad \text{with } n = \dim(X) \text{ and } c_1 = c_1(T_F).$$

Writing $\operatorname{Todd}(F) = \sum_k \operatorname{Todd}(F)_k$, with $\operatorname{Todd}(F)_k \in H^{2k}(F,\mathbb{Q})$, we find

$$\int_F \operatorname{Todd}(F)(\operatorname{ch}\wedge T_F)^{-1} = 2^{-n}\sum_k \int_F \frac{(-c_1)^k}{k!} 2^{n-k}\operatorname{Todd}(F)_{n-k} = \int_F \operatorname{Todd}(F)\, e^{-c_1/2},$$

hence $\hat{A}(F) = 1$. □

Note that the argument applies also when $\dim(X) \equiv 2 \pmod 4$ but gives the trivial equality $\hat{A}(F) = 0$.

2. *Symplectic four-folds*

2.1. When $\dim(X) = 4$, the fixed locus F is a surface (not necessarily connected). In that case $\hat{A}(F)$ is equal to $-\frac{1}{8}\operatorname{sign}(F)$, where $\operatorname{sign}(F)$ is the signature of the intersection form on $H^2(F,\mathbb{R})$ (see [7, 1.5, 1.6, and 8.2.2]); we have

$$\operatorname{sign}(F) = \tfrac{1}{3}(K_F^2 - 2e(F)) = K_F^2 - 8\chi(\mathcal{O}_F),$$

where $e(F)$ is the topological Euler characteristic of F, and we put $K_F^2 = \sum_i K_{F_i}^2$ if $F_1,\ldots,F_p$ are the connected components of F.

Therefore, Theorem 1 gives

$$\operatorname{sign}(F) = K_F^2 - 8\chi(\mathcal{O}_F) = -8 \quad \text{and} \quad K_F^2 - 2e(F) = -24.$$

We will be able to say more when the action of σ on $H^2(X)$ controls the action on $H^4(X)$, that is, when the canonical map $\mathrm{Sym}^2 H^2(X) \to H^4(X)$ is an isomorphism. By [6] this happens if and only if $b_2(X) = 23$. This is the case for one of the two families of symplectic four-folds known so far, namely the family of Hilbert schemes $S^{[2]}$ of a K3 surface S (and their deformations).

THEOREM 2. *Let X be a symplectic four-fold with $b_2(X) = 23$, σ be an antisymplectic involution of X and F be its fixed surface. Let t denote the trace of σ^* acting on $H^{1,1}(X)$.*

(a) *We have*

$$K_F^2 = t^2 - 1, \quad \chi(\mathcal{O}_F) = \tfrac{1}{8}(t^2+7), \quad e(F) = \tfrac{1}{2}(t^2+23).$$

(b) *The local deformation space of (X,σ) is smooth of dimension $\frac{1}{2}(21-t)$.*

(c) *The integer t can take any odd value with $-19 \leqslant t \leqslant 21$.*

Proof. The classical Lefschetz formula reads

$$e(F) = \sum_i (-1)^i \operatorname{Tr} \sigma^*_{|H^i(X)},$$

where we put $H^*(X) := H^*(X,\mathbb{Q})$. In the case $b_2 = 23$, the odd degree cohomology vanishes, and the natural map $\mathrm{Sym}^2 H^2(X) \to H^4(X)$ is an isomorphism [6]. Let a and b be, respectively, the dimensions of the $(+1)$- and (-1)-eigenspaces of σ^* on $H^2(X)$. We have $a+b = 23$ and $a - b = t - 2$. Then

$$\operatorname{Tr} \sigma^*_{|H^4(X)} = \tfrac{1}{2}a(a+1) + \tfrac{1}{2}b(b+1) - ab = \tfrac{1}{2}(t-2)^2 + \tfrac{23}{2}$$

and

$$e(F) = 2 + 2\operatorname{Tr}\sigma^*_{|H^2(X)} + \operatorname{Tr}\sigma^*_{|H^4(X)} = 2 + 2(t-2) + \tfrac{1}{2}(t-2)^2 + \tfrac{23}{2} = \tfrac{1}{2}(t^2+23);$$

using (2.1) we deduce the other formulas of (a).

We have $H^2(X,T_X) \cong H^2(X,\Omega^1_X) = 0$, hence the versal deformation space Def_X of X is smooth and can be locally identified with $H^1(X,T_X)$; the involution σ gives rise to an involution of Def_X, which under the above identification corresponds to σ^* acting on $H^1(X,T_X)$. Thus, the deformation space of (X,σ) is identified with the $(+1)$-eigenspace of σ^*. As $\sigma^*\varphi = -\varphi$, this eigenspace is mapped by the isomorphism

$$H^1(X,T_X) \xrightarrow{i(\varphi)} H^1(X,\Omega^1_X),$$

to the (-1)-eigenspace of σ^* in $H^1(X,\Omega^1_X)$. With the previous notation, the dimension of this eigenspace is $b - 2 = \frac{1}{2}(21-t)$, which proves (b).

Let us prove (c). As σ preserves some Kähler class, we have $a = \frac{1}{2}(t+21) \geqslant 1$, hence $t \geqslant -19$; as $\sigma^*\varphi = -\varphi$, we have $b = \frac{1}{2}(25-t) \geqslant 2$, hence $t \leqslant 21$. We construct in §§ 3.2–3.4 below examples with all possible values of t. □

COROLLARY 1. *The pair $(K_F^2, \chi(\mathcal{O}_F))$ can take any of the values* $(0,1), (8,2), (24,4), (48,7), (80,11), (120,16), (168,22), (224,29), (288,37), (360,46)$ *and* $(440,56)$.

3. *Examples*

3.1. Let S be a K3 surface and σ be an antisymplectic involution of S; it extends to an antisymplectic involution $\sigma^{[2]}$ of the Hilbert scheme $X = S^{[2]}$, which preserves the exceptional divisor E (the locus of non-reduced subschemes). We have $H^{1,1}(X) = H^{1,1}(S) \oplus \mathbb{C}[E]$, hence

$t = \operatorname{Tr} \sigma^*_{|H^{1,1}(S)} + 1$. The fixed locus of σ is a curve Γ on S (not necessarily connected); the Lefschetz formula for σ gives $t = e(\Gamma) + 1$. The list of all possibilities for Γ can be found in [**8**].

The fixed surface F of $\sigma^{[2]}$ is the union of the symmetric square $\Gamma^{(2)}$ and the quotient surface S/σ.

3.2. Let C be an irreducible plane curve of degree 6, with s ordinary double points ($0 \leqslant s \leqslant 10$) and no other singularities. Let $\pi : S' \to \mathbb{P}^2$ be the double covering of $\mathbb{P}^2$ branched along C, S be the minimal resolution of S' and σ be the involution of S that exchanges the sheets of π. The fixed locus Γ of σ is the normalization of C; thus, $e(\Gamma) = -18 + 2s$ and $t = -17 + 2s$.

3.3. For each integer r with $1 \leqslant r \leqslant 10$, there exists a K3 surface S and an involution of S whose fixed locus is the disjoint union of r rational curves [**8**]. Then $e(\Gamma) = 2r$ and $t = 2r + 1$. Together with the previous example, this gives all integers t appearing in Theorem 2(c), except $t = -19$.

3.4. The case $t = -19$ is particularly interesting, because, when it holds, the deformation space of (X, σ) has maximal dimension twenty (Theorem 2(b)). The space $H^2(X, \mathbb{Q})^+$ is one-dimensional, generated by an ample class (1.2); the deformation space of (X, σ) coincides locally with the deformation space of X as a polarized variety. We know only one example of this situation: O'Grady has constructed a twenty-dimensional family of projective symplectic four-folds, which are double coverings of certain sextic hypersurfaces in $\mathbb{P}^5$, called EPW sextics [**9**]. The corresponding involution is antisymplectic and must satisfy $t = -19$ by Theorem 2(b). The fixed surface F is connected, and from Theorem 2(a) we recover the invariants $K_F^2 = 360$, $\chi(\mathcal{O}_F) = 46$ already obtained in [**10**].

3.5. As explained in [**5**] (I am indebted to O'Grady for pointing out this paper to me, thus correcting an inaccurate remark in the first version of this note.), the above pairs (X, σ) specialize to $(S^{[2]}, \tau)$, where S is a smooth quartic surface in $\mathbb{P}^3$ that contains no line and τ associates to a length 2 subscheme $z \in S^{[2]}$, the residual subscheme in the intersection of S, and the line spanned by z. The fixed locus becomes the surface B of bitangents to S; this explains why B has the same invariants $K_B^2 = 360$, $\chi(\mathcal{O}_B) = 46$, as already observed by Welters [**11**].

3.6. There are many other examples, which give rise to interesting exercises. Here is one: we start with the involution ι of $\mathbb{P}^5$ given by $\iota(X_0, \ldots, X_5) = (-X_0, X_1, \ldots, X_5)$. Let $V \subset \mathbb{P}^5$ be a smooth cubic three-fold invariant under ι: its equation must be of the form $X_0^2 L(X_1, \ldots, X_5) + G(X_1, \ldots, X_5) = 0$, where L is linear and G cubic. The Fano variety X of lines contained in V is a symplectic four-fold [**3**], and ι defines an involution σ of X.

The fixed points of ι in $\mathbb{P}^5$ are $p = (1, 0, \ldots, 0)$ and the hyperplane H given by $X_0 = 0$. A line $\ell \in X$ is preserved by ι if and only if it contains at least two fixed points; this means that either ℓ contains p, or it is contained in H. The lines passing through p are parametrized by the cubic surface $S \subset H$ given by $L = G = 0$; the lines contained in H form the Fano surface T of the cubic three-fold $G = 0$ in H. Thus, the fixed surface F of σ is the disjoint union of S and T.

Using the canonical isomorphism $H^{1,1}(X) \xrightarrow{\sim} H^{2,2}(V)$ (see [**3**]) and Griffiths' description of the cohomology of the hypersurface V, one finds easily $t = -7$. Then Theorem 2(a) gives $K_F^2 = 48$ and $\chi(\mathcal{O}_F) = 7$. As $K_S^2 = 3$ and $\chi(\mathcal{O}_S) = 1$, we recover the values $K_T^2 = 45$ and $\chi(\mathcal{O}_T) = 6$ (see [**4**]).

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