The Verlinde formula for $\mathbf{PGL}_p$

Arnaud Beauville [1]

To the memory of
Claude ITZYKSON

Introduction

The Verlinde formula expresses the number of linearly independent conformal blocks in any rational conformal field theory. I am concerned here with a quite particular case, the Wess-Zumino-Witten model associated to a complex semi-simple group [2] G. In this case the space of conformal blocks can be interpreted as the space of holomorphic sections of a line bundle on a particular projective variety, the moduli space $\mathrm{M_G}$ of holomorphic G-bundles on the given Riemann surface. The fact that the dimension of this space of sections can be explicitly computed is of great interest for mathematicians, and a number of rigorous proofs of that formula (usually called by mathematicians, somewhat incorrectly, the "Verlinde formula") have been recently given (see e.g. [F], [B-L], [L-S]).

These proofs deal only with simply-connected groups. In this paper we treat the case of the projective group $\mathbf{PGL}_r$ when r is prime.

Our approach is to relate to the case of $\mathbf{SL}_r$, using standard algebro-geometric methods. The components $\mathrm{M}^d_{\mathbf{PGL}_r}$ $(0\le d<r)$ of the moduli space $\mathrm{M}_{\mathbf{PGL}_r}$ can be identified with the quotients $\mathrm{M}^d_r/\mathrm{J}_r$, where M^d_r is the moduli space of vector bundles on X of rank r and fixed determinant of degree d, and J_r the finite group of holomorphic line bundles α on X such that $\alpha^{\otimes r}$ is trivial. The space we are looking for is the space of J_r-invariant global sections of a line bundle $\mathcal{L}$ on M^d_r; its dimension can be expressed in terms of the character of the representation of J_r on $\mathrm{H}^0(\mathrm{M}^d_r,\mathcal{L})$. This is given by the Lefschetz trace formula, with a subtlety for $d=0$, since M^0_r is not smooth. The key point (already used in [N-R]) which makes the computation quite easy is that the fixed point set of any non-zero element of J_r is an abelian variety – this is where the assumption on the group is essential. Extending the method to other cases would require a Chern classes computation on the moduli space $\mathrm{M_H}$ for some semi-simple subgroups H of G; this may be feasible, but goes far beyond the scope of the present paper. Note that the case of $\mathrm{M}^1_{\mathbf{PGL}_2}$ has been previously worked out in [P] (with an unfortunate misprint in the formula).

[1] Partially supported by the European HCM project "Algebraic Geometry in Europe" (AGE).

[2] This group is the complexification of the compact semi-simple group considered by physicists.

In the last section we check that our formulas agree with the predictions of Conformal Field Theory, as they appear for instance in [S-Y]. Note that our results are slightly more precise (in this particular case): we get a formula for $\dim \mathrm{H}^0(\mathrm{M}^d_{\mathbf{PGL}_r}, \mathcal{L})$ for every d, while CFT only predicts the sum of these dimensions (see Remark 4.3).

1. The moduli space $\mathrm{M}_{\mathbf{PGL}_r}$

(1.1) Throughout the paper we denote by X a compact (connected) Riemann surface, of genus $g \geq 2$; we fix a point p of X. Principal $\mathbf{PGL}_r$-bundles on X correspond in a one-to-one way to projective bundles of rank $r-1$ on X, i.e. bundles of the form $\mathbf{P}(\mathrm{E})$, where E is a rank r vector bundle on X; we say that $\mathbf{P}(\mathrm{E})$ is semi-stable if the vector bundle E is semi-stable. The semi-stable projective bundles of rank $r-1$ on X are parameterized by a projective variety, the moduli space $\mathrm{M}_{\mathbf{PGL}_r}$.

Two vector bundles E, F give rise to isomorphic projective bundles if and only if F is isomorphic to $\mathrm{E} \otimes \alpha$ for some line bundle α on X. Thus a projective bundle can always be written as $\mathbf{P}(\mathrm{E})$ with $\det \mathrm{E} = \mathcal{O}_{\mathrm{X}}(dp)$, $0 \leq d < r$; the vector bundle E is then determined up to tensor product by a line bundle α with $\alpha^r = \mathcal{O}_{\mathrm{X}}$. In particular, the moduli space $\mathrm{M}_{\mathbf{PGL}_r}$ has r connected components $\mathrm{M}^d_{\mathbf{PGL}_r}$ $(0 \leq d < r)$. Let us denote by M^d_r the moduli space of semi-stable vector bundles on X of rank r and determinant $\mathcal{O}_{\mathrm{X}}(dp)$, and by J_r the kernel of the multiplication by r in the Jacobian JX of X; it is a finite group, canonically isomorphic to $\mathrm{H}^1(\mathrm{X}, \mathbf{Z}/(r))$. The group J_r acts on M^d_r, by the rule $(\alpha, \mathrm{E}) \mapsto \mathrm{E} \otimes \alpha$; it follows from the above remarks that the component $\mathrm{M}^d_{\mathbf{PGL}_r}$ is isomorphic to the quotient $\mathrm{M}^d_r/\mathrm{J}_r$.

(1.2) We will need a precise description of the line bundles on $\mathrm{M}_{\mathbf{PGL}_r}$. Let me first recall how line bundles on M^d_r can be constructed [D-N]: a simple way is to mimic the classical definition of the theta divisor on the Jacobian of X (i.e. in the rank 1 case). Put $\delta = (r, d)$; let A be a vector bundle on X of rank r/δ and degree $(r(g-1)-d)/\delta$. These conditions imply $\chi(\mathrm{E} \otimes \mathrm{A}) = 0$ for all E in M^d_r; if A is general enough, it follows that the condition $\mathrm{H}^0(\mathrm{X}, \mathrm{E} \otimes \mathrm{A}) \neq 0$ defines a (Cartier) divisor Θ_{A} in M^d_r. The corresponding line bundle $\mathcal{L}_d := \mathcal{O}(\Theta_{\mathrm{A}})$ does not depend on the choice of A, and generates the Picard group $\mathrm{Pic}(\mathrm{M}^d_r)$.

(1.3) The quotient map $q : \mathrm{M}^d_r \to \mathrm{M}^d_{\mathbf{PGL}_r}$ induces a homomorphism $q^* : \mathrm{Pic}(\mathrm{M}^d_{\mathbf{PGL}_r}) \to \mathrm{Pic}(\mathrm{M}^d_r)$, which is easily seen to be injective. Its image is determined in [B-L-S]: it is generated by $\mathcal{L}_d^{\delta}$ if r is odd, by $\mathcal{L}_d^{2\delta}$ if r is even.

(1.4) Let $\mathcal{L}'$ be a line bundle on $\mathrm{M}^d_{\mathbf{PGL}_r}$. The line bundle $\mathcal{L} := q^*\mathcal{L}'$ on

M_r^d admits a natural action of J_r, compatible with the action of J_r on M_r^d (this is often called a J_r-linearization of $\mathcal{L}$). This action is characterized by the property that every element α of J_r acts trivially on the fibre of $\mathcal{L}$ at a point of M_r^d fixed by α. In the sequel we will always consider line bundles on M_r^d of the form $q^*\mathcal{L}'$, and endow them with the above J_r-linearization.

This linearization defines a representation of J_r on the space of global sections; essentially by definition, the global sections of $\mathcal{L}'$ correspond to the J_r-invariant sections of $\mathcal{L}$. Therefore our task will be to compute the dimension of the space of invariant sections; as indicated in the introduction, we will do that by computing, for any $\alpha \in J_r$ of order r, the trace of α acting on $H^0(M_r^d, \mathcal{L})$.

2. The action of J_r on $H^0(M_r^d, \mathcal{L}_d^k)$

We start with the case when r and d are coprime, which is easier to deal with because the moduli space is smooth.

Proposition 2.1 .– *Assume r and d are coprime. Let k be an integer; if r is even we assume that k is even. Let α be an element of order r in* JX *. Then the trace of α acting on $H^0(M_r^d, \mathcal{L}_d^k)$ is $(k+1)^{(r-1)(g-1)}$.*

Proof: The Lefschetz trace formula reads [A-S]

$$\mathrm{Tr}(\alpha \,|\, H^0(M_r^d, \mathcal{L}_d^k)) = \int_P \mathrm{Todd}(T_P)\ \lambda(N_{P/M_r^d}, \alpha)^{-1}\ \widetilde{\mathrm{ch}}(\mathcal{L}^k_{d\,|P}, \alpha)\ .$$

Here P is the fixed subvariety of α; whenever F is a vector bundle on P and φ a diagonalizable endomorphism of F, so that F is the direct sum of its eigen-sub-bundles F_λ for $\lambda \in \mathbf{C}$, we put

$$\widetilde{\mathrm{ch}}(F, \varphi) = \sum \lambda\, \mathrm{ch}(F_\lambda) \quad ; \quad \lambda(F, \varphi) = \prod_\lambda \sum_{p \geq 0} (-\lambda)^p\, \mathrm{ch}(\boldsymbol{\Lambda}^p F_\lambda^*)\ .$$

We have a number of informations on the right hand side thanks to [N-R]:

(2.1 *a*) Let $\pi : \widetilde{X} \to X$ be the étale r-sheeted covering associated to α; put $\xi = \alpha^{r(r-1)/2} \in \mathrm{JX}$. The map $L \mapsto \pi_*(L)$ identifies any component of the fibre of the norm map $\mathrm{Nm} : J^d\widetilde{X} \to J^dX$ over $\xi(dp)$ with P. In particular, P is isomorphic to an abelian variety, hence the term $\mathrm{Todd}(T_P)$ is trivial.

(2.1 *b*) Let $\theta \in H^2(P, \mathbf{Z})$ be the restriction to P of the class of the principal polarization of $J^d\widetilde{X}$. The term $\lambda(N_{P/M_r^d}, \alpha)$ is equal to $r^{r(g-1)} e^{-r\theta}$.

(2.1 *c*) The dimension of P is $N = (r-1)(g-1)$, and the equality $\int_P \frac{\theta^N}{N!} = r^{g-1}$ holds.

With our convention the action of α on $\mathcal{L}^k_{d\,|\mathrm{P}}$ is trivial. The class $c_1(\mathcal{L}_{d\,|\mathrm{P}})$ is equal to $r\theta$: the pull back to P of the theta divisor Θ_{A} (1.2) is the divisor of line bundles L in P with $\mathrm{H}^0(\mathrm{L}\otimes\pi^*\mathrm{A})\neq 0$; to compute its cohomology class we may replace $\pi^*\mathrm{A}$ by any vector bundle with the same rank and degree, in particular by a direct sum of r line bundles of degree $r(g-1)-d$, which gives the required formula.

Putting things together, we find

$$\mathrm{Tr}(\alpha\,|\,\mathrm{H}^0(\mathrm{M}^d_r,\mathcal{L}^k_d)) = \int_{\mathrm{P}} r^{-r(g-1)}e^{r\theta}e^{kr\theta} = (k+1)^{(r-1)(g-1)}\ . \quad \blacksquare$$

We now consider the degree 0 case:

Proposition 2.2 .– *Let k be a multiple of r, and of $2r$ if r is even; let α be an element of order r in JX. Then the trace of α acting on $\mathrm{H}^0(\mathrm{M}^0_r,\mathcal{L}^k_0)$ is $(\frac{k}{r}+1)^{(r-1)(g-1)}$.*

Proof: We cannot apply directly the Lefschetz trace formula since it is manageable only for smooth projective varieties; instead we use another well-known tool, the Hecke correspondence (this idea appears for instance in [B-S]). For simplicity we write M_d instead of M^d_r. There exists a Poincaré bundle $\mathcal{E}$ on $\mathrm{X}\times\mathrm{M}_1$, i.e. a vector bundle whose restriction to $\mathrm{X}\times\{\mathrm{E}\}$, for each point E of M_1, is isomorphic to E. Such a bundle is determined up to tensor product by a line bundle coming from M_1; we will see later how to normalize it. We denote by $\mathcal{E}_p$ the restriction of $\mathcal{E}$ to $\{p\}\times\mathrm{M}_1$, and by $\mathcal{P}$ the projective bundle $\mathbf{P}(\mathcal{E}^*_p)$ on M_1. A point of $\mathcal{P}$ is a pair (E,φ) where E is a vector bundle in M_1 and $\varphi:\mathrm{E}\to\mathbf{C}_p$ a non-zero homomorphism, defined up to a scalar; the kernel of φ is then a vector bundle $\mathrm{F}\in\mathrm{M}_1$, and we can view equivalently a point of $\mathcal{P}$ as a pair of vector bundles (F,E) with $\mathrm{F}\in\mathrm{M}_0$, $\mathrm{E}\in\mathrm{M}_1$ and $\mathrm{F}\subset\mathrm{E}$. The projections p_d on M_d $(d=0,1)$ give rise to the "Hecke diagram"

$$\begin{array}{ccc} & \mathcal{P} & \\ {}^{p_1}\swarrow & & \searrow^{p_0} \\ \mathrm{M}_1 & & \mathrm{M}_0 \end{array}\ .$$

Lemma 2.3 .– *The Poincaré bundle $\mathcal{E}$ can be normalized (in a unique way) so that $\det\mathcal{E}_p=\mathcal{L}_1$; then $\mathcal{O}_{\mathcal{P}}(1)\cong p_0^*\mathcal{L}_0$.*

Proof: Let $\mathrm{E}\in\mathrm{M}_1$. The fibre $p_1^{-1}(\mathrm{E})$ is the projective space of non-zero linear forms $\ell:\mathrm{E}_p\to\mathbf{C}$, up to a scalar. The restriction of $p_0^*\mathcal{L}_0$ to this projective space is $\mathcal{O}(1)$ (choose a line bundle L of degree $g-1$ on X; if E is general enough, $\mathrm{H}^0(\mathrm{X},\mathrm{E}\otimes\mathrm{L})$ is spanned by a section s with $s(p)\neq 0$, and the condition that the bundle F corresponding to ℓ belongs to $\Theta_{\rm L}$ is the vanishing of $\ell(s(p))$). Therefore $p_0^*\mathcal{L}_0$ is of the form $\mathcal{O}_{\mathcal{P}}(1)\otimes p_1^*\mathcal{N}$ for some line bundle $\mathcal{N}$ on M_1. Replacing $\mathcal{E}$ by $\mathcal{E}\otimes\mathcal{N}$ we ensure $\mathcal{O}_{\mathcal{P}}(1)\cong p_0^*\mathcal{L}_0$.

An easy computation gives $\mathrm{K}_{\mathcal{P}}=p_1^*\mathcal{L}_1^{-1}\otimes p_0^*\mathcal{L}_0^{-r}$ ([B-L-S], Lemma 10.3). On the other hand, since $\mathcal{P}=\mathbf{P}(\mathcal{E}_p^*)$, we have $\mathrm{K}_{\mathcal{P}}=p_1^*(\mathrm{K}_{\mathrm{M}_1}\otimes\det\mathcal{E}_p)\otimes\mathcal{O}_{\mathcal{P}}(-r)$; using $\mathrm{K}_{\mathrm{M}_1}=\mathcal{L}_1^{-2}$ [D-N], we get $\det\mathcal{E}_p=\mathcal{L}_1$. ∎

We normalize $\mathcal{E}$ as in the lemma; this gives for each $k\geq 0$ a canonical isomorphism $p_{1*}p_0^*\mathcal{L}_0^k\cong\mathsf{S}^k\mathcal{E}_p$. Let α be an element of order r of JX. It acts on the various moduli spaces in sight; with a slight abuse of language, I will still denote by α the corresponding automorphism. There exists an isomorphism $\alpha^*\mathcal{E}\xrightarrow{\sim}\mathcal{E}\otimes\alpha$, unique up to a scalar ([N-R], lemma 4.7); the induced isomorphism $u:\alpha^*\mathcal{E}_p\xrightarrow{\sim}\mathcal{E}_p$ induces the action of α on $\mathcal{P}$. Imposing $u^r=\mathrm{Id}$ determines u up to a r-th root of unity, hence determines completely $\mathsf{S}^k u$ when k is a multiple of r. Since the Hecke diagram is equivariant with respect to α, it gives rise to a diagram of isomorphisms

$$\begin{array}{ccccc} & & \mathrm{H}^0(\mathcal{P},p_0^*\mathcal{L}_0^k) & & \\ & \overset{p_1^*}{\nearrow} & & \overset{p_0^*}{\nwarrow} & \\ \mathrm{H}^0(\mathrm{M}_1,\mathsf{S}^k\mathcal{E}_p) & & & & \mathrm{H}^0(\mathrm{M}_0,\mathcal{L}_0^k) \end{array}$$

which is compatible with the action of α; in particular, the trace we are looking for is equal to the trace of α on $\mathrm{H}^0(\mathrm{M}_1,\mathsf{S}^k\mathcal{E}_p)$.

We are now in the situation of Prop. 2.1, and the Lefschetz trace formula gives:

$$\mathrm{Tr}(\alpha\,|\,\mathrm{H}^0(\mathrm{M}_1,\mathsf{S}^k\mathcal{E}_p))=\int_{\mathrm{P}}\mathrm{Todd}(\mathrm{T}_{\mathrm{P}})\ \lambda(\mathrm{N}_{\mathrm{P}/\mathrm{M}_1},\alpha)^{-1}\ \widetilde{\mathrm{ch}}(\mathsf{S}^k\mathcal{E}_{p\,|\mathrm{P}},\alpha)\ .$$

The only term we need to compute is $\widetilde{\mathrm{ch}}(\mathsf{S}^k\mathcal{E}_{p\,|\mathrm{P}},\alpha)$. Let $\mathcal{N}$ be the restriction to $\widetilde{\mathrm{X}}\times\mathrm{P}$ of a Poincaré line bundle on $\widetilde{\mathrm{X}}\times\mathrm{J}^1\widetilde{\mathrm{X}}$; let us still denote by $\pi:\widetilde{\mathrm{X}}\times\mathrm{P}\rightarrow\mathrm{X}\times\mathrm{P}$ the map $\pi\times\mathrm{Id}_{\mathrm{P}}$. The vector bundles $\pi_*(\mathcal{N})$ and $\mathcal{E}_{|\mathrm{X}\times\mathrm{P}}$ have the same restriction to $\mathrm{X}\times\{\gamma\}$ for all $\gamma\in\mathrm{P}$, hence after tensoring $\mathcal{N}$ by a line bundle on P we may assume they are isomorphic ([R], lemma 2.5). Restricting to $\{p\}\times\mathrm{P}$ we get $\mathcal{E}_{p\,|\mathrm{P}}=\bigoplus_{\pi(q)=p}\mathcal{N}_q$, with $\mathcal{N}_q=\mathcal{N}_{|\{q\}\times\mathrm{P}}$.

We claim that the $\mathcal{N}_q$'s are the eigen-sub-bundles of $\mathcal{E}_{p\,|\mathrm{P}}$ relative to α. By (2.1 *a*), a pair $(\mathrm{E},\mathrm{F})\in\mathcal{P}$ is fixed by α if and only if $\mathrm{E}=\pi_*\mathrm{L}$, $\mathrm{F}=\pi_*\mathrm{L}'$, with $\mathrm{Nm}(\mathrm{L})=\xi(p)$, $\mathrm{Nm}(\mathrm{L}')=\xi$; because of the inclusion $\mathrm{F}\subset\mathrm{E}$ we may take L' of the form $\mathrm{L}(-q)$, for some point $q\in\pi^{-1}(p)$. In other words, the fixed locus of α acting on $\mathcal{P}$ is the disjoint union of the sections $(\sigma_q)_{q\in\pi^{-1}(p)}$ of the fibration $p_1^{-1}(\mathrm{P})\rightarrow\mathrm{P}$

characterized by $\sigma_q(\pi_*\mathrm{L}) = (\pi_*\mathrm{L}, \pi_*(\mathrm{L}(-q)))$. Viewing $\mathcal{P}$ as $\mathbf{P}(\mathcal{E}^*_{p\,|\mathrm{P}})$, the section σ_q corresponds to the exact sequence

$$0 \to \pi_*(\mathcal{N}(-q))_{\,|\{p\}\times\mathrm{P}} \longrightarrow \pi_*(\mathcal{N})_{\,|\{p\}\times\mathrm{P}} \cong \mathcal{E}_{\,|\{p\}\times\mathrm{P}} \longrightarrow \mathcal{N}_q \to 0\ .$$

Therefore on each fibre $\mathbf{P}(\mathrm{E}_p)$, for $\mathrm{E} \in \mathrm{P}$, the automorphism α has exactly r fixed points, corresponding to the r sub-spaces $\mathcal{N}_{(q,\mathrm{E})}$ for $q \in \pi^{-1}(p)$; this proves our claim.

The line bundles $\mathcal{N}_q$ for $q \in \widetilde{\mathrm{X}}$ are algebraically equivalent, and therefore have the same Chern class. We thus have $c_1(\mathcal{E}_{p\,|\mathrm{P}}) = r\,c_1(\mathcal{N}_q)$. On the other hand we know that $\det \mathcal{E}_p = \mathcal{L}_1$ (lemma 2.3), and that $c_1(\mathcal{L}_{1\,|\mathrm{P}}) = r\theta$ (proof of Prop. 2.1). By comparison we get $c_1(\mathcal{N}_q) = \theta$. Putting things together we obtain

$$\widetilde{\mathrm{ch}}(\mathsf{S}^k\mathcal{E}_{p\,|\mathrm{P}}, \alpha) = \int_{\mathrm{P}} \mathrm{Tr}\,\mathsf{S}^k\mathrm{D}_r\ e^{k\theta} r^{-r(g-1)} e^{r\theta}$$

where D_r is the diagonal r-by-r matrix with entries the r distinct r-th roots of unity.

Lemma 2.4.– *The trace of* $\mathsf{S}^k\mathrm{D}_r$ *is* 1 *if* r *divides* k *and* 0 *otherwise.*

Consider the formal series $s(\mathrm{T}) := \sum_{i\geq 0} \mathrm{T}^i\,\mathrm{Tr}\,\mathsf{S}^i u$ and $\lambda(\mathrm{T}) := \sum_{i\geq 0} \mathrm{T}^i\,\mathrm{Tr}\,\boldsymbol{\Lambda}^i u$. The formula $s(\mathrm{T})\lambda(-\mathrm{T}) = 1$ is well-known (see e.g. [Bo], § 9, formula (11)). But

$$\lambda(-\mathrm{T}) = \sum_{i=0}^{r} (-\mathrm{T})^i\,\mathrm{Tr}\,\boldsymbol{\Lambda}^i u = \prod_{\zeta^r=1} (1-\zeta\mathrm{T}) = 1 - \mathrm{T}^r\ ,$$

hence the lemma. Using (2.1 c) the Proposition follows. ∎

3. Formulas

In this section I will apply the above results to compute the dimension of the space of sections of the line bundle $\mathcal{L}_d^k$ on the moduli space $\mathrm{M}^d_{\mathbf{PGL}_r}$. Let me first recall the corresponding Verlinde formula for the moduli spaces M^d_r. Let $\delta = (r,d)$; we write $\mathcal{L}_d = \mathcal{D}^{r/\delta}$, with the convention that we only consider powers of $\mathcal{D}$ which are multiple of r/δ (the line bundle $\mathcal{D}$ actually makes sense on the *moduli stack* $\mathcal{M}^d_r$, and generates its Picard group). We denote by $\boldsymbol{\mu}_r$ the center of $\mathbf{SL}_r$, i.e. the group of scalar matrices ζI_r with $\zeta^r = 1$.

Proposition 3.1.– *Let* T_k *be the set of diagonal matrices* $t = \mathrm{diag}(t_1, \ldots, t_r)$ *in* $\mathbf{SL}_r(\mathbf{C})$ *with* $t_i \neq t_j$ *for* $i \neq j$, *and* $t^{k+r} \in \boldsymbol{\mu}_r$; *for* $t \in \mathrm{T}_k$, *let* $\delta(t) = \prod_{i<j}(t_i - t_j)$.

Then

$$\dim \mathrm{H}^0(\mathrm{M}_r^d, \mathcal{D}^k) = r^{g-1}(k+r)^{(r-1)(g-1)} \sum_{t\in \mathrm{T}_k/\mathfrak{S}_r} \frac{((-1)^{r-1}t^{k+r})^{-d}}{|\delta(t)|^{2g-2}} \quad .$$

Proof: According to [B-L], Thm. 9.1, the space $\mathrm{H}^0(\mathrm{M}_r^d, \mathcal{D}^k)$ for $0<d<r$ is canonically isomorphic to the space of conformal blocks in genus g with the representation $\mathrm{V}_{k\varpi_{r-d}}$ of $\mathbf{SL}_r$ with highest weight $k\varpi_{r-d}$ inserted at one point. The Verlinde formula gives therefore (see [B], Cor. 9.8 [1]):

$$\dim \mathrm{H}^0(\mathrm{M}_r^d, \mathcal{D}^k) = r^{g-1}(k+r)^{(r-1)(g-1)} \sum_{t\in \mathrm{T}_k/\mathfrak{S}_r} \frac{\mathrm{Tr}_{\mathrm{V}_{k\varpi_{r-d}}}(t)}{|\delta(t)|^{2g-2}} \ ;$$

this is still valid for $d=0$ with the convention $\varpi_r = 0$.

The character of the representation $\mathrm{V}_{k\varpi_{r-d}}$ is given by the Schur formula (see e.g. [F-H], Thm. 6.3):

$$\mathrm{Tr}_{\mathrm{V}_{k\varpi_{r-d}}}(t) = \frac{1}{\delta(t)} \begin{vmatrix} t_1^{k+r-1} & t_2^{k+r-1} & \ldots & t_r^{k+r-1} \\ t_1^{k+r-2} & t_2^{k+r-2} & \ldots & t_r^{k+r-2} \\ \vdots & \vdots & & \vdots \\ t_1^{k+d} & t_2^{k+d} & \ldots & t_r^{k+d} \\ t_1^{d-1} & t_2^{d-1} & \ldots & t_r^{d-1} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{vmatrix} \ .$$

Writing $t^{k+r} = \zeta I_r \in \boldsymbol{\mu}_r$, the big determinant reduces to $\zeta^{r-d}(-1)^{d(r-d)}\det(t_j^{d-i})$, and finally, since $\prod t_i = 1$, to $((-1)^{r-1}\zeta)^{-d}\delta(t)$, which gives the required formula. ∎

Corollary 3.2.– *Let* T'_k *be the set of matrices* $t=\mathrm{diag}(t_1,\ldots,t_r)$ *in* $\mathbf{SL}_r(\mathbf{C})$ *with* $t_i \neq t_j$ *if* $i\neq j$, *and* $t^{k+r} = (-1)^{r-1}I_r$. *Then*

$$\sum_{d=0}^{r-1} \dim \mathrm{H}^0(\mathrm{M}_r^d, \mathcal{D}^k) = r^{g}(k+r)^{(r-1)(g-1)} \sum_{t\in \mathrm{T}'_k/\mathfrak{S}_r} \frac{1}{|\delta(t)|^{2g-2}} \cdot \quad ∎$$

We now consider the moduli space $\mathrm{M}_{\mathbf{PGL}_r}$. We know that the line bundle $\mathcal{D}^k$ on M_r^d descends to $\mathrm{M}^d_{\mathbf{PGL}_r} = \mathrm{M}_r^d/\mathrm{J}_r$ exactly when k is a multiple of r if r is odd, or of $2r$ if r is even (1.3). When this is the case we obtain a line bundle

[1] There is a misprint in the first equality of that corollary, where one should read $\mathrm{T}_\ell^{\mathrm{reg}}/\mathrm{W}$ instead of $\mathrm{T}_\ell^{\mathrm{reg}}$; the second equality (and the proof!) are correct.

on $\mathrm{M}^d_{\mathbf{PGL}_r}$, that we will still denote by $\mathcal{D}^k$; its global sections correspond to the J_r-invariant sections of $\mathrm{H}^0(\mathrm{M}^d_r,\mathcal{D}^k)$.

We will assume that r is *prime*, so that every non-zero element α of J_r has order r. Then Prop. 2.1 and 2.2 lead immediately to a formula for the dimension of the J_r-invariant subspace of $\mathrm{H}^0(\mathrm{M}^d_r,\mathcal{D}^k)$ as the average of the numbers $\mathrm{Tr}(\alpha)$ for α in J_r. Using Prop. 3.1 we conclude:

Proposition 3.3.– *Assume that* r *is prime. Let* k *be a multiple of* r *; if* $r=2$ *assume* $4\mid k$. *Then*

$$\dim \mathrm{H}^0(\mathrm{M}^d_{\mathbf{PGL}_r},\mathcal{D}^k) = r^{-2g}\ \dim \mathrm{H}^0(\mathrm{M}^d_r,\mathcal{D}^k)\ +\ (1-r^{-2g})(\frac{k}{r}+1)^{(r-1)(g-1)}$$
$$= r^{-2g}\,(\frac{k}{r}+1)^{(r-1)(g-1)}\ \Big(r^{r(g-1)}\sum_{t\in \mathrm{T}_k/\mathfrak{S}_r}\frac{((-1)^{r-1}t^{k+r})^{-d}}{|\delta(t)|^{2g-2}}\ +\ r^{2g}-1\Big)\ .$$

Summing over d and plugging in Cor. 3.2 gives the following rather complicated formula:

Corollary 3.4.–

$$\dim \mathrm{H}^0(\mathrm{M}_{\mathbf{PGL}_r},\mathcal{D}^k) = r^{1-2g}\,(\frac{k}{r}+1)^{(r-1)(g-1)}\ \Big(r^{r(g-1)}\sum_{t\in \mathrm{T}'_k/\mathfrak{S}_r}\frac{1}{|\delta(t)|^{2g-2}}+r^{2g}-1\Big)\ .$$

As an example, if k is an integer divisible by 4, we get

$$(3.5)\quad \dim \mathrm{H}^0(\mathrm{M}_{\mathbf{PGL}_2},\mathcal{D}^k) = 2^{1-2g}\,(\frac{k}{2}+1)^{g-1}\Big(\sum_{\substack{l\ \mathrm{odd}\\ 0<l<k+2}}\frac{1}{(\sin\frac{l\pi}{k+2})^{2g-2}}\ +\ 2^{2g}-1\Big)\ .$$

4. Relations with Conformal Field Theory

(4.1) According to Conformal Field Theory, the space $\mathrm{H}^0(\mathrm{M}_{\mathbf{PGL}_r},\mathcal{D}^k)$ should be canonically isomorphic to the space of conformal blocks for a certain Conformal Field Theory, the WZW model associated to the projective group. This implies in particular that its dimension should be equal to $\sum_j |\mathrm{S}_{0j}|^{2-2g}$, where (S_{ij}) is a unitary symmetric matrix. For instance in the case of the WZW model associated to $\mathbf{SL}_2$, we have

$$\mathrm{S}_{0j} = \frac{\sin\frac{(j+1)\pi}{k+2}}{\sqrt{\frac{k}{2}+1}}\qquad \text{, with}\quad 0\le j\le k\ ,$$

where the index j can be thought as running through the set of irreducible representations $\mathsf{S}^1,\ldots,\mathsf{S}^k$ of $\mathbf{SL}_2$ (or equivalently $\mathbf{SU}_2$), with $\mathsf{S}^j:=\mathsf{S}^j(\mathbf{C}^2)$.

We deduce from (3.5) an analogous expression for $\mathbf{PGL}_2$: we restrict ourselves to even indices and write

$$S'_{0j} = 2\,S_{0j} \qquad \text{for} \quad j \text{ even } < k/2 \quad ; \qquad S'_{0,\frac{k}{2}^{(1)}} = S'_{0,\frac{k}{2}^{(2)}} = S_{0\frac{k}{2}} \quad .$$

In other words, we consider only those representations of $\mathbf{SL}_2$ which factor through $\mathbf{PGL}_2$ and we identify the representation S^{2j} with S^{k-2j} , doubling the coefficient S_{0j} when these two representations are distinct, and counting twice the representation which is fixed by the involution (this process is well-known, see e.g. [M-S]).

(4.2) The case of $\mathbf{SL}_r$ is completely analogous; we only need a few more terminology from representation theory (we follow the notation of [B]). The primary fields are indexed by the set P_k of dominant weights λ with $\lambda(H_\theta) \le k$, where H_θ is the matrix $\operatorname{diag}(1,0,\ldots,0,-1)$. For $\lambda \in P_k$, we put $t_\lambda = \exp 2\pi i \,\dfrac{\lambda+\rho}{k+r}$ (we identify the Cartan algebra of diagonal matrices with its dual using the standard bilinear form); the map $\lambda \mapsto t_\lambda$ induces a bijection of P_k onto $T_k/\mathfrak{S}_r$ ([B], lemma 9.3 *c*)). In view of Prop. 3.1, the coefficient $S_{0\lambda}$ for $\lambda \in P_k$ is given by

$$S_{0\lambda} = \frac{\delta(t_\lambda)}{\sqrt{r}(k+r)^{(r-1)/2}} \ .$$

Passing to $\mathbf{PGL}_r$, we first restrict the indices to the subset P'_k of elements $\lambda \in P_k$ such that t_λ belongs to T'_k ; this means that λ belongs to the root lattice, i.e. that the representation V_λ factors through $\mathbf{PGL}_r$. The center $\boldsymbol{\mu}_r$ acts on T_k by multiplication; this action preserves T'_k , and commutes with the action of $\mathfrak{S}_r$. The corresponding action on P_k is deduced, via the bijection $\lambda \mapsto \frac{\lambda+\rho}{k+r}$, from the standard action of $\boldsymbol{\mu}_r$ on the fundamental alcove A with vertices $\{0,\varpi_1,\ldots,\varpi_{r-1}\}$. [1]

We identify two elements of P'_k if they are in the same orbit with respect to this action. The action has a unique fixed point, the weight $\frac{k}{r}\rho$, which corresponds to the diagonal matrix D_r (2.4); we associate to this weight r indices $\nu^{(1)},\ldots,\nu^{(r)}$, and put

$$S'_{0\lambda} = r\,S_{0\lambda} \quad \text{for } \lambda \in P'_k/\boldsymbol{\mu}_r \ , \ \lambda \neq \frac{k}{r}\rho \ ; \qquad S'_{0,\nu^{(i)}} = S_{0,\frac{k}{r}\rho} \quad \text{for } i = 1,\ldots,r \ .$$

From Cor. 3.4 follows easily the formula $\dim H^0(M_{\mathbf{PGL}_r},\mathcal{D}^k) = \sum |S'_{0\lambda}|^{2-2g}$, where λ runs over $P'_k/\boldsymbol{\mu}_r \cup \{\nu^{(1)},\ldots,\nu^{(r)}\}$.

Remark 4.3 .– It is not clear to me what is the physical meaning of the space $H^0(M^d_{\mathbf{PGL}_r},\mathcal{D}^k)$, in particular if its dimension can be predicted in terms of the S-matrix. It is interesting to observe that the number $N(g)$ given by Prop. 3.3, which

[1] The element $\exp\varpi_1$ of the center gives the rotation of A which maps 0 to ϖ_1 , ϖ_1 to ϖ_2 , $\ldots$, and ϖ_{r-1} to 0 .

is equal to $\dim \mathrm{H}^0(\mathrm{M}^d_{\mathbf{PGL}_r}, \mathcal{D}^k)$ for $g \geq 2$, *is not necessarily an integer* for $g = 1$: for $d = 0$ we find $\mathrm{N}(1) = 1 + \dfrac{(k+1)^{r-1} - 1}{r^2}$, which is not an integer unless $r^2 \mid k$.

REFERENCES

[A-S] M.F. Atiyah, I.M. Singer: *The index of elliptic operators III.* Ann. of Math. **87**, 546-604 (1968).

[B] A. Beauville: *Conformal blocks, Fusion rings and the Verlinde formula.* Proc. of the Hirzebruch 65 Conf. on Algebraic Geometry, Israel Math. Conf. Proc. **9**, 75-96 (1996).

[B-L] A. Beauville, Y. Laszlo: *Conformal blocks and generalized theta functions.* Comm. Math. Phys. **164**, 385-419 (1994).

[B-L-S] A. Beauville, Y. Laszlo, Ch. Sorger: *The Picard group of the moduli of* G*-bundles on a curve.* Preprint alg-geom/9608002.

[B-S] A. Bertram, A. Szenes: *Hilbert polynomials of moduli spaces of rank* 2 *vector bundles II.* Topology **32**, 599-609 (1993).

[Bo] N. Bourbaki: *Algèbre*, Chap. X (Algèbre homologique). Masson, Paris (1980).

[D-N] J.M. Drezet, M.S. Narasimhan: *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques.* Invent. math. **97**, 53-94 (1989).

[F] G. Faltings: *A proof for the Verlinde formula.* J. Algebraic Geometry **3**, 347-374 (1994).

[F-H] W. Fulton, J. Harris: *Representation theory.* GTM **129**, Springer-Verlag, New York Berlin Heidelberg (1991).

[L-S] Y. Laszlo, Ch. Sorger: *The line bundles on the moduli of parabolic* G*-bundles over curves and their sections.* Annales de l'ENS, to appear; preprint alg-geom/9507002.

[M-S] G. Moore, N. Seiberg: *Taming the conformal zoo.* Phys. Letters B **220**, 422-430 (1989).

[N-R] M.S. Narasimhan, S. Ramanan: *Generalized Prym varieties as fixed points.* J. of the Indian Math. Soc. **39**, 1-19 (1975).

[P] T. Pantev: *Comparison of generalized theta functions.* Duke Math. J. **76**, 509-539 (1994).

[R] S. Ramanan: *The moduli spaces of vector bundles over an algebraic curve.* Math. Ann. **200**, 69-84 (1973).

[S-Y] A.N. Schellekens, S. Yankielowicz: *Field identification fixed points in the coset construction.* Nucl. Phys. **B 334**, 67 (1990).

Arnaud Beauville
DMI – École Normale Supérieure
(URA 762 du CNRS)
45 rue d'Ulm
F-75230 Paris Cedex 05