Moduli of cubic surfaces and Hodge theory

[*After Allcock, Carlson, Toledo*]

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Introduction

This is a detailed version of three lectures given at the annual meeting of the Research Group "Complex Algebraic Geometry" at Luminy in October 2005. The aim was to explain, in a way as elementary as possible, the work of Allcock, Carlson, Toledo [ACT] which describes, using Hodge theory, the moduli space of cubic surfaces in $\mathbf{P}^3$ as a quotient of the complex ball in $\mathbf{C}^4$. That work uses a number of different techniques which are quite basic in algebraic geometry: Hodge theory of course, monodromy, differential study of the period map, geometric invariant theory, Torelli theorem for the cubic threefold ... One of our aims is to explain these techniques by illustrating how they work in a concrete and relatively simple situation.

As a result, these notes are quite different from the original paper [ACT]. While that paper contains a wealth of interesting and difficult results (on the various moduli spaces which can be considered, the corresponding monodromy group, their description by generators and relations), we have concentrated on the main theorem and the basic methods involved, at the cost of being sometimes sketchy on the technical details of the proof. We hope that these notes may serve as an introduction to this nice subject.

In the next section we will motivate the construction by discussing a more complicated but more classical case, namely *quartic* surfaces in $\mathbf{P}^3$. In § 2 we will explain the main result; at the end of that section we will explain the strategy of the proof, and at the same time the plan of these notes.

1. Motivation: the case of quartic surfaces

As announced, we start by recalling briefly the description of the moduli space of quartic surfaces in $\mathbf{P}^3$. References include [BHPV], [X], or [B2] for a short introduction.

(1.1) A quartic surface $\mathrm{S} \subset \mathbf{P}^3$ is a K3 surface, which means that it admits a unique (up to a scalar) holomorphic 2-form ω, which is non-zero at every point. The only interesting cohomology of S is the lattice $\mathrm{H}^2(\mathrm{S},\mathbf{Z})$, endowed with the unimodular symmetric bilinear form defined by the cup-product. Moreover the vector space $\mathrm{H}^2(\mathrm{S},\mathbf{C}) = \mathrm{H}^2(\mathrm{S},\mathbf{Z}) \otimes \mathbf{C}$ admits a Hodge decomposition

$$\mathrm{H}^2(\mathrm{S},\mathbf{C}) = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2} \ ,$$

which is determined by the position of the line $H^{2,0} = \mathbf{C}\omega$ in $H^2(S, \mathbf{C})$ (we have $H^{0,2} = \overline{H^{2,0}}$, and $H^{1,1}$ is the orthogonal of $H^{2,0} \oplus H^{0,2}$). The point is that $H^2(S, \mathbf{C})$ depends only on the topology of S, while the position of $H^{2,0} = \mathbf{C}\omega$ depends heavily on the complex structure.

To be more precise, we denote by L a lattice isomorphic to $H^2(S, \mathbf{Z})$ for all S. We fix a vector $h_0 \in L$ of square 4 (they are all conjugate under $O(L)$). A *marked* quartic surface is a pair (S, σ) of a quartic S and an isometry $\sigma : L \xrightarrow{\sim} H^2(S, \mathbf{Z})$ such that $\sigma(h_0) = h$, the class in $H^2(S, \mathbf{Z})$ of a plane section. We denote by $\widetilde{\mathcal{M}}$ the moduli space of marked quartic surfaces; it is not difficult to see that it is a complex manifold. The group Γ of automorphisms of L which fix h_0 acts on $\widetilde{\mathcal{M}}$ by $\gamma \cdot (S, \sigma) = (S, \sigma \circ \gamma^{-1})$; the quotient $\mathcal{M} := \widetilde{\mathcal{M}}/\Gamma$ is the usual moduli space of quartic surfaces, that is, the open subset of $\mathbf{P}(H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(4))$ parameterizing smooth quartic surfaces modulo the action of the linear group $PGL(4)$.

(1.2) The advantage of working with $\widetilde{\mathcal{M}}$ is that we can now compare the Hodge structures of different surfaces. Given (S, σ), we extend σ to an isomorphism $L_{\mathbf{C}} \xrightarrow{\sim} H^2(S, \mathbf{C})$ and put

$$\tilde{\wp}(S, \sigma) = \sigma^{-1}(H^{2,0}) = \sigma^{-1}([\omega]) \in \mathbf{P}(L_{\mathbf{C}}) \ .$$

The map $\tilde{\wp}$ is called the *period map*, for the following reason: choose a basis $(e_1, \ldots, e_{22})$ of L^*, so that $L_{\mathbf{C}}$ is identified with $\mathbf{C}^{22}$. Put $\gamma_i = {}^t\sigma^{-1}(e_i)$, viewed as an element of $H_2(S, \mathbf{Z})$; then

$$\tilde{\wp}(S, \sigma) = \Big(\int_{\gamma_1} \omega \ : \ \ldots \ : \ \int_{\gamma_{22}} \omega \Big) \ \in \mathbf{P}^{21} \ .$$

The numbers $\int_{\gamma_i} \omega$ are classically called the "periods" of ω.

Since ω is holomorphic we have $\omega \wedge \omega = 0$, and $\int_S \omega \wedge \bar{\omega} > 0$; moreover, since ω is of type (2,0) and h of type (1,1), we have $\omega . h = 0$ in $H^2(S, \mathbf{C})$. In other words, $\tilde{\wp}(S, \sigma)$ lies in the subvariety Ω of $\mathbf{P}(L_{\mathbf{C}})$, called the *period domain*, defined by

$$\Omega = \{ [x] \in \mathbf{P}(L_{\mathbf{C}}) \mid x^2 = x.h_0 = 0 \ , \ x.\bar{x} > 0 \} \ .$$

The action of Γ on $L_{\mathbf{C}}$ preserves Ω, and the map $\tilde{\wp}$ is Γ-equivariant. Thus we have a commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\tilde{\wp}} & \Omega \\ \big\downarrow & & \big\downarrow \\ \mathcal{M} & \xrightarrow{\wp} & \Omega/\Gamma \end{array} \ .$$

Theorem 1.3 .– *The maps* $\tilde{\wp}$ *and* $\wp$ *are open embeddings.*

Moreover we have an explicit description of the image of $\tilde{\wp}$ (and therefore of that of $\wp$): its complement is a countable, locally finite union of hyperplanes $(\mathrm{H}_\delta)_{\delta\in\Delta}$, where Δ is a certain subset of L and $\mathrm{H}_\delta = \{[x]\in\Omega \mid (x.\delta)=0\}$.

This description of the moduli space $\mathcal{M}$ (and the analogous one for all families of projective K3 surfaces) has many geometric applications: automorphisms of the surfaces, structure of the Picard group, geometry of the moduli space, etc.

(1.4) In degree greater than 4, things become much more complicated. We still have a period map $\wp : \mathcal{M} \to \Omega/\Gamma$, where Ω is an open subset of an orthogonal grassmannian and Γ a discrete subgroup; but the dimension of the period domain becomes much larger than that of $\mathcal{M}$. The map $\wp$ is known to be generically injective [Do], but this is not enough to extract interesting geometric information – e.g. what are the possible Picard groups, automorphism groups, etc.

We now go back to the case of interest to us – *cubic* surfaces.

2. Statement of the main result

(2.1) For a cubic surface $\mathrm{S}\subset \mathbf{P}^3$, the cohomology does not carry any interesting structure: there are no holomorphic 2-forms, so we have $\mathrm{H}^2(\mathrm{S},\mathbf{C}) = \mathrm{H}^{1,1}$.

We will consider instead the Hodge structure of another variety, canonically associated to S: the triple cyclic covering V of $\mathbf{P}^3$ branched along S. If S is defined by an equation $\mathrm{F}(\mathrm{X}_0,\ldots,\mathrm{X}_3)=0$, V is the cubic hypersurface in $\mathbf{P}^4$ with equation $\mathrm{X}_4^3 = \mathrm{F}(\mathrm{X}_0,\ldots,\mathrm{X}_3)$. We denote by σ the automorphism $(\mathrm{X}_0,\ldots,\mathrm{X}_4)\mapsto(\mathrm{X}_0,\ldots,\mathrm{X}_3,\rho\mathrm{X}_4)$ of V, with $\rho = e^{2\pi i/3}$. Its acts on $\mathrm{H}^3(\mathrm{V},\mathbf{Z})$, with no fixed vector except 0 (a class fixed by σ in $\mathrm{H}^3(\mathrm{V},\mathbf{Q})$ comes from $\mathrm{V}/\langle\sigma\rangle = \mathbf{P}^3$, hence is 0); so we can view $\mathrm{H}^3(\mathrm{V},\mathbf{Z})$ as a module over the ring $\mathbf{Z}[\rho]$. We denote by $\langle\ ,\ \rangle$ the alternate form on $\mathrm{H}^3(\mathrm{V},\mathbf{Z})$ deduced from the cup-product, as well as its $\mathbf{C}$-bilinear extension to $\mathrm{H}^3(\mathrm{V},\mathbf{C})$. We associate to it the $\mathbf{Z}$-bilinear form h on $\mathrm{H}^3(\mathrm{V},\mathbf{Z})$, with values in $\mathbf{Z}[\rho]$, given by

$$h(a,b) = \langle a,\rho b\rangle - \rho\langle a,b\rangle\ . \tag{2.1}$$

It is an easy exercise to check that h is a $\mathbf{Z}[\rho]$-hermitian unimodular form on $\mathrm{H}^3(\mathrm{V},\mathbf{Z})$.

(2.2) The automorphism σ acts on $\mathrm{H}^3(\mathrm{V},\mathbf{C})$ with eigenvalues ρ and ρ^2; thus we have a direct sum decomposition

$$\mathrm{H}^3(\mathrm{V},\mathbf{C}) = \mathrm{H}^3(\mathrm{V})_\rho \oplus \mathrm{H}^3(\mathrm{V})_{\rho^2} \tag{2.2}$$

into eigenspaces for ρ and ρ^2. These eigenspaces are conjugate, and are both isotropic with respect to the alternate form. We define a hermitian form h' on $\mathrm{H}^3(\mathrm{V},\mathbf{C})$ (and therefore on $\mathrm{H}^3(\mathrm{V})_\rho$) by $h'(a,b) = -i\sqrt{3}\,\langle a,\bar{b}\rangle$.

Let $j : \mathrm{H}^3(\mathrm{V},\mathbf{Z}) \to \mathrm{H}^3(\mathrm{V})_\rho$ be the composition of the canonical injection $\mathrm{H}^3(\mathrm{V},\mathbf{Z}) \hookrightarrow \mathrm{H}^3(\mathrm{V},\mathbf{C})$ with the projection from $\mathrm{H}^3(\mathrm{V},\mathbf{C})$ onto $\mathrm{H}^3(\mathrm{V})_\rho$; by construction j is $\mathbf{Z}[\rho]$-linear.

Proposition 2.3 .– *The homomorphism* $j_{\mathbf{C}} : \mathrm{H}^3(\mathrm{V},\mathbf{Z}) \otimes_{\mathbf{Z}[\rho]} \mathbf{C} \longrightarrow \mathrm{H}^3(\mathrm{V})_\rho$ *deduced from* j *is an isometric isomorphism.*

Proof : Let $a \in \mathrm{H}^3(\mathrm{V},\mathbf{Z})$; its expression in the decomposition (2.2) is

$$a = j(a) + \bar{j}(a) \ .$$

Since $\mathrm{H}^3(\mathrm{V})_\rho$ and $\mathrm{H}^3(\mathrm{V})_{\rho^2}$ are isotropic for the cup-product, we have

$$h(a,a) = \langle a, \rho a\rangle = \langle j(a)+\bar{j}(a)\, ,\, \rho j(a)+\rho^2\bar{j}(a)\rangle = (\rho^2-\rho)\langle j(a),\bar{j}(a)\rangle = h'(j(a),j(a)) \ .$$

Thus $j_{\mathbf{C}}$ is isometric; since $\dim \mathrm{H}^3(\mathrm{V})_\rho = \frac{1}{2}\dim \mathrm{H}^3(\mathrm{V},\mathbf{C}) = \dim \mathrm{H}^3(\mathrm{V},\mathbf{Z}) \otimes_{\mathbf{Z}[\rho]} \mathbf{C}$, it is an isomorphism. ∎

(2.4) Finally we introduce Hodge theory into the picture. Since σ is holomorphic, it is compatible with the Hodge decomposition $\mathrm{H}^3(\mathrm{V},\mathbf{C}) = \mathrm{H}^{2,1} \oplus \mathrm{H}^{1,2}$; thus we have a decomposition

$$\mathrm{H}^3(\mathrm{V})_\rho = \mathrm{H}^{2,1}_\rho \oplus \mathrm{H}^{1,2}_\rho \ .$$

Proposition 2.5 .– *This decomposition is orthogonal with respect to the hermitian form* h' *; the space* $\mathrm{H}^{2,1}_\rho$ *is* 4*-dimensional positive, while* $\mathrm{H}^{1,2}_\rho$ *is a negative line.*

Proof : We will compute the dimensions of the subspaces $\mathrm{H}^{2,1}_\rho$ and $\mathrm{H}^{1,2}_\rho$ in the next section. It suffices then to show that h' is positive definite on $\mathrm{H}^{2,1}$ and negative definite on $\mathrm{H}^{1,2}$. This is straightforward: compute locally in a coordinate system, using the fact that the form $\omega = dx \wedge dy \wedge \overline{dz}$ on $\mathbf{C}^3$ satisfies $-i\int \omega \wedge \bar{\omega} > 0$ on any ball. ∎

It follows that $(\mathrm{H}^3(\mathrm{V})_\rho, h')$ is isomorphic to $\mathbf{C}^{4,1}$, that is, the vector space $\mathbf{C}^5$ with the hermitian form $h_{4,1}$ given by $h_{4,1}(x,y) = -x_0\bar{y}_0 + \sum_{i=1}^4 x_i\bar{y}_i$. In fact we have a more precise result:

Proposition 2.6 .– *As a lattice over* $\mathbf{Z}[\rho]$ *,* $\mathrm{H}^3(\mathrm{V},\mathbf{Z})$ *is isomorphic to* $\mathbf{Z}[\rho]^{4,1}$ *, that is* $\mathbf{Z}[\rho]^5$ *with the hermitian form* $h_{4,1}$ *.*

Proof : This is a general result: *a unimodular hermitian lattice* L *over* $\mathbf{Z}[\rho]$ *of signature* (p,q) *with* $p,q>0$ *is isomorphic to* $\mathbf{Z}[\rho]^{p,q}$. Note that this is simpler than the classification of indefinite unimodular lattices over $\mathbf{Z}$, where not only the signature but also the parity (that is, whether the quadratic form takes only even values or not) is needed. The point is that in our case every unimodular lattice is odd, because if two vectors a,b satisfy $h(a,b)=\rho$, one of the vectors a , b and $a+b$ is odd.

The proof follows closely the analogous result for odd indefinite unimodular lattices over $\mathbf{Z}$ (see e.g. [S], ch. V). We first observe that L *contains an isotropic* (non-zero) *vector*. If $\operatorname{rk} L \geq 3$ this follows from Meyer's theorem ([S], ch. IV, § 3, cor. 2); if $\operatorname{rk} L = 2$, the $\mathbf{Q}(\rho)$-hermitian vector space $L \otimes_{\mathbf{Z}} \mathbf{Q}$ admits an orthogonal basis (e, f) with $h(e,e) = a$, $h(f,f) = b$, $a, b \in \mathbf{Q}$. Since L is unimodular we have $ab = \pm|\lambda|^2$ for some $\lambda \in \mathbf{Q}(\rho)$, and actually $ab = -|\lambda|^2$ because L is indefinite. Then the vector $be + \lambda f$ is isotropic, and some multiple of it belongs to L.

Let x be a primitive isotropic vector in L; there exists $y \in L$ with $h(x,y) = 1$. Then $h(y + k\rho x, y + k\rho x) = h(y,y) - k$ takes any integral value for $k \in \mathbf{Z}$, in particular we find $e \in L$ with $h(e,e) = \pm 1$. Then $L = \mathbf{Z}[\rho]e \oplus e^{\perp}$, and the $\mathbf{Z}[\rho]$-lattice $e^{\perp}$ is unimodular, and indefinite (if $\operatorname{rk} L > 2$) for an appropriate choice of the sign of $h(e,e)$. The result follows by induction. ∎

(2.7) We are now in position to state the main result of [ACT]. Let $\mathcal{M}$ be the moduli space of cubic surfaces; as in section 1, this is simply the quotient of the open subset of $\mathbf{P}(\mathrm{H}^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(4))$ parameterizing smooth cubic surfaces by the linear group $\mathrm{PGL}(4)$. We define a *framing* of a cubic surface S as a $\mathbf{Z}[\rho]$-linear isometry $\lambda : \mathbf{Z}[\rho]^{4,1} \xrightarrow{\sim} \mathrm{H}^3(\mathrm{V}, \mathbf{Z})$, with the convention that two isometries differing by a unity of $\mathbf{Z}[\rho]$ (that is, a 6-th root of unity) give the same framing. An isomorphism $u : \mathrm{S}' \xrightarrow{\sim} \mathrm{S}$ of cubic surfaces is induced by an isomorphism $v : \mathrm{V}' \xrightarrow{\sim} \mathrm{V}$ of the corresponding threefolds, *well-defined up to multiplication by* $\sigma^{\pm 1}$; v induces an isometry $\mathbf{Z}[\rho]$-linear $v^* : \mathrm{H}^3(\mathrm{V}, \mathbf{Z}) \xrightarrow{\sim} \mathrm{H}^3(\mathrm{V}', \mathbf{Z})$, well-defined modulo multiplication by $\mathbf{Z}[\rho]^*$. If λ is a framing of S, $\lambda' := v^* \circ \lambda$ is a well-defined framing of S′, and we say that the framed cubics (S, λ) and (S', λ') are isomorphic.

The isomorphism classes of framed cubic surfaces are parametrized by an analytic space $\widetilde{\mathcal{M}}$ (we will actually see in (3.5) that it is a manifold). The projective unitary group $\Gamma := \mathrm{PU}(\mathbf{Z}[\rho]^{4,1}) = \mathrm{U}(\mathbf{Z}[\rho]^{4,1})/\mathbf{Z}[\rho]^*$ acts on $\widetilde{\mathcal{M}}$ by $\gamma \cdot (\mathrm{S}, \lambda) = (\mathrm{S}, \lambda \circ \gamma^{-1})$, and the moduli space $\mathcal{M}$ of cubic surfaces is the quotient of $\widetilde{\mathcal{M}}$ by Γ.

(2.8) Given a framed cubic (S, λ), the isometric isomorphism

$$\tau : \mathbf{C}^{4,1} \xrightarrow{\lambda_{\mathbf{C}}} \mathrm{H}^3(\mathrm{V}, \mathbf{Z}) \otimes_{\mathbf{Z}[\rho]} \mathbf{C} \xrightarrow{j_{\mathbf{C}}} \mathrm{H}^3(\mathrm{V})_\rho$$

(Prop. 2.3) provides a positive definite hyperplane $\tau^{-1}(\mathrm{H}^{2,1}_\rho)$ of $\mathbf{C}^{4,1}$ (Prop. 2.5). Such a subspace is defined by an equation $\sum_{i=0}^4 a_i z_i = 0$ in $\mathbf{C}^{4,1}$; its positivity is easily seen to be equivalent to $\sum_{i=1}^4 |a_i|^2 < |a_0|^2$. Normalizing so that $a_0 = 1$ we see that *the positive hyperplanes of* $\mathbf{C}^{4,1}$ *are parameterized by the complex* 4-*dimensional ball* $\mathbf{B}_4 \subset \mathbf{P}(\mathbf{C}^{4,1})$.

We have thus defined a period map

$$\tilde{\wp} : \widetilde{\mathcal{M}} \longrightarrow \mathbf{B}_4$$

which associates to (S,λ) the hyperplane $\tau^{-1}(\mathrm{H}^{2,1}_\rho)$ of $\mathbf{C}^{4,1}$. The group Γ acts faithfully on $\mathbf{B}_4$ (viewed as the variety of 4-dimensional positive definite subspaces of $\mathbf{C}^{4,1}$), and the map $\tilde{\wp}$ is equivariant, so that as in § 1 we have a commutative diagram:

$$\begin{array}{ccc}
\widetilde{\mathcal{M}} & \xrightarrow{\ \tilde{\wp}\ } & \mathbf{B}_4 \\
\big\downarrow & & \big\downarrow \\
\mathcal{M} & \xrightarrow{\ \wp\ } & \mathbf{B}_4/\Gamma
\end{array} \quad .$$

For each $\delta \in \mathbf{Z}[\rho]^{4,1}$ with $h_{4,1}(\delta,\delta)=1$, we denote by H_δ the hypersurface in $\mathbf{B}_4$ consisting of 4-planes $\mathrm{P}\subset\mathbf{C}^{4,1}$ with $\delta\in\mathrm{P}$; it is the trace on $\mathbf{B}_4$ of the hyperplane $\sum\delta_i z_i=0$ in $\mathbf{P}(\mathbf{C}^{4,1})$. We will show that these hyperplanes form a locally finite family, so that $\mathcal{H}=\bigcup\limits_{h(\delta,\delta)=1}\mathrm{H}_\delta$ is a closed analytic subset of $\mathbf{B}_4$.

Theorem 2.9.– *The period maps define isomorphisms* $\tilde{\wp}:\widetilde{\mathcal{M}}\xrightarrow{\sim}\mathbf{B}_4-\mathcal{H}$ *and* $\wp:\mathcal{M}\xrightarrow{\sim}(\mathbf{B}_4-\mathcal{H})/\Gamma$.

Remarks 2.10 .– a) We have not followed here the conventions of [ACT]: they choose (essentially) to associate to (S,λ) the negative line $\tau^{-1}(\mathrm{H}^{1,2}_\rho)$, which unfortunately varies *antiholomorphically* with (S,λ) – the point is that for a hermitian form the passage from a space to its orthogonal is antiholomorphic.

b) The ball $\mathbf{B}_4$ is called in [ACT] the *complex hyperbolic space* $\mathbf{C}\mathrm{H}_4$. We will briefly discuss why in (8.8).

(2.11) *Strategy of the proof*

We will prove successively that:

1) The period map $\tilde{\wp}$ is a local isomorphism (§ 3); this is based on Griffiths' description of the differential of the period map.

2) $\wp$ and $\tilde{\wp}$ are open embeddings; this follows from 1) and the Torelli theorem for cubic threefolds of Clemens and Griffiths.

3) The image of $\tilde{\wp}$ is contained in $\mathbf{B}_4-\mathcal{H}$ (§ 5).

4) Geometric invariant theory provides natural enlargements of $\mathcal{M}$, namely open embeddings $\mathcal{M}\subset\mathcal{M}_s\subset\mathcal{M}_{ss}$. $\mathcal{M}_s$ is the moduli space of *stable* cubic surfaces, that is, cubics with (at most) ordinary double points. The boundary $\partial\mathcal{M}_s:=\mathcal{M}_s-\mathcal{M}$, which parameterizes singular cubics, is irreducible. $\mathcal{M}_{ss}$ is a normal projective variety, and $\mathcal{M}_{ss}-\mathcal{M}_s$ is a point (§ 6).

5) $\wp$ extends to a map $\wp_s:\mathcal{M}_s\to\mathbf{B}_4/\Gamma$, and $\wp_s$ extends to $\wp_{ss}:\mathcal{M}_{ss}\to\widehat{\mathbf{B}_4}/\Gamma$, where the target is a normal projective variety, the *Satake compactification* of $\mathbf{B}_4/\Gamma$,

which in our case is again a one-point compactification of $\mathbf{B}_4/\Gamma$.

Then we are done. Indeed $\wp_{ss}$ is proper, hence so is $\wp_s$. Now $\wp_s^{-1}(\mathcal{H}/\Gamma)$ is a divisor of $\mathcal{M}_s$, contained in $\partial\mathcal{M}_s$ by 3), hence equal to $\partial\mathcal{M}_s$. Thus $\wp$ is the restriction of $\wp_s$ to $\wp_s^{-1}((\mathbf{B}_4-\mathcal{H})/\Gamma)$, hence is proper, and therefore surjective. It follows that $\tilde{\wp}$ also is surjective; in view of 2) both are isomorphisms. ∎

3. The differential of the period map

(3.1) To compute the differential of the period map we use a method invented by Griffiths [G], which gives a concrete way of describing the Hodge structure of a hypersurface. We recall briefly the idea in the case of a cubic threefold $\mathrm{V}\subset\mathbf{P}^4$. The Gysin exact sequence provides a canonical isomorphism

$$\mathrm{Res}:\mathrm{H}^4(\mathbf{P}^4-\mathrm{V},\mathbf{C})\xrightarrow{\sim}\mathrm{H}^3(\mathrm{V},\mathbf{C})\ ;$$

since the variety $\mathbf{P}^4-\mathrm{V}$ is affine, its complex cohomology can be expressed in terms of algebraic de Rham cohomology: specifically $\mathrm{H}^4(\mathbf{P}^4-\mathrm{V},\mathbf{C})$ is represented by classes of algebraic 4-forms on $\mathbf{P}^4-\mathrm{V}$ modulo exact ones. But algebraic forms on $\mathbf{P}^4-\mathrm{V}$ are rational forms on $\mathbf{P}^4$ with poles along V only; we obtain a filtration on $\mathrm{H}^4(\mathbf{P}^4-\mathrm{V},\mathbf{C})$ by the order of the pole, and the fundamental result is that the isomorphism Res carries this filtration onto the Hodge filtration of $\mathrm{H}^3(\mathrm{V})$.

To be more explicit, let G be a cubic form defining V. Put

$$\Omega:=\sum_{i=0}^{4}(-1)^i\,\mathrm{X}_i\,d\mathrm{X}_0\wedge\ldots\wedge\widehat{d\mathrm{X}_i}\wedge\ldots\wedge d\mathrm{X}_4\ .$$

Rational forms on $\mathbf{P}^4$ with a pole of order $\le p$ along V are of the form $\omega=\frac{\mathrm{P}\Omega}{\mathrm{G}^p}$, where P is a form of degree $3p-5$ (so that ω is homogeneous of degree 0). Write $\mathrm{S}=\bigoplus\limits_{d\geq 0}\mathrm{S}_d$ for the graded ring $\mathbf{C}[\mathrm{X}_0,\ldots,\mathrm{X}_4]$, and J_G for the Jacobian ideal $(\mathrm{G}'_{\mathrm{X}_0},\ldots,\mathrm{G}'_{\mathrm{X}_4})$ of G. The general result of [G] gives in that case:

Proposition 3.2.– 1) *The map* $\varphi_1:\mathrm{L}\mapsto\mathrm{Res}\,\dfrac{\mathrm{L}\Omega}{\mathrm{G}^2}$ *induces an isomorphism* $\mathrm{S}_1\xrightarrow{\sim}\mathrm{H}^{2,1}$.

2) *The map* $\mathrm{P}\mapsto\mathrm{Res}\,\dfrac{\mathrm{P}\Omega}{\mathrm{G}^3}$ *from* S_4 *into* $\mathrm{H}^3(\mathrm{V},\mathbf{C})$ *is surjective, and induces an isomorphism* $(\mathrm{S}/\mathrm{J}_\mathrm{G})_4\xrightarrow{\sim}\mathrm{H}^3(\mathrm{V},\mathbf{C})/\mathrm{H}^{2,1}$.

Let us apply this to our cubic threefold V obtained from the surface $\mathrm{S}\subset\mathbf{P}^3$.

Corollary 3.3.– *There is a canonical isomorphism* $\mathrm{H}^0(\mathrm{S},\mathcal{O}_\mathrm{S}(1))\xrightarrow{\sim}\mathrm{H}^{2,1}_\rho$ *; the space* $\mathrm{H}^{1,2}_\rho$ *has dimension* 1.

Proof: Since G is invariant under σ and $\sigma^*\Omega=\rho\Omega$, we see that φ_1 maps the invariant part of S_1 (which is identified with $\mathrm{H}^0(\mathrm{S},\mathcal{O}_\mathrm{S}(1))$) onto $\mathrm{H}^{2,1}_\rho$, and the ρ-eigenspace $\mathbf{C}\,\mathrm{X}_4$ onto $\mathrm{H}^{2,1}_{\rho^2}$, which is conjugate to $\mathrm{H}^{1,2}_\rho$. ∎

Corollary 3.4 .– *Any automorphism of a framed cubic surface* (S,λ) *is trivial.*

Proof : Let u be an automorphism of S, and v an automorphism of V which induces u on S. Then v commutes with σ, and therefore preserves the decomposition (2.2). It also preserves the Hodge decomposition, hence the subspace $\mathrm{H}^{2,1}_{\rho}$. If $v^*\circ\lambda$ gives the same framing as λ, v^* acts on $\mathrm{H}^3(\mathrm{V},\mathbf{Z})$ by multiplication by a unity of $\mathbf{Z}[\rho]$; therefore it induces a homothety of $\mathrm{H}^{2,1}_{\rho}$, hence of $\mathrm{H}^0(\mathrm{S},\mathcal{O}_{\mathrm{S}}(1))$ in view of the preceding corollary. ■

(3.5) Let $\mathcal{C}$ be the open subset of $\mathrm{H}^0(\mathbf{P}^3,\mathcal{O}(3))$ consisting of cubic forms for which the corresponding surface is smooth. We define as in (2.7) the moduli space $\widetilde{\mathcal{C}}$ of pairs (F,λ) where $\mathrm{F}\in\mathcal{C}$ and λ is a framing of the surface $\mathrm{F}=0$; the map $\widetilde{\mathcal{C}}\rightarrow\mathcal{C}$ is an étale Galois covering with Galois group Γ.

The group $\mathrm{GL}(4)$ acts on $\mathcal{C}$, and the moduli space $\mathcal{M}$ is the quotient $\mathcal{C}/\mathrm{GL}(4)$ (we will discuss what this means in § 6). This action lifts to $\widetilde{\mathcal{C}}$, and commutes with that of Γ. The preceding Corollary means that this action is *free*, so the quotient $\widetilde{\mathcal{M}}=\widetilde{\mathcal{C}}/\mathrm{GL}(4)$ is *smooth*, of dimension 4.

(3.6) To compute the differential of the period map we will need the following general fact. Let L be a complex vector space, and $t\mapsto\mathrm{E}_t\subset\mathrm{L}$ a holomorphic map of the unit disk D in the Grassmannian $\mathbf{G}(\mathrm{L})$. Let us recall the expression of the derivative $\dot{\mathrm{E}}_0$ of that map at the origin. The tangent space to $\mathbf{G}(\mathrm{L})$ at E_0 is canonically identified with $\mathrm{Hom}(\mathrm{E}_0,\mathrm{L}/\mathrm{E}_0)$. Let $e\in\mathrm{E}_0$; for any holomorphic map $t\mapsto e(t)\in\mathrm{E}_t$ such that $e(0)=e$, $\dot{\mathrm{E}}_0(e)$ is the class of $\dot{e}(0)$ in L/E_0. In particular, *the differential is* 0 *if and only if* $\dot{e}(0)\in\mathrm{E}_0$ *for all* $e\in\mathrm{E}_0$.

After these preliminaries we arrive to the main result of this section:

Proposition 3.7 .– *The map* $\tilde{\wp}:\widetilde{\mathcal{M}}\rightarrow\mathbf{B}_4$ *is a local isomorphism.*

Proof : Since both spaces are smooth of the same dimension, it will be enough to prove that the tangent map is injective. Let φ be a holomorphic map from the unit disk D into $\widetilde{\mathcal{C}}$. We want to show that if the differential of $\tilde{\wp}\circ\varphi$ is 0, the vector $\dot{\varphi}(0)$ is tangent to the $\mathrm{GL}(4)$-orbit of $\varphi(0)$ in $\widetilde{\mathcal{C}}$, and thus goes to 0 in $\widetilde{\mathcal{M}}$.

The map φ corresponds to a family $(\mathrm{F}_t)_{t\in\mathrm{D}}$ of cubic forms on $\mathbf{P}^3$, together with a framing of the corresponding surfaces. We consider the family of cubic threefolds $(\mathrm{V}_t)_{t\in\mathrm{D}}$ defined by $\mathrm{G}_t:=\mathrm{X}_4^3-\mathrm{F}_t(\mathrm{X}_0,\ldots,\mathrm{X}_3)=0$, and the spaces $\mathrm{E}_t=\mathrm{H}^{2,1}_{\rho}(\mathrm{V}_t)\subset\mathrm{H}^3(\mathrm{V}_t)_{\rho}$. We will write $\mathrm{F},\mathrm{G},\mathrm{V}$ instead of $\mathrm{F}_0,\mathrm{G}_0,\mathrm{V}_0$, and $\dot{\mathrm{F}}:=\big(\dfrac{d\mathrm{F}_t}{dt}\big)_{t=0}$.

The local system $\big(\mathrm{H}^3(\mathrm{V}_t)_{\rho}\big)_{t\in\mathrm{D}}$ is trivialized by the framing, so we can apply (3.6). The space E_t is spanned by the classes of the forms $\mathrm{Res}\,\frac{\mathrm{L}\Omega}{\mathrm{G}_t^2}$, where L is a linear form in $X_0,\ldots,X_3$ (Cor. 3.3). We have

$$\frac{d}{dt}\big(\mathrm{Res}\,\frac{L\Omega}{G_t^2}\big)_{t=0}=\mathrm{Res}\,\frac{2\dot{F}L\Omega}{G^3}\ ;$$

the class of this form belongs to $H^3(V)_\rho$, and it is in $H^{2,1}_\rho$ if and only if $L\dot{F}$ belongs to the ideal jacobian J_G (Prop. 3.2). But J_G is spanned by J_F and X_4^2; making $X_4=0$ we find $L\dot{F}\in J_F$ in the ring $\mathbf{C}[X_0,\ldots,X_3]$. If the differential $\dot{\varphi}(0)$ vanishes, this must hold for all linear forms L. The ring $R:=\mathbf{C}[X_0,\ldots,X_3]/J_F$ is a complete intersection graded artinian ring, with socle in degree 4; this implies that the multiplication $R_1\times R_3\to R_4\cong\mathbf{C}$ is a perfect pairing (see e.g. [V], Thm. 18.19). Therefore $\dot{F}$ belongs to J_F, that is, can be written $\sum_i L_iF'_{X_i}$ for some linear forms L_i; in other words, $\dot{F}=\mathfrak{X}\cdot F$ where $\mathfrak{X}$ is the vector field $\sum L_i\frac{\partial}{\partial X_i}$ on $\mathbf{P}^3$. This means that $\dot{\varphi}(0)$ is tangent to the $GL(4)$-orbit of F in $\widetilde{\mathcal{C}}$ as required. ∎

4. Injectivity of the period map

(4.1) We start with some linear algebra. Let W be a complex vector space with a real structure (given by a $\mathbf{C}$-antilinear involution $x\mapsto\bar{x}$). The formula $h(x,y)=-i\langle x,\bar{y}\rangle$ establishes a bijective correspondence between:

- non-degenerate $\mathbf{C}$-bilinear alternate forms $\langle\ ,\ \rangle$ on W defined over $\mathbf{R}$;
- non-degenerate hermitian forms h on W satisfying $h(\bar{x},\bar{y})=-\overline{h(x,y)}$.

Fix a pair of such forms on W, and put $\dim W=2n$. For a n-dimensional subspace L of W, the following conditions are equivalent:

- L is isotropic w.r.t. $\langle\ ,\ \rangle$, and positive definite w.r.t. h (we will say that L is *positive Lagrangian*);
- L is positive definite w.r.t. h, and there is an orthogonal decomposition $W=L\oplus\overline{L}$ ("polarized Hodge structure of weight 1").

These subspaces are parameterized by an open subset of the Grassmannian of maximal isotropic subspaces of W, which is homogeneous under the symplectic group $Sp(W)$. As is well-known, this homogeneous space is isomorphic to the *Siegel upper-half space* $\mathbf{H}_n$, that is, the open subset of the space of complex symmetric $n\times n$ matrices whose imaginary part is positive definite. Let us recall briefly why:

We choose two isotropic vector subspaces A and A' of W, defined over $\mathbf{R}$, such that $W=A\oplus A'$. Let $L\subset W$ be a positive Lagrangian subspace; then $L\cap A=L\cap A'=(0)$ because A and A' are also isotropic w.r.t. h, so L is the graph of an isomorphism $u_L:A\to A'$. The formula $b_L(x,y)=\langle x,u_L(y)\rangle$ defines a $\mathbf{C}$-bilinear symmetric form b_L on A, which determines u_L and therefore $L=\{x+u_L(x)\mid x\in A\}$. It is an exercise to check that the positivity of L is equivalent to that of the imaginary part of b_L.

(4.2) We will now define a natural morphism from the complex ball $\mathbf{B}_n$ into $\mathbf{H}_{n+1}$. Let U be a complex vector space with a hermitian form h_{U} of signature $(n,1)$. Put $\mathrm{W}=\mathrm{U}\oplus\overline{\mathrm{U}}$, with the hermitian form h which coincides with h_{U} on U, for which U and $\overline{\mathrm{U}}$ are orthogonal, and such that $h(\bar{x},\bar{y})=-\overline{h_{\mathrm{U}}(x,y)}$ for $x,y\in\mathrm{U}$. By (4.1) the form h is associated to a symplectic form $\langle\ ,\ \rangle$ on W.

The ball $\mathbf{B}_n$ parameterizes positive hyperplanes $\mathrm{P}\subset\mathrm{U}$. We associate to such a hyperplane the subspace $\mathrm{P}\oplus\overline{\mathrm{P}}^{\perp}$ of W. This is a positive Lagrangian subspace, which defines a point of $\mathbf{H}_{n+1}$; it is readily seen that it varies holomorphically with P (because we take the *conjugate* of $\mathrm{P}^{\perp}$, see Remark 2.10 a).

Let L be the hermitian $\mathbf{Z}[\rho]$-module $\mathbf{Z}[\rho]^{n,1}$; it carries a natural symplectic form $\langle\ ,\ \rangle$ such that (2.1) holds. Choose an isometry $\mathrm{U}\xrightarrow{\sim}\mathrm{L}\otimes_{\mathbf{Z}[\rho]}\mathbf{C}$. Then W is canonically identified with $\mathrm{L}\otimes_{\mathbf{Z}}\mathbf{C}$. The group $\mathrm{U(L)}$ acts on W, preserving the symplectic form, and the above morphism is compatible with these actions. Thus it factors as

$$\tau:\mathbf{B}_n/\mathrm{U(L)}\to\mathbf{H}_{n+1}/\,\mathrm{Sp(L)}\ .$$

(4.3) We now go back to our situation. Let V be a smooth cubic threefold; choosing a symplectic isomorphism $\mu:\mathbf{Z}^{10}\xrightarrow{\sim}\mathrm{H}^3(\mathrm{V},\mathbf{Z})$ (where $\mathbf{Z}^{10}$ is endowed with the standard symplectic form), we deduce from the Hodge decomposition $\mathrm{H}^3(\mathrm{V},\mathbf{C})=\mathrm{H}^{2,1}\oplus\mathrm{H}^{1,2}$ a polarized Hodge structure of weight 1 on $\mathbf{C}^{10}$, that is, a point $\pi(\mathrm{V},\mu)\in\mathbf{H}_5$. A change of the marking μ amounts to an action of the group $\mathrm{Sp}(10,\mathbf{Z})$, so we get a well-defined period map $\pi:\mathcal{V}\to\mathbf{H}_5/\,\mathrm{Sp}(10,\mathbf{Z})$, where $\mathcal{V}$ is the moduli space of smooth cubic threefolds.

Let $t:\mathcal{M}\to\mathcal{V}$ be the map which associates to a cubic surface $\mathrm{S}\subset\mathbf{P}^3$ the triple covering of $\mathbf{P}^3$ branched along S.

Claim 4.4 .– *The diagram*

$$\begin{array}{ccc}\mathcal{M} & \xrightarrow{\ \wp\ } & \mathbf{B}_4/\Gamma \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathcal{V} & \xrightarrow{\ \pi\ } & \mathbf{H}_5/\,\mathrm{Sp}(10,\mathbf{Z})\end{array}$$

is commutative.

Proof : Let $\mathrm{S}\in\mathcal{M}$, and let V be the associated threefold. Choose an isometry $\lambda:\mathbf{Z}[\rho]^{4,1}\xrightarrow{\sim}\mathrm{H}^3(\mathrm{V},\mathbf{Z})$; this is also a symplectic isomorphism. The images of S in $\mathbf{H}_5/\,\mathrm{Sp}(10,\mathbf{Z})$ under $\pi\circ t$ and $\tau\circ\wp$ correspond both to certain 5-dimensional subspaces of $\mathrm{H}^3(\mathrm{V},\mathbf{C})$, pulled back to $\mathbf{C}^{10}$ via λ. The first one is simply $\mathrm{H}^{2,1}(\mathrm{V})$, while the second one is obtained from $\mathrm{H}^{2,1}_{\rho}\subset\mathrm{H}^3(\mathrm{V})_{\rho}$ by construction (4.2). But the orthogonal of $\mathrm{H}^{2,1}_{\rho}$ in $\mathrm{H}^3(\mathrm{V})_{\rho}$ is $\mathrm{H}^{1,2}_{\rho}$, whose conjugate in $\mathrm{H}^3(\mathrm{V},\mathbf{C})$ is $\mathrm{H}^{2,1}_{\rho^2}$. Thus we find $\mathrm{H}^{2,1}_{\rho}\oplus\mathrm{H}^{2,1}_{\rho^2}=\mathrm{H}^{2,1}(\mathrm{V})$ again. ∎

The injectivity of $\wp$ is a consequence of a celebrated theorem of Clemens and Griffiths [CG]:

Theorem 4.5 .– *The period map π is injective.*

In other words, the cubic threefold V can be recovered from the (polarized, weight 1) Hodge structure on $\mathrm{H}^3(\mathrm{V},\mathbf{Z})$. The data of this Hodge structure is equivalent to that of the *intermediate Jacobian* $\mathrm{JV}=\mathrm{H}^{1,2}/\mathrm{H}^3(\mathrm{V},\mathbf{Z})$, an abelian variety with a principal polarization – that is, an ample divisor $\Theta\subset\mathrm{JV}$, defined up to translation, with $\dim\mathrm{H}^0(\mathrm{JV},\mathcal{O}(\Theta))=1$. The problem is thus to recover V from the data (JV,Θ); one way [B1] is to prove that Θ has only one singular point, of multiplicity 3, and that the tangent cone to Θ at this point is canonically identified to the cone over the cubic threefold $\mathrm{V}\subset\mathbf{P}^4$. ∎

Theorem 4.6 .– *The period maps $\wp$ and $\tilde{\wp}$ are open embeddings.*

Proof : Let us prove first that $\wp$ is generically injective. In view of the diagram (4.4) and of thm. 4.5, it is enough to prove that $t:\mathcal{M}\to\mathcal{V}$ is generically injective – that is, that we can recover the surface S from V when S is general enough.

Let F be a cubic form defining S, so that V is defined by the form $\mathrm{G}=\mathrm{X}_4^3-\mathrm{F}(\mathrm{X}_0,\ldots,\mathrm{X}_3)$. The Hessian $\mathrm{H}(\mathrm{G}):=\det\bigl(\frac{\partial^2\mathrm{G}}{\partial\mathrm{X}_i\partial\mathrm{X}_j}\bigr)$ equals $6\mathrm{X}_4\mathrm{H}(\mathrm{F})$. If $\mathrm{H}(\mathrm{F})$ is irreducible, the hyperplane $\mathrm{X}_4=0$ is the only degree 1 component of $\mathrm{H}(\mathrm{G})$, and is therefore determined by V; and so is S, which is the intersection of V with this hyperplane. Now the fact that $\mathrm{H}(\mathrm{F})$ is irreducible for a general cubic F is classical – in fact, $\mathrm{H}(\mathrm{F})$ is a quartic surface with 10 ordinary nodes. This can be seen easily for a cubic given in "Sylvester form" $\mathrm{F}=\mathrm{L}_0^3+\ldots+\mathrm{L}_4^3$, where the L_i 's are general linear forms. The reader can prove it as an exercise, or look at [D-K, 5.15]. This proves that $\wp$ is generically injective.

Let s, s' be two points of $\widetilde{\mathcal{M}}$ such that $\tilde{\wp}(s)=\tilde{\wp}(s')$ and s is general. By the preceding result there exists an element γ of Γ such that $s'=\gamma s$; therefore $\tilde{\wp}(s)$ is fixed by γ. But for $\gamma\neq 1$, the fixed locus of γ in $\mathbf{B}_4$ is a closed analytic subvariety of $\mathbf{B}_4$, strictly smaller than $\mathbf{B}_4$; hence the points of $\mathbf{B}_4$ with trivial stabilizer (in Γ) form a dense subset of $\mathbf{B}_4$. Since s is a general point, we have $\gamma=1$ and $s'=s$, so $\tilde{\wp}$ is also generically injective.

Since $\tilde{\wp}$ is étale (Prop. 3.7), it is an open embedding, and induces an isomorphism of $\widetilde{\mathcal{M}}$ onto an open subset $\mathrm{U}\subset\mathbf{B}_4$ stable under Γ. Then $\wp$ induces an isomorphism of $\mathcal{M}=\widetilde{\mathcal{M}}/\Gamma$ onto $\mathrm{U}/\Gamma\subset\mathbf{B}_4/\Gamma$. ∎

5. The image of $\tilde{\wp}$

It remains to identify the image of $\tilde{\wp}$ with $\mathbf{B}_4 - \mathcal{H}$. In this section we will do the easy part, namely prove that this image is contained in $\mathbf{B}_4 - \mathcal{H}$.

Proposition 5.1 .– *Let* $\delta \in \mathrm{H}^3(\mathrm{V},\mathbf{Z})$ *with* $h(\delta,\delta)=1$ *. Then* $j(\delta) \notin \mathrm{H}^{2,1}_\rho$.

Proof: Suppose $j(\delta) \in \mathrm{H}^{2,1}_\rho$; then $\bar{j}(\delta) \in \mathrm{H}^{1,2}_{\rho^2}$ is the component of δ in $\mathrm{H}^{1,2}$. Thus the $\mathbf{C}$-linear map $\mathbf{C} \to \mathrm{H}^{1,2}$ which maps 1 to $\bar{j}(\delta)$ factors as

$$\varphi : \mathbf{C}/\mathbf{Z}[\rho] \longrightarrow \mathrm{H}^{1,2}/\operatorname{Im}\mathrm{H}^3(\mathrm{V},\mathbf{Z}) = \mathrm{JV} \ .$$

The principal polarization of JV is given by the hermitian form $h''(a,b) = 2i\langle a,\bar{b}\rangle$ on $\mathrm{H}^{1,2}$, that is, $h'' = -\frac{2}{\sqrt{3}}h'$; that of the elliptic curve $\mathrm{E} := \mathbf{C}/\mathbf{Z}[\rho]$ by the form $h_{\mathrm{E}}(x,y) = \frac{2}{\sqrt{3}}x\bar{y}$ (the unique positive hermitian form on $\mathbf{C}$ whose imaginary part is unimodular on $\mathbf{Z}[\rho]$). Since j is isometric, we have

$$(\varphi^*h')(1,1) = h'(\bar{j}(\delta),\bar{j}(\delta)) = -\overline{h'(j(\delta),j(\delta))} = -h(\delta,\delta) = -1 \ , \ \text{hence} \ \ \varphi^*h'' = h_{\mathrm{E}} \ .$$

This means that JV can be written as a product $\mathrm{E}\times\mathrm{A}$ of principally polarized abelian varieties (see e.g. [C-G], 3.6). But then the theta divisor Θ of JV is the sum of the pull backs of the theta divisors of each factor; this is impossible, for instance because it implies that Θ is singular in codimension 2, while we have seen that it has a unique singular point (4.5). ■

Let Δ_1 be the set of vectors $\delta \in \mathbf{Z}[\rho]^{4,1}$ such that $h_{4,1}(\delta,\delta)=1$. Recall that we have defined $\mathcal{H} = \bigcup_{\delta\in\Delta_1} \mathrm{H}_\delta$, where H_δ is the hypersurface in $\mathbf{B}_4$ consisting of 4-planes $\mathrm{P} \subset \mathbf{C}^{4,1}$ containing δ.

Corollary 5.2 .– *The image of* $\tilde{\wp}$ *is contained in* $\mathbf{B}_4 - \mathcal{H}$ *.*

Proof : Let (S,λ) be a framed cubic surface, V the associated cubic threefold, and $\tau : \mathbf{C}^{4,1} \xrightarrow{\sim} \mathrm{H}^3(\mathrm{V})_\rho$ the corresponding isomorphism (2.8). Let $\delta\in\Delta_1$. We have $\tau(\delta) = j(\lambda(\delta))$, so the Proposition means that δ does not belong to $\tau^{-1}(\mathrm{H}^{2,1}_\rho) = \tilde{\wp}(\mathrm{S},\lambda)$. ■

The following lemma shows that $\mathcal{H}$ is a closed analytic subvariety of $\mathbf{B}_4$:

Lemma 5.3 .– *The family of hyperplanes* $(\mathrm{H}_\delta)_{\delta\in\Delta_1}$ *is locally finite.*

Proof: Let $z\in\mathbf{B}_4$; we want to show that for ε small enough, the ball $\mathrm{B}(z,\varepsilon)$ meets only finitely many of the hyperplanes H_δ. Let us fix some notation first: we write h instead of $h_{4,1}$; for $x,y\in\mathbf{C}^{4,1}$, we put $h_+(x,y) = x_1\bar{y}_1 + \ldots + x_4\bar{y}_4$, so that $h(x,y) = h_+(x,y) - x_0\bar{y}_0$. We put $\|x\| = \sqrt{h_+(x,x)}$. We identify $\mathbf{C}^4$ with the affine hyperplane $z_0=1$ in $\mathbf{C}^{4,1}$. Note that $\|\ \|$ induces the standard hermitian norm on $\mathbf{C}^4$.

We choose ε so that $\|z\| < 1 - 2\varepsilon$. Suppose that the hyperplane H_δ meets $\mathrm{B}(z,\varepsilon)$; let z' a point in the intersection. We have

$$|h(\delta,\bar z)| = |h(\delta,\bar z - \bar z')| = |h_+(\delta,\bar z - \bar z')| \le \varepsilon\|\delta\|$$

On the other hand we have $h(\delta,\bar z) = h_+(\delta,\bar z) - \delta_0$ and $|h_+(\delta,\bar z)| \le \|\delta\|\,\|z\|$. Since $\|\delta\|^2 = 1 + |\delta_0|^2$, there exists $\mathrm{M} > 0$ such that $\|\delta\| > \mathrm{M}$ implies $|\delta_0| \ge (1-\varepsilon)\|\delta\|$; then $|h_+(\delta,\bar z)| \le |\delta_0|$, and therefore

$$\varepsilon\|\delta\| \ge |h(\delta,\bar z)| \ge |\delta_0| - \|\delta\|\,\|z\| > \varepsilon\|\delta\| \ ,$$

a contradiction. Thus $|\delta_0| \le \|\delta\| \le \mathrm{M}$, so the set of elements $\delta \in \Delta_1$ such that $\mathrm{H}_\delta \cap \mathrm{B}(z,\varepsilon) \neq \varnothing$ is bounded, and therefore finite. This proves our assertion. ∎

(5.4) Though this is not strictly necessary for what follows, let us observe that *the subvariety* $\mathcal{H}/\Gamma \subset \mathbf{B}_4/\Gamma$ *is irreducible*, or in other words that Γ acts transitively on the set of hypersurfaces H_δ. This amounts to say that Γ acts transitively on Δ_1. Let $\delta_1, \delta' \in \Delta_1$; the orthogonal $\delta_i^\perp$ is a unimodular $\mathbf{Z}[\rho]$-hermitian lattice of signature $(3,1)$, thus isomorphic to $\mathbf{Z}[\rho]^{3,1}$ (2.6). Any isometry u of $\delta^\perp$ onto $\delta'^\perp$ extends to an isometry of $\mathbf{Z}[\rho]^{4,1}$ which maps δ to δ', hence our assertion.

6. Stable and semi-stable cubic surfaces

(6.1) To go further we will compactify our situation, that is, embed $\widetilde{\mathcal{M}}$ in a larger moduli space $\widetilde{\mathcal{M}}_s$ such that $\tilde\wp$ extends to a *proper* map from $\widetilde{\mathcal{M}}_s$ to $\mathbf{B}_4$ – as explained in 2.11, this will imply the main result.

Such a compactification is provided by Mumford's geometric invariant theory [M], which we now briefly recall. Let G_0 be a semi-simple algebraic group, acting linearly on a vector space E; put $\mathrm{G} = \mathbf{C}^*\mathrm{G}_0$. We are interested in the quotient E/G – the case we have in mind is $\mathrm{E} = \mathrm{H}^0(\mathbf{P}^3, \mathcal{O}(3))$, $\mathrm{G}_0 = \mathrm{SL}(4)$. More precisely, we are looking for an open G-invariant subset $\mathrm{E}' \subset \mathrm{E}$, as large as possible, and a good quotient map $\pi : \mathrm{E}' \to \mathrm{E}'/\mathrm{G}$. Depending on what we call "good" there are two possible answers:

Definition 6.2 .– *A vector* e *of* E *is:*

– stable *if its orbit* $\mathrm{G}_0 e$ *is closed and its stabilizer finite;*

– semi-stable *if* $0 \notin \overline{\mathrm{G}_0 e}$.

(6.3) Let $\mathrm{E}_s \subset \mathrm{E}_{ss}$ denote the open G-invariant subsets of E consisting of stable and semi-stable points. There exists a good quotient E_s/G, and a "reasonable" quotient $\mathrm{E}_{ss}/\mathrm{G}$, which is normal and projective and contains E_s/G as an open subset. The points of E_s/G correspond to the orbits of G in E_s, while the points of $\mathrm{E}_{ss}/\mathrm{G}$ correspond to the closed orbits in E_{ss} – given such a closed orbit $\mathrm{G}e$, all semi-stable points whose orbit closure contains $\mathrm{G}e$ go to the same class as e in $\mathrm{E}_{ss}/\mathrm{G}$.

Mumford gives a very efficient criterion to check whether a vector $e \in \mathrm{E}$ is stable:

Criterion 6.4.– a) *A vector $e \in \mathrm{E}$ is not semi-stable if and only if there exists a homomorphism*[1] $\lambda : \mathbf{C}^* \to \mathrm{G}_0$ *such that* $\lambda(t)e \to 0$ *when* $t \to 0$.

b) e *is not stable if and only if there exists a homomorphism* $\lambda : \mathbf{C}^* \to \mathrm{G}_0$ *such that* $\lambda(t)e$ *admits a limit* $\notin \mathrm{G}_0 e$ *when* $t \to 0$.

We will now apply this criterion to the case of cubic surfaces, with $\mathrm{G}_0 = \mathrm{SL}(4)$. We say that a surface $\mathrm{F} = 0$ is stable, or semi-stable, if so is F.

Proposition 6.5.– a) *A cubic surface is stable if and only if it is smooth or has only ordinary double points.*

b) *There is only one closed orbit of semi-stable, non stable cubic surfaces, namely that of the surface* $\mathrm{X}_0^3 = \mathrm{X}_1\mathrm{X}_2\mathrm{X}_3$.

Proof : We will prove that:

a) Cubic surfaces with only ordinary double points or ordinary cusps[2] are semi-stable.

b) Cubic surfaces with only ordinary double points are stable.

c) Cubic surfaces with a singularity worse than an ordinary cusp are not semi-stable.

d) Cubic surfaces with at least one ordinary cusp (and perhaps some ordinary double points) are semi-stable not stable; they all contain in their orbit closure the orbit of the cubic $\mathrm{X}_0^3 - \mathrm{X}_1\mathrm{X}_2\mathrm{X}_3 = 0$, which is closed.

This will imply the assertions of the Proposition.

a) Let S be a cubic surface, defined by a form F. Assume that S is not semi-stable; let $\lambda : \mathbf{C}^* \to \mathrm{SL}(4)$ be a homomorphism such that $\lambda(t)\mathrm{F} \to 0$ when $t \to 0$. In an appropriate system of coordinates, $\lambda(t)$ is the diagonal matrix with entries $(t^{r_0}, \ldots, t^{r_3})$, with $r_0 + \ldots + r_3 = 0$; we can assume $r_0 \le \ldots \le r_3$. Write $\mathrm{F} = \sum a_{\boldsymbol{\alpha}} \mathrm{X}^{\boldsymbol{\alpha}}$, with $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_3)$, $\sum \alpha_i = 3$. Then $\lambda(t)\mathrm{F} = \sum t^{\mathbf{r}\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathrm{X}^{\boldsymbol{\alpha}}$, with $\mathbf{r}\boldsymbol{\alpha} := \sum r_i \alpha_i$, and we must have $\mathbf{r}\boldsymbol{\alpha} > 0$ whenever $a_{\boldsymbol{\alpha}} \neq 0$.

If S is irreducible, this implies $r_2 > 0$: otherwise every monomial appearing in F must be divisible by X_3. Then F cannot contain a monomial divisible by X_0^2 or $\mathrm{X}_0\mathrm{X}_1$, because

$$2r_0 + r_i \le r_0 + r_1 + r_i \le r_0 + r_1 + r_3 = -r_2 < 0 \ .$$

[1] This means of course a morphism of algebraic groups.

[2] That is a singularity of type A_2, defined in local analytic coordinates by $x^2 + y^2 + z^3 = 0$.

Thus S has a double point at $(1,0,0,0)$, with a tangent cone of rank ≤ 2. If this rank is exacly 2, we must have $r_0+r_2+r_3>0$, that is, $r_1<0$. But then the monomial X_1^3 does not appear in F, so S cannot have an ordinary cusp. We conclude that *cubic surfaces with only ordinary double points or cusps are semi-stable.*

b) When F is only assumed to be non-stable, the condition becomes $\mathbf{r}\boldsymbol{\alpha} \geq 0$ when $a_{\boldsymbol{\alpha}} \neq 0$. If $r_2>0$ the same analysis shows that S cannot have an ordinary double point at $(1,0,0,0)$, but we may now have $r_2=0$. If $r_0<r_1$ the only change is that F can contain the monomial $X_0X_1X_3$; on the other hand it cannot contain $X_0X_2^2$, and therefore the tangent cone has again rank ≤ 2. Finally if $r_0=r_1$, so that $\mathbf{r}=(-1,-1,0,2)$, it is easy to check that S has at least one cusp. So *cubic surfaces with at most ordinary double points are stable.*

c) Conversely, suppose S has a singularity worse than an ordinary double point. In an appropriate system of coordinates its equation can be written

$$X_0Q(X_1,X_2,X_3)+H(X_1,X_2,X_3) \quad \text{with} \quad \deg Q=2,\ \deg H=3\ .$$

The tangent cone $Q=0$ has rank ≤ 2. If the rank is ≤ 1, we can assume $Q=aX_3^2$; the homomorphism $\lambda:\mathbf{C}^*\to \mathrm{SL}(4)$ associated to $\mathbf{r}=(-5,1,1,3)$ takes F to 0, so S is not semi-stable. If $\operatorname{rk} Q=2$, we can assume $Q=X_2X_3$; we take the homomorphism $\lambda:\mathbf{C}^*\to \mathrm{SL}(4)$ associated to $\mathbf{r}=(-2,0,1,1)$. Then $\lim_{t\to 0}\lambda(t)F=X_0X_2X_3+cX_1^3$, where c is the coefficient of X_1^3 in H. If the point $(1,0,0,0)$ is not an ordinary cusp, we have $c=0$, and the limit is the union of 3 planes which is not semi-stable. We conclude that *cubic surfaces with a singularity worse than an ordinary cusp are not semi-stable.*

d) Now if S has an ordinary cusp at $(1,0,0,0)$ (that is, $c\neq 0$), we find $\lim_{t\to 0}\lambda(t)F=F_0:=X_0X_2X_3+cX_1^3$. The surface S_0 defined by that equation has 3 ordinary cusps, and therefore is semi-stable; every orbit closure of a cuspidal cubic contains S_0, so its orbit is closed. Finally S_0 is not stable because its stabilizer in $\mathrm{SL}(4)$ contains the matrices $\mathrm{diag}(\lambda,1,\mu,\nu)$ with $\lambda,\mu,\nu\in\mathbf{C}^*$, $\lambda\mu\nu=1$. ∎

(6.6) We will denote by $\mathcal{C}_s\subset H^0(\mathbf{P}^3,\mathcal{O}(3))$ the open subset of stable cubic forms, and $\mathcal{C}$ the open subset of $\mathcal{C}_s$ corresponding to smooth surfaces. According to (6.3) there exists a good quotient $\mathcal{M}_s:=\mathcal{C}_s/\mathrm{GL}(4)$ which contains $\mathcal{M}=\mathcal{C}/\mathrm{GL}(4)$ as an open subset, and which parametrizes isomorphism classes of cubic surfaces with at most ordinary double points. It admits a normal, projective one-point compactification $\mathcal{M}_{ss}$.

Let $\Delta:=\mathcal{C}_s-\mathcal{C}$ be the subvariety of $\mathcal{C}_s$ parametrizing singular surfaces.

Proposition 6.7 .– *Δ is an irreducible divisor in $\mathcal{C}_s$ with local normal crossings.*

Recall that this means that at each point of Δ, there exists a system of local coordinates $z_1, \ldots, z_N$ on $\mathcal{C}_s$ and an integer $k \le \mathrm{N}$ such that Δ is given by $z_1 \ldots z_k = 0$.

Proof: In $\mathbf{P}^3 \times \mathcal{C}_s$, consider the incidence variety

$$\mathcal{I} = \{(p, \mathrm{F}) \mid p \in \mathrm{Sing}(\mathrm{F})\}$$

The fibre of $\mathcal{I}$ over $p \in \mathbf{P}^3$ is an open subset of the linear subspace of cubics singular at x, hence $\mathcal{I}$ is smooth, irreducible, of codimension 4. The projection $q : \mathcal{I} \to \mathcal{C}_s$ induces a finite, birational morphism $\mathcal{I} \to \Delta$; its fibre above $\mathrm{F} \in \Delta$ is the number of singular points of the cubic $\mathrm{F} = 0$. This implies in particular that Δ is an irreducible divisor.

Let $(p, \mathrm{F}) \in \mathcal{I}$. Choosing a plane at infinity in $\mathbf{P}^3$ away from p, we view p in $\mathbf{C}^3$ and F as a polynomial on $\mathbf{C}^3$. A tangent vector to $\mathbf{C}^3 \times \mathcal{C}_s$ at (p, F) is given by a pair (v, G) with $v \in \mathbf{C}^3$, $\mathrm{G} \in \mathrm{H}^0(\mathbf{P}^3, \mathcal{O}(3))$; it is tangent to $\mathcal{I}$ when $(\mathrm{F} + \varepsilon\mathrm{G})(p + \varepsilon v) = (\mathrm{F}' + \varepsilon\mathrm{G}')(p + \varepsilon v) = 0$ (with $\varepsilon^2 = 0$), that is:

$$\mathrm{G}(p) = \mathrm{G}'(p) + \mathrm{F}''(p) \cdot v = 0$$

Since the hessian matrix $\mathrm{F}''(p)$ is invertible, the second equation determines uniquely v once G is known. In other words, *the tangent map* $\mathrm{T}q : \mathrm{T}_{p,\mathrm{F}}(\mathcal{I}) \to \mathrm{T}_{\mathrm{F}}(\mathcal{C}_s)$ *is injective, and its image is the hyperplane* $\mathcal{C}_p$ *of cubic forms passing through* p.

Suppose S has k ordinary double points $p_1, \ldots, p_k$. Then Δ is locally isomorphic to the union of the hyperplanes $\mathcal{C}_{p_i}$; what remains to be proved is that these hyperplanes are linearly independent, that is, that the points p_i impose independent conditions to cubic surfaces. But a cubic surface has at most 4 ordinary double points (the maximum is attained only for the *Cayley cubic* $\sum_{i<j<k} \mathrm{X}_i\mathrm{X}_j\mathrm{X}_k = 0$). It is immediate to check that 4 points or less impose independent conditions on cubic surfaces. ■

7. Extension of the period map

(7.1) Our aim now is to extend the covering $\widetilde{\mathcal{M}} \to \mathcal{M}$ to a (ramified) covering $\widetilde{\mathcal{M}}_s \to \mathcal{M}_s$. To avoid the difficulties due to the singularities of $\mathcal{M}$ we will rather work with $\mathcal{C}$ and $\mathcal{C}_s$. As in (2.8) we define a moduli space $\widetilde{\mathcal{C}}$ consisting of cubic forms F together with a framing of the threefold V defined by $\mathrm{X}_4^3 - \mathrm{F} = 0$; forgetting the framing gives an étale Galois covering $\widetilde{\mathcal{C}} \to \mathcal{C}$, with Galois group Γ. It extends to a ramified covering $\widetilde{\mathcal{C}}_s \to \mathcal{C}_s$ for purely topological reasons, which we now explain.

Suppose given a (connected) manifold X, and an open subset $U \subset X$ such that $\Delta := X - U$ is a divisor with local normal crossings. Let $\pi : \widetilde{U} \to U$ be an étale covering. Let $x \in \Delta$. Locally around x X is isomorphic to D^n, and U to $(D^*)^k \times D^{n-k}$. In particular we obtain k commuting classes in $\pi_1(U)$, and therefore k commuting transformations of the fibre $\pi^{-1}(u)$ for $u \in U$, well-defined up to conjugacy; these are the monodromy transformations at x.

Lemma 7.2 .– *Assume that the monodromy transformations are of finite order. There exists a manifold* $\widetilde{X}$ *and a branched covering* $\widetilde{X} \to X$ *extending* π. *Any covering with these properties is isomorphic to* $\tilde{\pi}$.

Proof : The result is well-known when the covering is finite: then $\widetilde{X}$ is simply the normalization of X in the function field of $\widetilde{U}$. This is almost the case here: locally over U, the covering $\widetilde{U} \to U$ is a disjoint union of finite coverings. Thus it admits a normal extension, unique up to isomorphism. Because of the unicity these local coverings glue together to define $\widetilde{X}$.

It remains to check that $\widetilde{X}$ is smooth. We can suppose $X = D^n$, $U = (D^*)^k \times D^{n-k}$. Then any component of $\widetilde{U}$ is again isomorphic to $(D^*)^k \times D^{n-k}$ (mapped to U by $(z_1, \ldots, z_n) \mapsto (z_1^{r_1}, \ldots, z_k^{r_k}; z_{k+1}, \ldots, z_n)$), and the corresponding component of $\widetilde{X}$ is $D^k \times D^{n-k}$. ∎

(7.3) To apply this lemma to the covering $\widetilde{\mathcal{C}} \to \mathcal{C}$, we need to know that the monodromy transformations are of finite order. For this we will need some basic facts about monodromy; a possible reference is [D], § 1, see also [B3].

Over $\mathcal{C}$ we have a universal family of cubic threefolds $f : \mathcal{V} \to \mathcal{C}$, and therefore a local system $R^3 f_*(\mathbf{Z})$. Let D be a disk in $\mathcal{C}_s$, with $D^* \subset \mathcal{C}$ while 0 corresponds to a surface with nodes. This give rise to a monodromy transformation T of $H^3(\mathcal{V}_\varepsilon, \mathbf{Z})$, for $\varepsilon \in D^*$, which is computed as follows. Suppose first that the cubic surface acquires one double point at 0, with equation in local coordinates $x^2 + y^2 + z^2 = 0$. Then $\mathcal{V}_0$ acquires a singularity of type A_2, given locally by $w^3 = x^2 + y^2 + z^2$. There are two *vanishing cycles* δ, η in $H^3(\mathcal{V}_\varepsilon, \mathbf{Z})$, with $\langle \delta, \eta \rangle = 1$. The monodromy is the composition of the symplectic transvections T_δ and T_η w.r.t. to δ and η. Thus T is the identity on the orthogonal of δ and η, while in the plane spanned by δ, η it is represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

which is of order 6.

If the surface acquires k nodes, $\mathcal{V}_0$ acquires k singular points of type A_2; we get k pairs (δ_i, η_i) which are orthogonal to each other (intuitively, the vanishing cycles live near each singularity, and thus do not mix). Therefore the transformations $\mathrm{T}_{\delta_i}\mathrm{T}_{\eta_i}$ commute, and their product T is still of order 6.

This gives the monodromy transformation T for the local system $\mathrm{R}^3 f_*(\mathbf{Z})$; the covering $\widetilde{\mathcal{C}} \to \mathcal{C}$ corresponds to the local system $\mathrm{Isom}(\mathbf{Z}[\rho]^{4,1}, \mathrm{R}^3 f_*(\mathbf{Z}))$, where the monodromy is given by $\lambda \mapsto \mathrm{T} \circ \lambda$. This is still of order 6, hence we can apply the lemma. We conclude that *the covering* $\widetilde{\mathcal{C}} \to \mathcal{C}$ *extends to a branched covering* $\widetilde{\mathcal{C}}_s \to \mathcal{C}_s$ *, with* $\widetilde{\mathcal{C}}_s$ *smooth.*

Because of the unicity, the (commuting) actions of $\mathrm{GL}(4)$ and Γ on $\widetilde{\mathcal{C}}$ both extend to $\widetilde{\mathcal{C}}_s$. The quotient $\widetilde{\mathcal{M}}_s = \widetilde{\mathcal{C}}_s/\mathrm{GL}(4)$ is a ramified Galois cover of $\mathcal{M}_s$, with group Γ.

Now comes the reward:

Proposition 7.4.– *The period maps* $\tilde{\wp} : \widetilde{\mathcal{M}} \to \mathbf{B}_4$ *and* $\wp : \mathcal{M} \to \mathbf{B}_4/\Gamma$ *extend to* $\tilde{\wp}_s : \widetilde{\mathcal{M}}_s \to \mathbf{B}_4$ *and* $\wp_s : \mathcal{M}_s \to \mathbf{B}_4/\Gamma$ *.*

Proof: The composite map $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}} \xrightarrow{\wp} \mathbf{B}_4$ extends to $\widetilde{\mathcal{C}}_s$ by the Riemann extension theorem [G-H, p. 9]. The map obtained is equivariant with respect to $\mathrm{GL}(4)$, hence factors through a map $\tilde{\wp}_s : \widetilde{\mathcal{M}}_s \to \mathbf{B}_4$ which extends $\tilde{\wp}$. This map is Γ-equivariant (because it is so on $\widetilde{\mathcal{M}}$), hence provides a map $\wp_s : \mathcal{M}_s \to \mathbf{B}_4/\Gamma$ extending $\wp$. ∎

(7.5) The last step is to extend $\wp_s$ to the one-point compactification $\mathcal{M}_{ss}$ of $\mathcal{M}_s$ (6.6). For this we need to compactify the variety $\mathbf{B}_4/\Gamma$. There is a general way of doing that, called the Satake, or Baily-Borel, compactification. It is particularly simple in the case of the ball $\mathbf{B}_n$ (and well-known in the case of $\mathbf{B}_1 = \mathbf{H}_1$, which is used to compactify modular curves). One adds to $\mathbf{B}_n$ the *rational points* of the boundary $\mathbf{S}_{n-1}$ of $\mathbf{B}_n$; a basis of neighborhoods for such a point $r \in \mathbf{S}_{n-1}(\mathbf{Q})$ is formed by the balls inside $\mathbf{B}_n$ tangent to $\mathbf{B}_n$ at r (to which r itself is added). The group $\mathrm{PU}(n,1;\mathbf{Q})$ acts on $\widehat{\mathbf{B}}_n := \mathbf{B}_n \cup \mathbf{S}_{n-1}(\mathbf{Q})$; given a subgroup Γ of $\mathrm{PU}(n,1;\mathbf{Q})$ such that $\mathbf{B}_n/\Gamma$ has finite volume, one shows that the quotient $\widehat{\mathbf{B}}_n/\Gamma$ has a natural structure of projective variety.

In our case, one proves [ACT, 7.22] that Γ $(= \mathrm{PU}(\mathbf{Z}[\rho]^{4,1}))$ acts transitively on $\mathbf{S}_4(\mathbf{Q})$, so that $\widehat{\mathbf{B}}_4/\Gamma$ is again a one-point compactification of $\mathbf{B}_4/\Gamma$. We can now state:

Proposition 7.6.– *The period map* $\wp_s : \mathcal{M}_s \to \mathbf{B}_4/\Gamma$ *extends to a map* $\wp_{ss} : \mathcal{M}_{ss} \to \widehat{\mathbf{B}}_4/\Gamma$ *, which maps the unique non-stable point of* $\mathcal{M}_{ss}$ *to the boundary point of* $\widehat{\mathbf{B}}_4/\Gamma$ *.*

This requires a somewhat detailed analysis of the behaviour of the period map under degeneration, for which we refer to [ACT]. As explained in (2.11), this implies the main result. ∎

8. Complements

Some other results of [ACT]

(8.1) As mentioned in the introduction, there are many other results in [ACT]. We will only discuss briefly two of them, because they are natural complements to what we have done so far. The first one shows that the various extensions of the period map that we have defined are well-behaved:

Proposition 8.2 .– *The period maps* $\tilde{\wp}_s : \widetilde{\mathcal{M}}_s \to \mathbf{B}_4$, $\wp_s : \mathcal{M}_s \to \mathbf{B}_4/\Gamma$ *and* $\wp_{ss} : \mathcal{M}_{ss} \to \widehat{\mathbf{B}}_4/\Gamma$ *are isomorphisms.* ∎

The proof requires a thorough analysis of the behaviour of these maps along the divisor of nodal surfaces.

(8.3) The space $\widetilde{\mathcal{M}}$ of framed cubic surfaces may appear somewhat artificial; it would be more natural, in view of § 1, to study the space $\mathcal{M}'$ of *marked* cubic surfaces, that is, of pairs (S, σ) where $\sigma : \mathbf{Z}^{1,6} \xrightarrow{\sim} \mathrm{H}^2(\mathrm{S}, \mathbf{Z})$ is an isometry mapping the class $h_0 = (3, -1, \ldots, -1)$ onto the class of a hyperplane section. This space turns out to be related to $\widetilde{\mathcal{M}}$ as follows. Consider the homomorphism $\mathbf{Z}[\rho] \to \mathbf{F}_3$ which maps ρ to 1 . We have $\mathbf{Z}[\rho]^{4,1} \otimes_{\mathbf{Z}[\rho]} \mathbf{F}_3 = \mathbf{F}_3^5$, and the hermitian form $h_{4,1}$ induces the quadratic form q on $\mathbf{F}_3^5$ such that $q(x) = -x_0^2 + x_1^2 + \ldots + x_4^2$. Thus we have a homomorphism $\mathrm{U}(\mathbf{Z}[\rho]^{4,1}) \to \mathrm{O}(q, \mathbf{F}_3)$, which induces a homomorphism

$$\varphi : \Gamma = \mathrm{PU}(\mathbf{Z}[\rho]^{4,1}) \longrightarrow \mathrm{PO}(q, \mathbf{F}_3) \ .$$

Proposition 8.4 .– *The homomorphism* φ *is surjective; let* $\Gamma' \subset \Gamma$ *be its kernel. The moduli space* $\mathcal{M}'$ *is isomorphic to* $\widetilde{\mathcal{M}}/\Gamma'$ *(and therefore to* $(\mathbf{B}_4 - \mathcal{H})/\Gamma')$.

Thus we have a tower of (ramified) Galois coverings:

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \longrightarrow & \widetilde{\mathcal{M}}/\Gamma' & \longrightarrow & \widetilde{\mathcal{M}}/\Gamma \\ & & \shortparallel & & \shortparallel \\ & & \mathcal{M}' & \longrightarrow & \mathcal{M} \end{array}$$

with Galois groups Γ' , Γ and $\Gamma/\Gamma' = \mathrm{PO}(q, \mathbf{F}_3)$. The group $\mathrm{PO}(q, \mathbf{F}_3)$ is isomorphic to the Weyl group $\mathrm{W}(\mathrm{E}_6)$ ([Bo], § 4, exerc. 2), and a marking of a cubic surface is easily seen to be equivalent to fixing the configuration of its 27 lines (with its incidence relation). Thus we recover the classical fact that the automorphism group of that configuration is $\mathrm{W}(\mathrm{E}_6)$.

The proof of the Proposition is somewhat indirect: we do not know how to deduce a marking of a cubic surface from a framing.

Further developpments

(8.5) Allcock and Freitag have used Borcherds' method to construct automorphic forms on $\mathbf{B}_4$ w.r.t. the group Γ'. These forms embed $\mathbf{B}_4/\Gamma'$, and therefore the moduli space $\mathcal{M}'$, into $\mathbf{P}^9$; the image is defined by cubic equations [AF]. This embedding is analyzed from a different point of view in [vG].

(8.6) A completely different approach, leading to very analogous results, is proposed in [DGK]. The authors associate to a cubic surface a particular K3 surface with an automorphism of order 3. The periods of K3 surfaces of this type turn out to be parameterized again by the ball $\mathbf{B}_4$, thus providing another uniformization of the moduli spaces $\mathcal{M}$, $\mathcal{M}'$, etc. by the ball.

Other situations

(8.7) As explained in [ACT], the idea of considering the periods of a branched covering goes back at least to Picard: in [P], he associates to a 5-point set $\mathrm{S} \subset \mathbf{P}^1$ the cyclic triple cover $\mathrm{C} \to \mathbf{P}^1$ branched along S; this is a curve of genus 4 with an automorphism of order 3, so we can mimic the constructions of § 2 in this set-up. One finds that the moduli space of 5-point sets in $\mathbf{P}^1$ is isomorphic to an open subset of $\mathbf{B}_2/\Gamma$, where Γ is some arithmetic subgroup of $\mathrm{PU}(2,1)$. A more general situation is studied in [DM]; in particular, they realize the moduli space of 6-point (resp. 8-point) sets as quotient of $\mathbf{B}_3$ (resp. $\mathbf{B}_5$), using the triple (resp. quadruple) cover of $\mathbf{P}^1$ branched along these sets.

The same idea has been used by Kondō to describe the moduli space of non-hyperelliptic curves of genus 3 and 4 as quotients of the ball [K1, K2].

An analogous description of the moduli space of cubic *threefolds* has been obtained recently in [ACT2], see also [LS2]. In [LS1], the authors give a set of conditions (rather restrictive of course) which guarantee that the period map gives an open embedding into a bounded domain.

Hyperbolic geometry

(8.8) Finally let us say a few words about the "complex hyperbolic geometry" which appears in the title of [ACT]. Let $\mathrm{K} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$; let $\mathrm{K}^{n,1}$ be the K-vector space K^{n+1} with the standard hermitian form of signature $(n,1)$. The set of negative lines in $\mathbf{P}(\mathrm{K}^{n,1})$ is parametrized by the ball $\mathbf{B}_n(\mathrm{K}) \subset \mathrm{K}^n \subset \mathbf{P}(\mathrm{K}^{n,1})$ (see (2.8)). The group $\mathrm{U}(n,1;\mathrm{K})$ acts transitively on the ball, and the stabilizer of 0 is the maximal compact subgroup $\mathrm{U}(n,\mathrm{K}) \times \mathrm{U}(1,\mathrm{K})$. Thus the ball $\mathbf{B}_n(\mathrm{K})$ is identified to the homogeneous space $\mathrm{KH}_n := \mathrm{U}(n,1;\mathrm{K})/(\mathrm{U}(n,\mathrm{K}) \times \mathrm{U}(1,\mathrm{K}))$, the K-*hyperbolic space* of dimension n. These spaces have a rich geometry which has been extensively studied (see for instance [Go] for complex hyperbolic spaces), and which is used in [ACT] e.g. for the detailed study of the group Γ.

[ACT] D. Allcock, J. Carlson, D. Toledo: *The complex hyperbolic geometry of the moduli space of cubic surfaces*. J. Algebraic Geom. **11** (2002), no. 4, 659–724.

[ACT2] D. Allcock, J. Carlson, D. Toledo: *The moduli space of cubic threefolds as a ball quotient*. Preprint `math.AG/0608287`.

[AF] D. Allcock, E. Freitag: *Cubic surfaces and Borcherds products*. Comm. Math. Helv. **77** (2002), 270–296.

[B1] A. Beauville: *Les singularités du diviseur Thêta de la jacobienne intermédiaire de l'hypersurface cubique dans* $\mathbf{P}^4$. Algebraic threefolds (Proc. Varenna 1981), LN **947**, 190–208; Springer-Verlag (1982).

[B2] A. Beauville: *Surfaces K3*. Exp. 609 du Séminaire Bourbaki; Astérisque **105–106**, 217–229 (1983).

[B3] A. Beauville: *Le groupe de monodromie des familles universelles d'hypersurfaces et d'intersections complètes*. Complex analysis and Algebraic Geometry, LN **1194**, 195–207; Springer-Verlag (1986).

[Bo] N. Bourbaki: *Groupes et algèbres de Lie.*, Chap. VI. Hermann, Paris (1968).

[BHPV] W. Barth, K. Hulek, C. Peters, A. Van de Ven: *Compact complex surfaces*. 2nd edition. Ergebnisse der Math., **4**. Springer-Verlag, Berlin, 2004.

[CG] H. Clemens, P. Griffiths: *The intermediate Jacobian of the cubic threefold*. Ann. of Math. (2) **95** (1972), 281–356.

[D] M. Demazure: *Classification des germes à point critique isolé et à nombres de modules* 0 *ou* 1 *(d'après V. I. Arnold)*. Exp. 443 du Séminaire Bourbaki; Lecture Notes in Math. **431**, 124–142. Springer, Berlin (1975).

[Do] R. Donagi: *Generic Torelli for projective hypersurfaces*. Compositio Math. **50** (1983), no. 2-3, 325–353.

[D-K] I. Dolgachev, V. Kanev: *Polar covariants of plane cubics and quartics*. Adv. Math. **98** (1993), no. 2, 216–301.

[DGK] I. Dolgachev, B. van Geemen, S. Kondō: *A complex ball uniformization of the moduli space of cubic surfaces via periods of* K3 *surfaces*. J. Reine Angew. Math. **588** (2005), 99–148.

[DM] P. Deligne, G. Mostow: *Monodromy of hypergeometric functions and non-lattice integral monodromy*. Publ. Math. I.H.E.S. **63** (1986) 5–89.

[G] P. Griffiths: *On the periods of certain rational integrals*. Ann. of Math. (2) **90** (1969), 460–541.

[Go] W. Goldman: *Complex hyperbolic geometry*. Oxford University Press, New York, 1999.

[vG] B. van Geemen: *A linear system on Naruki's moduli space of marked cubic surfaces*. Internat. J. Math. **13** (2002), no. 2, 183–208.

[K1] S. Kondō: *A complex hyperbolic structure for the moduli space of curves of genus three*. J. Reine Angew. Math. **525** (2000), 219–232.

[K2] S. Kondō: *The moduli space of curves of genus 4 and Deligne-Mostow's complex reflection groups*. Algebraic geometry 2000, Azumino (Hotaka), 383–400, Adv. Stud. Pure Math. **36**, Math. Soc. Japan, Tokyo, 2002.

[LS1] E. Looijenga, R. Swierstra: *On period maps that are open embeddings*. Preprint `math.AG/0512489`.

[LS2] E. Looijenga, R. Swierstra: *The period map for cubic threefolds*. Preprint `math.AG/0608279`.

[M] D. Mumford, J. Fogarty, F. Kirwan: *Geometric invariant theory*. 3rd ed. Springer-Verlag, Berlin-New York, 1994.

[P] E. Picard: *Sur les fonctions de deux variables indépendantes analogues aux fonctions modulaires*. Acta Math. **2** (1883) 114–135.

[S] J.-P.Serre: *Cours d'arithmétique*. Presses Universitaires de France, Paris, 1977.

[V] C. Voisin: *Théorie de Hodge et géométrie algébrique complexe*. Cours Spécialisés **10**. SMF, Paris, 2002.

[X] *Géométrie des surfaces K3: modules et périodes* (séminaire Palaiseau 81-82). Astérisque **126** (1985).

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