Algebraic Geometry
The Coble hypersurfaces
Arnaud Beauville

Abstract
Let $A$ be an indecomposable principally polarized abelian variety of dimension $g$. Third order theta functions embed $A$ in a projective space $\mathbb{P}(V_3)$ of dimension $3g - 1$, while second order theta functions embed the Kummer variety $X = A/\{\pm 1\}$ in a projective space $\mathbb{P}(V_2)$ of dimension $2g - 1$. Coble observed that for $g = 2$ there is a unique cubic hypersurface in $\mathbb{P}(V_3)$ that is singular along $A$, and for $g = 3$ a unique quartic hypersurface in $\mathbb{P}(V_2)$ singular along $X$. We explain these facts by a simple analysis of the representations of the corresponding Heisenberg group.

Résumé
Les hypersurfaces de Coble. Soit $A$ une variété abélienne principalement polarisée indécomposable, de dimension $g$. Les fonctions thêta d’ordre 3 plongent $A$ dans un espace projectif $\mathbb{P}(V_3)$ de dimension $3g - 1$, tandis que les fonctions thêta d’ordre 2 plongent la variété de Kummer $X = A/\{\pm 1\}$ dans un espace projectif $\mathbb{P}(V_2)$ de dimension $2g - 1$. Coble a observé que pour $g = 2$ il existe une unique hypersurface cubique dans $\mathbb{P}(V_3)$ qui est singulière le long de $A$, et pour $g = 3$ une unique hypersurface quartique dans $\mathbb{P}(V_2)$ singulière le long de $X$. Nous expliquons ces faits par une analyse élémentaire des représentations du groupe de Heisenberg correspondant.

Version française abrégée
Soit $(A, L)$ une variété abélienne principalement polarisée, indécomposable, de dimension $g$. Pour $v$ entier $\geq 1$, on pose $V_v = H^0(A, L^v)$. Le morphisme naturel $\varphi_v : A \to \mathbb{P}(V_v)$ est un plongement pour $v \geq 3$ ; pour $v = 2$ il induit un plongement de la variété de Kummer $A/\{\pm 1\}$ dans $\mathbb{P}(V_2)$. Coble [3,4] a observé que pour $g = 2$ il existe une unique hypersurface cubique dans $\mathbb{P}(V_3)$ qui est singulière le long de $\varphi_3(A)$, et pour $g = 3$ une unique hypersurface quartique dans $\mathbb{P}(V_2)$ singulière le long de $\varphi_2(A)$ ; plus récemment ces hypersurfaces ont été interprétées en termes de fibrés vectoriels sur des courbes [10,11].

Nous allons montrer que ces faits sont une conséquence d’un résultat général et élémentaire sur les représentations du groupe de Heisenberg. Notons $A_v$ le noyau de la multiplication par $v$ dans $A$. Ce groupe agit sur
A par translations en préservant le fibré $\mathcal{L}^\nu$, donc agit sur $\mathbb{P}(V_\nu)$; cette action se relève en une action sur $V_\nu$ d’une extension centrale $A_\nu$ de $A_\nu$ par $\mathbb{C}^*$. Posons $n = v$ si $v$ est impair, $n = 2v$ si $v$ est pair. Pour tout $\gamma \in A_\nu$, l’élément $\gamma^n$ appartient au centre $\mathbb{C}^*$ de $A_\nu$, et l’application $\gamma \mapsto \gamma^n$ est un homomorphisme de $A_\nu$ sur $\mathbb{C}^*$. Notons $H_n$ son noyau ; c’est une extension centrale

$$1 \to \mu_n \to H_n \to A_\nu \to 0$$

de $A_\nu$ par le groupe $\mu_n$ des racines $n$-ièmes de l’unité dans $\mathbb{C}$.

Notons $V = V_\nu$, et choisissons un système de coordonnées $(T_1, \ldots, T_N)$ sur $\mathbb{P}(V)$ ($N = v^g$).

**Proposition 0.1.** On suppose $n = 3$ ou $4$. Soit $W$ un sous-$H_n$-module irréductible de $S^{n-1}V$. Il existe une forme $H_n$-invariante $F \in S^nV$, unique à un scalaire près, telle que $(\partial F/\partial T_1, \ldots, \partial F/\partial T_N)$ soit une base de $W$.

**Idée de la démonstration.** La représentation de $H_n$ sur $V$ est l’unique représentation irréductible de $H_n$ dans laquelle le centre $\mu_n$ agit par homothéties. Il en résulte que la représentation de $H_n$ sur $S^{n-1}V$ est la somme de $k$ copies de $V^*$, avec

$$k = \dim S^{n-1}V/\dim V^* = \frac{1}{N} \left( \frac{N + n - 2}{n - 1} \right).$$

L’espace $\text{Hom}_{H_n}(V^*, S^{n-1}V)$ est de dimension $k$ ; il paramètre les sous-$H_n$-modules simples de $S^{n-1}V$.

Considérons l’application $H_n$-équivariante injective

$$h : S^nV \to \text{Hom}(V^*, S^{n-1}V)$$

donnée par $h(F)(\partial) = \partial F$ (on identifie $V^*$ à l’espace des dérivées de degré $-1$ de $SV$). Elle induit une injection $(S^nV)^H \hookrightarrow \text{Hom}_{H_n}(V^*, S^{n-1}V)$ des sous-espaces $H_n$-invariants. Un calcul élémentaire prouve alors que ces deux espaces ont la même dimension, ce qui entraîne la proposition. □

Soit $X$ une sous-variété de $\mathbb{P}(V)$, invariante sous $A_\nu$, et soit $\mathcal{I}_X$ son faisceau d’idéaux dans $\mathbb{P}(V)$. Il résulte de la Proposition 0.1 que si $(F_1, \ldots, F_m)$ est une base de l’espace des formes $H_n$-invariantes de degré $n$ qui sont singulières le long de $X$, les dérivées partielles $(\partial F_i/\partial T_j)$ forment une base de $H^0(\mathbb{P}(V), \mathcal{I}_X(n - 1))$. En particulier, si $\dim H^0(\mathbb{P}(V), \mathcal{I}_X(n - 1)) = v^g$, il existe (à un scalaire près) une unique forme $H_n$-invariante de degré $n$ singulière le long de $X$. On voit facilement que c’est le cas dans les deux exemples de Coble. De plus, dans ces deux cas, un argument simple montre qu’il n’existe pas d’autre forme de degré $n$ singulière le long de $X$.

**1. Introduction**

The title of this Note refers to the following nice observations of Coble. Let $A$ be a complex abelian variety, of dimension $g$, and $\mathcal{L}$ a line bundle on $A$ defining a principal polarization (that is, $\mathcal{L}$ is ample and $\dim H^0(A, \mathcal{L}) = 1$). We will assume throughout that $(A, \mathcal{L})$ is indecomposable, that is, cannot be written as a product of principally polarized abelian varieties of lower dimension.

Fix an integer $v \geq 1$ and put $V_\nu = H^0(A, \mathcal{L}^\nu)$. We consider the morphism $\varphi_\nu : A \to \mathbb{P}(V_\nu)$ defined by the global sections of $\mathcal{L}^\nu$. Recall that $\varphi_\nu$ is an embedding for $v \geq 3$, and that $\varphi_2$ induces an embedding of the Kummer variety $A/\{\pm 1\}$ in $\mathbb{P}(V_2)$. Let $A_\nu$ be the kernel of the multiplication by $v$ in $A$; the group $A_\nu$ acts on $A$ and on $\mathbb{P}(V_\nu)$ in such a way that $\varphi_\nu$ is $A_\nu$-equivariant.

---

1 We use Grothendieck’s notation: $\mathbb{P}(V_\nu)$ is the space of hyperplanes of $V_\nu$. 
Proposition 1.1 (Coble). (1) Let \( g = 2 \). There exists a unique \( A_2 \)-invariant cubic hypersurface in \( \mathbb{P}(V_3) \) (\( \cong \mathbb{P}^8 \)) that is singular along \( \varphi_3(A) \). The polars of this cubic span the space of quadrics in \( \mathbb{P}(V_3) \) containing \( \varphi_3(A) \).

(2) Let \( g = 3 \). There exists a unique \( A_2 \)-invariant quartic hypersurface in \( \mathbb{P}(V_2) \) (\( \cong \mathbb{P}^7 \)) that is singular along \( \varphi_2(A) \). The polars of this quartic span the space of cubic hypersurfaces in \( \mathbb{P}(V_2) \) containing \( \varphi_2(A) \).

The proof of (2) appears in [4], and that of (1) in [3] (actually the cubic is not explicitly mentioned in that paper, but it is easily deduced from the equations for the quadrics containing \( \varphi_3(A) \). I am indebted to I. Dolgachev for this reference). Both results are proved by explicit computations. These hypersurfaces have a beautiful interpretation in terms of vector bundles on curves (see [10] for the quartic and [11] for the cubic).

An analogous statement appears in [12], this time for the moduli space \( SU_C(2) \) of semi-stable rank 2 vector bundles with trivial determinant on a curve of genus 4 with no vanishing theta-constant (this moduli space is naturally embedded in \( \mathbb{P}(V_3) \)). Oxbury and Pauly prove that it is contained in a unique \( A_2 \)-invariant quartic hypersurface, whose polars span the space of cubic hypersurfaces containing \( SU_C(2) \).

The main observation of this note is that these facts follow from a general (and elementary) result about representations of the Heisenberg group (Proposition 2.1 below). Let us just mention here a geometric consequence of that result:

Proposition 1.2. Let \( n = 3 \) or \( 4 \); put \( v = 3 \) if \( n = 3 \), \( v = 2 \) if \( n = 4 \). Let \( (T_1, \ldots, T_N) \) be a coordinate system on \( \mathbb{P}(V_v) \). Let \( X \) be an \( A_v \)-invariant subvariety of \( \mathbb{P}(V_v) \). Then the space of hypersurfaces of degree \( n - 1 \) containing \( X \) admits a basis \( (\partial F_i/\partial T_j) \), where \( F_1, \ldots, F_m \) are forms of degree \( n \) on \( \mathbb{P}(V_v) \), such that the hypersurfaces \( F_i = 0 \) are \( A_v \)-invariant (and singular along \( X \)).

2. Heisenberg submodules of \( S^{n-1}V \)

Let \( n \) be an integer; we put \( v = n \) if \( n \) is odd, \( v = n/2 \) if \( n \) is even. We write for brevity \( V \) instead of \( V_v \). We will occasionally pick a coordinate system \( (T_1, \ldots, T_N) \) on \( \mathbb{P}(V_v) \), to make some of our statements more concrete.

The action of \( A_v \) on \( \mathbb{P}(V) \) lifts to an action on \( V \) of a central extension \( \tilde{A}_v \) of \( A_v \) by \( \mathbb{C}^* \). For all \( y \in \tilde{A}_v \), the element \( y^n \) belongs to the center \( \mathbb{C}^* \) of \( \tilde{A}_v \), and the map \( y \mapsto y^n \) is a homomorphism of \( \tilde{A}_v \) onto \( \mathbb{C}^* \) (this is where we need to take \( n = 2v \) instead of \( v \) when \( v \) is even). We denote by \( H_n \) its kernel; it is a central extension

\[
1 \rightarrow \mu_n \rightarrow H_n \rightarrow A_v \rightarrow 0
\]

of \( A_v \) by the group \( \mu_n \) of \( n \)th roots of unity in \( \mathbb{C} \).

Proposition 2.1. Assume \( n = 3 \) or \( 4 \). Let \( W \) be an irreducible sub-
\( H_n \)-module of \( S^{n-1}V \). There exists a \( H_n \)-invariant form \( F \in S^2V \), unique up to a scalar, such that \( (\partial F/\partial T_1, \ldots, \partial F/\partial T_N) \) form a basis of \( W \).

Proof. Put \( N = \dim V (\cong v^N) \). The group \( H_n \) acts irreducibly on \( V \), and this is the unique irreducible representation of \( H_n \) on which the center \( \mu_n \) acts by homotheties. It follows that the representation of \( H_n \) on \( S^{n-1}V \) is isomorphic to the direct sum of \( k \) copies of \( V^* \), with

\[
k = \dim S^{n-1}V / \dim V^* = \frac{1}{N} \left( \frac{N + n - 2}{n - 1} \right).
\]

The space \( \text{Hom}_{H_n}(V^*, S^{n-1}V) \) has dimension \( k \); it parametrizes the irreducible sub-
\( H_n \)-modules of \( S^{n-1}V \).

Consider the \( H_n \)-equivariant injective map

\[
h : S^2V \rightarrow \text{Hom}(V^*, S^{n-1}V)
\]
given by \( h(F)(\bar{\partial}) = \partial F \) (we identify \( V^* \) with the space of degree \(-1\) derivations of \( SV \)). It induces an injection \((S^aV)^{H_n} \hookrightarrow \text{Hom}_{H_n}(V^*, S^{a-1}V)\) of the \( H_n \)-invariant subspaces. The assertion of the proposition is that this map is onto, or equivalently that \( \dim(S^aV)^{H_n} = \frac{1}{n}(N+n-2) \).

The action of \( H_n \) on \( S^aV \) factors through the abelian quotient \( A_v \), hence is the direct sum of 1-dimensional representations \( V_{\chi} \) corresponding to characters \( \chi \) of \( A_v \). We claim that all non-trivial characters of \( A_v \) appear with the same multiplicity. To see this, consider the group \( \text{Aut}(H_n, \mu_n) \) of automorphisms of \( H_n \) which induce the identity on \( \mu_n \). Because of the unicity property of the representation \( \rho : H_n \to GL(V) \), for every \( \phi \in \text{Aut}(H_n, \mu_n) \) the representation \( \rho \circ \phi \) is isomorphic to \( \rho \), thus \((S^a\rho) \circ \phi \) is isomorphic to \( S^a\rho \). This implies that the characters appearing in the decomposition of \( S^aV \) are exchanged by the action of \( \text{Aut}(H_n, \mu_n) \). But the action of \( \text{Aut}(H_n, \mu_n) \) on \( A_v \) factors through a surjective homomorphism \( \text{Aut}(H_n, \mu_n) \to \text{Sp}(A_v) \) (see e.g. [2], Ch. 6, Lemma 6.6). Since \( v \) is prime, the symplectic group \( \text{Sp}(A_v) \) acts transitively on the set of nontrivial characters of \( A_v \), hence our claim.

Thus we have

\[
S^aV = \left( \bigoplus_{\chi \neq 1} V_{\chi} \right)^m \oplus (S^aV)^{H_n}
\]

for some integer \( m \geq 0 \). Counting dimensions yields

\[
\frac{(N + n - 1)}{n} = m(N^2 - 1) + \dim(S^aV)^{H_n}.
\]

On the other hand a simple computation gives

\[
\frac{(N + n - 1)}{n} = m(N^2 - 1) + \frac{1}{N} \left( \frac{(N + n - 2)}{n - 1} \right),
\]

with \( m = \frac{1}{N}(N + 3) \) for \( n = 3 \), and \( m = \frac{1}{N}(N + 2)(N + 4) \) for \( n = 4 \). Moreover we have \( \dim(S^aV)^{H_n} \leq \frac{1}{N} \left( \frac{(N + n - 2)}{n - 1} \right) < N^2 - 1 \). Thus \( \dim(S^aV)^{H_n} \) and \( \frac{1}{N} \left( \frac{(N + n - 2)}{n - 1} \right) \) are both equal to the rest of the division of \( \binom{N+n-1}{n} \) by \( N^2 - 1 \), hence they are equal. \( \square \)

**Remarks.** (1) Unfortunately the cases \( n = 3 \) and \( n = 4 \) seem to be the only ones for which the proposition holds. If for instance \( n \) is prime \( \geq 5 \), it is easy to check that the equality \( \dim(S^aV)^{H_n} = \frac{1}{N}(N+n-2) \) never holds.

(2) The case \( n = 4 \) could also easily be deduced from [7], Proposition 2.

(3) The result holds more generally in characteristic \( \neq p \), with the same proof (the representation theory of a \( p \)-group in characteristic \( \neq p \) is isomorphic to its theory of complex representations). Therefore the results below hold in characteristic \( \neq p \), with the possible exception of Proposition 3.1 which uses a result of Donagi whose proof requires the characteristic to be zero.

**Corollary 2.2.** Let \( X \) be a subvariety of \( \mathbb{P}(V) \), invariant under the action of \( A_v \); denote by \( I_X \) the ideal sheaf of \( X \) in \( \mathbb{P}(V) \). Let \( (F_1, \ldots, F_m) \) be a basis of the space of \( H_n \)-invariant forms in \( S^aV \) which are singular along \( X \). Then the partial derivatives \( (\partial F_i/\partial T_j) \) form a basis of \( H^0(\mathbb{P}(V), I_X(n-1)) \). In particular, if \( \dim H^0(\mathbb{P}(V), I_X(n-1)) = v^k \), there exists a unique \( H_n \)-invariant form in \( S^aV \) which is singular along \( X \).

Indeed \( H^0(\mathbb{P}(V), I_X(n-1)) \) is a sub-\( H_n \)-module of \( H^0(\mathbb{P}(V), O_{\mathbb{P}(V)}(n-1)) = S^{a-1}V \), and therefore isomorphic to a direct sum of simple modules. \( \square \)

In the next section we will apply the corollary to the abelian variety \( A \) embedded in \( \mathbb{P}(V) \). Another interesting case is when \( X \) is the moduli space of vector bundles of rank 2 and trivial determinant on a curve \( C \) of genus 4 with no vanishing theta-constant. Let \( A \) be the Jacobian of \( C \); then \( X \) has a natural \( A_2 \)-equivariant embedding in \( \mathbb{P}(V_2) \), and Oxbury and Pauly prove the equality \( \dim H^0(\mathbb{P}(V_2), I_X(3)) = 8 \) [12]. Therefore there exists a unique \( H_n \)-invariant quartic hypersurface singular along \( X \).
3. Application: equations for abelian varieties

(3.1) Let us apply Corollary 2.2 to \( X = \varphi_v(A) \) embedded in \( \mathbb{P}(V_v) \). If \( n = 4 \) we will assume that \( (A, \mathcal{L}) \) has no vanishing theta-constant (that is, no symmetric theta divisor singular at 0 – if \( g = 3 \) this simply means that \( (A, \mathcal{L}) \) is the Jacobian of a non-hyperelliptic curve). This implies that the Kummer variety \( \varphi_v(A) \subset \mathbb{P}(V_v) \) is projectively normal, while \( \varphi_1(A) \) is always projectively normal in \( \mathbb{P}(V_1) \) [8]. Thus the natural map \( H^0(\mathbb{P}(V_v),\mathcal{O}_v(n-1)) \rightarrow H^0(X,\mathcal{O}_X(n-1)) \) is surjective, and this allows us to compute the dimension of its kernel. We find that the space of \( H_n \)-invariant forms in \( \mathcal{S}^n V \) singular along \( X \) has dimension \( m_\nu(g) \) given by

\[
m_3(g) = \frac{1}{4}(3^g - 2^{g+1} + 1), \quad m_4(g) = \frac{1}{6}(2^g(2^g + 3) - 3^{g+1} - 1);
\]

for any basis \((F_1, \ldots, F_{m_\nu(g)})\) of this space, the derivatives \((\partial F_i/\partial T_j)\) form a basis of the space of forms of degree \( n - 1 \) vanishing along \( X \).

(3.2) Let us consider in particular the case \( g = n - 1 \) considered by Coble. Since \( m_3(2) = m_4(3) = 1 \) we recover Coble’s result: there is a unique \( H_n \)-invariant hypersurface of degree \( n \) singular along \( \varphi_v(A) \). In fact we have a slightly better result:

**Proposition 3.1.** Assume \( g = n - 1 \). The Coble hypersurface in \( \mathbb{P}(V_v) \) is the unique hypersurface of degree \( n \) singular along \( \varphi_v(A) \).

**Proof.** The case of the Coble quartic is explained in [9], and the proof works equally well for the cubic. Let us recall briefly the argument. Let \( F = 0 \) be the Coble hypersurface. The derivatives \( \partial F/\partial T_1, \ldots, \partial F/\partial T_N \) span the space \( I_{n-1} \) of forms of degree \( n - 1 \) vanishing along \( \varphi_v(A) \); the action of \( H_n \) on \( I_{n-1} \) is irreducible.

Let \( W \) be the space of forms of degree \( n \) which are singular along \( \varphi_v(A) \); it is a sub-\( H_n \)-module of \( \mathcal{S}^n V \), hence a sum of one-dimensional representations \( W_f \). Let \( G \neq 0 \) in \( W_f \). The derivatives \( \partial G/\partial T_1, \ldots, \partial G/\partial T_N \) vanish on \( \varphi_v(A) \), hence span a subspace of \( I_{n-1} \); since this subspace is stable under \( H_n \), it is equal to \( I_{n-1} \). By [5], §1, this implies that there exists an automorphism \( T \) of \( V_v \) such that \( G = F \circ T \).

Now the singular locus of the Coble hypersurface is exactly \( \varphi_v(A) \) (see (3.3) below); thus \( T \) must preserve \( \varphi_v(A) \). In the group of automorphisms of \( V_v \) preserving \( \varphi_v(A) \), the Heisenberg group \( H_n \) is normal – because the group of translations of \( A \) is normal inside the group of all automorphisms. Thus \( T \) normalizes \( H_n \); this implies that the form \( G = F \circ T \) is \( H_n \)-invariant, and therefore proportional to \( F \) by Coble’s result. \( \square \)

(3.3) For \( g = 2 \), Coble states in [3] that \( \varphi_3(A) \) is the set-theoretical intersection of the quadrics that contain it – in other words, \( \varphi_3(A) \) is the singular locus of the Coble cubic; this is proved even scheme-theoretically in [11]. When \( g = 3 \) and \( (A, \mathcal{L}) \) has no vanishing theta-constant, Narasimhan and Ramanan have proved that the Kummer variety \( \varphi_2(A) \) is set-theoretically the singular locus of the Coble quartic [10]; this holds also scheme-theoretically by [9]. It is tempting to conjecture that both statements hold in higher dimension as well, namely that the abelian variety \( \varphi_3(A) \) is a scheme-theoretical intersection of quadrics and that the Kummer variety \( \varphi_2(A) \) is a scheme-theoretical intersection of cubics. However these quadrics or cubics cannot generate the full ideal of \( \varphi_v(A) \):

**Proposition 3.2.** The graded ideal \( I \) of \( \varphi_v(A) \) in \( \mathbb{P}(V_v) \) is not generated by its elements of degree \( \leq n - 1 \).

(Recall that \( I \) is generated by its elements of degree \( \leq n \), see [2], Ch. 7 and [6].)

Note that the proposition is immediate in the case \( g = n - 1 \) considered by Coble, because then \( \dim(V \otimes I_{n-1}) < \dim I_n \). However this inequality does not hold any more in higher genus.
Proof. We will prove the inequality \( \dim (V \otimes I_{n-1})^{H_3} < \dim (I_3)^{H_3} \), which implies that the multiplication map \( V \otimes I_{n-1} \to I_n \) cannot be surjective. Let us treat first the case \( n = 3 \). From the exact sequence \( 0 \to I_3 \to S^3 V \to H^0(A, L^9) \to 0 \) (3.1) we get

\[
\dim I_3 = \left( \frac{N + 2}{3} \right) - N^2 = \frac{N - 3}{6} (N^2 - 1) + \frac{N - 1}{2};
\]

as in Proposition 2.1 we conclude that \( \dim (I_3)^{H_3} = (N - 1)/2 \).

Let \( K \subseteq S^3 V \) be the space of \( H_3 \)-invariant cubic forms singular along \( \varphi_3(A) \); by the proposition the natural map \( V^* \otimes K \to I_2 \) is an isomorphism. The action of \( H_3 \) on \( K \) is trivial, and the \( H_3 \)-module \( V \otimes V^* \) is the direct sum of a one-dimensional factor for each character of \( A_3 \); thus

\[
\dim (V \otimes I_2)^{H_3} = \dim K = \frac{1}{6} (3^g - 2^{g+1} + 1) < \frac{1}{6} (N - 1)
\]
(3.1), hence the result.

For \( n = 4 \) the same method gives \( \dim (I_4)^{H_2} = \frac{1}{6} (N - 1)(N - 2) \), which is larger than \( \dim (V \otimes I_3)^{H_2} = \frac{1}{6} (N(N + 3) - 3^{g+1} - 1) \). \( \square \)

References