NON-RATIONALITY OF THE $S_6$-SYMMETRIC QUARTIC THREEFOLDS

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ABSTRACT. We prove that the quartic hypersurfaces defined by $\sum x_i = t \sum x_i^4 - (\sum x_i^2)^2 = 0$ in $\mathbb{P}^5$ are not rational for $t \neq 0, 2, 4, 6, \frac{10}{7}$.

1. INTRODUCTION

Let $V$ be the standard representation of $S_6$ (that is, $V$ is the hyperplane $\sum x_i = 0$ in $\mathbb{C}^6$, with $S_6$ acting by permutation of the basis vectors). The quartic hypersurfaces in $\mathbb{P}(V)$ ($\cong \mathbb{P}^4$) invariant under $S_6$ form the pencil

$$X_t : t \sum x_i^4 - (\sum x_i^2)^2 = 0, \quad t \in \mathbb{P}^1.$$ 

This pencil contains two classical quartic hypersurfaces, the Burkhardt quartic $X_2$ and the Igusa quartic $X_4$ (see for instance [H]); they are both rational.

For $t \neq 0, 2, 4, 6$ and $\frac{10}{7}$, the quartic $X_t$ has exactly 30 nodes; the set of nodes $N$ is the orbit under $S_6$ of $(1, 1, \rho, \rho, \rho^2, \rho^2)$, with $\rho = e^{2\pi i/3}$ ([vdG], §4). We will prove:

**Theorem.** For $t \neq 0, 2, 4, 6, \frac{10}{7}$, $X_t$ is not rational.

The method is that of [B]: we show that the intermediate Jacobian of a desingularization of $X_t$ is 5-dimensional and that the action of $S_6$ on its tangent space at 0 is irreducible. From this one sees easily that this intermediate Jacobian cannot be a Jacobian or a product of Jacobians, hence $X_t$ is not rational by the Clemens-Griffiths criterion. We do not know whether $X_t$ is unirational.

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2. THE ACTION OF $S_6$ ON $T_0(JX)$

We fix $t \neq 0, 2, 4, 6, \frac{10}{7}$, and denote by $X$ the desingularization of $X_t$ obtained by blowing up the nodes. The main ingredient of the proof is the fact that the action of $S_6$ on $JX$ is non-trivial. To prove this we consider the action of $S_6$ on the tangent space $T_0(JX)$, which is by definition $H^2(X, \Omega^1_X)$.

**Lemma 1.** Let $C$ be the space of cubic forms on $\mathbb{P}(V)$ vanishing along $N$. We have an isomorphism of $S_6$-modules $C \cong V \oplus H^2(X, \Omega^1_X)$.

**Proof:** The proof is essentially contained in [C]; we explain how to adapt the arguments there to our situation. Let $b : P \to \mathbb{P}(V)$ be the blowing-up of $\mathbb{P}(V)$ along $N$. The threefold $X$ is the strict transform of $X_t$ in $P$. The exact sequence

$$0 \to N_{X/P} \to \Omega^1_P|_X \to \Omega^1_X \to 0$$

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gives rise to an exact sequence
\[ 0 \rightarrow H^2(X, O_X^1) \rightarrow H^3(X, N^*_X/P) \rightarrow H^3(X, O_P^1|_X) \rightarrow 0 \]
([C], proof of theorem 1), which is \( \mathcal{G}_6 \)-equivariant. We will compute the two last terms.

The exact sequence
\[ 0 \rightarrow \Omega^1_P(-X) \rightarrow \Omega^1_P \rightarrow \Omega^1_P|_X \rightarrow 0 \]
provides an isomorphism \( H^3(X, \Omega^1_P|_X) \cong H^4(P, \Omega^1_P(-X)) \), and the latter space is isomorphic to \( H^4(P(V), \Omega^1_{P(V)}(-4)) \) ([C], proof of Lemma 3). By Serre duality \( H^4(P(V), \Omega^1_{P(V)}(-4)) \) is dual to \( H^0(P(V), T_{P(V)}(-1)) \cong V \). Thus the \( \mathcal{G}_6 \)-module \( H^3(X, \Omega^1_P|_X) \) is isomorphic to \( V^* \), hence also to \( V \).

Similarly the exact sequence \( 0 \rightarrow \mathcal{O}_P(-2X) \rightarrow \mathcal{O}_P(-X) \rightarrow N^*_X/P \rightarrow 0 \) and the vanishing of \( H^4(P, \mathcal{O}_P(-X)) \) ([C], Corollary 2) provide an isomorphism of \( H^3(X, N^*_X/P) \) onto \( H^4(P, \mathcal{O}_P(-2X)) \), which is naturally isomorphic to the dual of \( C \) ([C], proof of Proposition 2). The lemma follows.

**Lemma 2.** The dimension of \( C \) is 10.

**Proof:** Recall that the defect of \( X_t \) is the difference between the dimension of \( C \) and its expected dimension, namely:
\[ \text{def}(X_t) := \dim C - \left( \dim H^0(P(V), \mathcal{O}_{P(V)}(3)) - \# N \right). \]
Thus our assertion is equivalent to \( \text{def}(X_t) = 5 \).

To compute this defect we use the formula of [D-S], Theorem 1.5. Let \( F = 0 \) be an equation of \( X_t \) in \( P^4 \); let \( R := \mathbb{C}[X_0, \ldots, X_4]/(F'_0, \ldots, F'_4) \) be the Jacobian ring of \( F \), and let \( R^{sm} \) be the Jacobian ring of a smooth quartic hypersurface in \( P^4 \). The formula is
\[ \text{def}(X_t) = \dim R_7 - \dim R^{sm}_7. \]
In our case we have \( \dim R^{sm}_7 = \dim R^{sm}_4 = 35 - 5 = 30 \); a simple computation with Singular (for instance) gives \( \dim R_7 = 35 \). This implies the lemma.

**Proposition.** The \( \mathcal{G}_6 \)-module \( H^2(X, O_X^1) \) is isomorphic to \( V \).

**Proof:** Consider the homomorphisms \( a \) and \( b \) of \( \mathbb{C}^6 \) into \( H^0(P(V), \mathcal{O}_{P(V)}(3)) \) given by \( a(e_i) = x^3_i \), \( b(e_i) = x_i \sum x^3_j \). They are both \( \mathcal{G}_6 \)-equivariant and map \( V \) into \( C \); the subspaces \( a(V) \) and \( b(V) \) of \( C \) do not coincide, so we have \( a(V) \cap b(V) = 0 \). By Lemma 2 this implies \( C = a(V) \oplus b(V) \), so \( H^2(X, O_X^1) \) is isomorphic to \( V \) by Lemma 1.

**Remark.** Suppose \( t = 2, 6 \) or \( 10 \). Then the singular locus of \( X_t \) is \( N' \cup N'' \), where \( N' \) is the \( \mathcal{G}_6 \)-orbit of the point \( (1, -1, 0, 0, 0, 0) \) for \( t = 2 \), \((1, -1, 1, -1, 1, -1) \) for \( t = 6 \), \((-5, 1, 1, 1, 1, 1) \) for \( t = 10 \). Since \( x^3_1 - x^3_0 \) does not vanish on \( N'' \), the space of cubics vanishing along \( N' \cup N'' \) is strictly contained in \( C \). By Lemma 1 it contains a copy of \( V \), hence it is isomorphic to \( V \); therefore \( H^2(X, O_X^1) \) and \( JX \) are zero in these cases. We have already mentioned that \( X_2 \) and \( X_4 \) are rational. The quartic \( X_{10} \) is rational: it is the image of the anticanonical map of \( P^3 \) blown up along 6 lines which are permuted by \( \mathcal{G}_6 \) (see [C-S], proof of Lemma 4.5, and the references given there). We do not know whether this is the case for \( X_6 \).
3. Proof of the theorem

To prove that $X$ is not rational, we apply the Clemens-Griffiths criterion ([C-G], Cor. 3.26): it suffices to prove that $JX$ is not a Jacobian or a product of Jacobians.

Suppose $JX \cong JC$ for some curve $C$ of genus 5. By the Proposition $S_6$ embeds into the group of automorphisms of $JC$ preserving the principal polarization; by the Torelli theorem this group is isomorphic to $\text{Aut}(C)$ if $C$ is hyperelliptic and $\text{Aut}(C) \times \mathbb{Z}/2$ otherwise. Thus we find $\# \text{Aut}(C) \geq \frac{1}{2}6! = 360$. But this contradicts the Hurwitz bound $\# \text{Aut}(C) \leq 84(5 - 1) = 336$.

Now suppose that $JX$ is isomorphic to a product of Jacobians $J_1 \times \ldots \times J_p$, with $p \geq 2$. Recall that such a decomposition is unique up to the order of the factors: it corresponds to the decomposition of the Theta divisor into irreducible components ([C-G], Cor. 3.23). Thus the group $S_6$ permutes the factors $J_i$, and therefore acts on $[1, p]$; by the Proposition this action must be transitive. But we have $p \leq \dim JX = 5$, so this is impossible.

References