Non-rationality of the symmetric sextic Fano threefold

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Pour Gerard Van der Geer, en l'honneur de son 60ème anniversaire

Abstract. We prove that the symmetric sextic Fano threefold, defined by the equations $\sum X_i = \sum X_i^2 = \sum X_i^3 = 0$ in $\mathbb{P}^6$, is not rational. In view of the work of Prokhorov [P], our result implies that the alternating group $\mathfrak{A}_7$ admits only one embedding into the Cremona group Cr_3 up to conjugacy.

Résumé. Nous prouvons que le solide de Fano d'équations $\sum X_i = \sum X_i^2 = \sum X_i^3 = 0$ dans $\mathbb{P}^6$ n'est pas rationnel. Grâce aux résultats de Prokhorov [P], cela entraine que le groupe alterné $\mathfrak{A}_7$ admet un seul plongement (à conjugaison près) dans le groupe de Cremona à 3 variables.

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Introduction

The symmetric sextic Fano threefold is the subvariety X of $\mathbb{P}^6$ defined by the equations

$$\sum X_i = \sum X_i^2 = \sum X_i^3 = 0 \ .$$

It is a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^5$, with an action of $\mathfrak{S}_7$. We will prove that it is not rational.

Any smooth complete intersection of a quadric and a cubic in $\mathbb{P}^5$ is unirational [E]. It is known that a *general* such intersection is not rational: this is proved in [B] (thm. 5.6) using the intermediate Jacobian, and in [Pu] using the group of birational automorphisms. But neither of these methods allows to prove the non-rationality of any particular such threefold. Our method gives the above explicit (and very simple) counter-example to the Lüroth problem.

Our motivation comes from the recent paper of Prokhorov [P], which classifies the simple finite subgroups of the Cremona group $\mathrm{Cr}_3 = \mathrm{Bir}(\mathbb{P}^3)$. In view of this work our result implies that the alternating group $\mathfrak{A}_7$ admits only one embedding into Cr_3 up to conjugacy.

Our proof uses the Clemens-Griffiths criterion ([C-G], Cor. 3.26): if X is rational, its intermediate Jacobian JX is the Jacobian of a curve, or a product of Jacobians. The presence of the automorphism group $\mathfrak{S}_7$, together with the celebrated bound $\#\,\mathrm{Aut}(C) \leq 84(g-1)$ for a curve C of genus g, immediately implies that JX is not isomorphic to the Jacobian of a curve. To rule out products of Jacobians we need some more information, which is provided by a simple analysis of the representation of $\mathfrak{S}_7$ on the tangent space $T_0(JX)$.

Proof of the result

Theorem. *The intermediate Jacobian JX is not isomorphic to a Jacobian or a product of Jacobians. As a consequence, X is not rational.*

The second assertion follows from the first by the Clemens-Griffiths criterion mentioned in the introduction. Since the Jacobians and their products form a closed subvariety of the moduli space of principally polarized abelian varieties, this gives an easy proof of the fact that a general intersection of a quadric and a cubic in $\mathbb{P}^5$ is not rational.

As mentioned in the introduction, the classification in [P] together with the theorem implies:

Corollary. *Up to conjugacy, there is only one embedding of $\mathfrak{A}_7$ into the Cremona group Cr_3, given by an embedding $\mathfrak{A}_7 \subset \mathrm{PGL}_4(\mathbb{C})$.*

(The embedding $\mathfrak{A}_7 \subset \mathrm{PGL}_4(\mathbb{C})$ is the composition of the standard representation $\mathfrak{A}_7 \to \mathrm{SO}_6(\mathbb{C})$ and the double covering $\mathrm{SO}_6(\mathbb{C}) \to \mathrm{PGL}_4(\mathbb{C})$.)

The intermediate Jacobian JX has dimension 20. The group $\mathfrak{S}_7$ acts on JX and therefore on the tangent space $T_0(JX)$; we will first determine this action.

Lemma. *As a $\mathfrak{S}_7$-module $T_0(JX)$ is the sum of two irreducible representations, of dimensions 6 and 14.*

Proof. Let V be the standard (6-dimensional) representation of $\mathfrak{S}_7$, and put $\mathbb{P} := \mathbb{P}(V)$; we will view X as a subvariety of $\mathbb{P}$, stable under $\mathfrak{S}_7$.

By definition $T_0(JX)$ is $\mathrm{H}^2(X, \Omega^1_X)$. Every $\mathfrak{S}_7$-module is isomorphic to its dual, so we can identify $T_0(JX)$ with $\mathrm{H}^1(X, T_X(-1))$ by Serre duality. The exact sequence

$$0 \to T_X \longrightarrow T_{\mathbb{P}|X} \longrightarrow \mathcal{O}_X(2) \oplus \mathcal{O}_X(3) \to 0$$

twisted by $\mathcal{O}_X(-1)$, gives a cohomology exact sequence

$$0 \to \mathrm{H}^0(X, T_{\mathbb{P}}(-1)_{|X}) \to \mathrm{H}^0(X, \mathcal{O}_X(1)) \oplus \mathrm{H}^0(X, \mathcal{O}_X(2)) \to \\ \to \mathrm{H}^1(X, T_X(-1)) \longrightarrow \mathrm{H}^1(X, T_{\mathbb{P}}(-1)_{|X}) \ .$$

From the Euler exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \otimes_{\mathbb{C}} V \to T_{\mathbb{P}|X} \to 0$ we deduce $\mathrm{H}^1(X, T_{\mathbb{P}}(-1)_{|X}) = 0$ and an isomorphism $V \xrightarrow{\sim} \mathrm{H}^0(X, T_{\mathbb{P}}(-1)_{|X})$. Thus we find an exact sequence

$$0 \to V \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X(1)) \oplus \mathrm{H}^0(X, \mathcal{O}_X(2)) \longrightarrow T_0(JX) \to 0 \ ,$$

which is equivariant with respect to the action of $\mathfrak{S}_7$. As representations of $\mathfrak{S}_7$, $\mathrm{H}^0(X, \mathcal{O}_X(1))$ is isomorphic to V and $\mathrm{H}^0(X, \mathcal{O}_X(2))$ to $\mathsf{S}^2V/\mathbb{C}.q$, where q corresponds to the quadric containing X. On the other hand $\mathsf{S}^2V = \mathbb{C} \oplus V \oplus V_{(5,2)}$, where $V_{(5,2)}$ is the irreducible representation of $\mathfrak{S}_7$ corresponding to the partition $(5,2)$ of 7 ([F-H], exercise 4.19). Thus we get $T_0(JX) \cong V \oplus V_{(5,2)}$. Since $\dim T_0(JX) = 20$ and $\dim(V) = 6$ we find $\dim V_{(5,2)} = 14$. □

Proof of the theorem. We first observe that $\mathfrak{A}_7$ cannot act non-trivially on the Jacobian JC of a curve of genus $g \leq 20$. Indeed by the Torelli theorem we have $\mathrm{Aut}(JC) \cong \mathrm{Aut}(C)$ if C is hyperelliptic and $\mathrm{Aut}(JC) \cong \mathrm{Aut}(C) \times \mathbb{Z}/2$ otherwise. Since $\mathfrak{A}_7$ is simple we find $\#\,\mathrm{Aut}(C) \geq \#\,\mathfrak{A}_7 = 2520$. On the other hand we have $\#\,\mathrm{Aut}(C) \leq 84(g-1) \leq 1596$, a contradiction.

Now assume that JX is a product $J_1 \times \ldots \times J_m$ of Jacobians. Such a decomposition is *unique* up to the order of the factors: it corresponds to the decomposition of the Theta divisor into irreducible components, see [C-G], Cor. 3.23. Thus the group $\mathfrak{A}_7$ acts on $[1,m]$ by permuting the factors. Let $O_1, \ldots, O_\ell$ be the orbits of this action. For $1 \leq k \leq \ell$ we put $J_{(k)} := J_{m_k}$ with $m_k = \min O_k$; then for each i in O_k J_i is isomorphic to $J_{(k)}$, so our decomposition can be written $JX \cong J_{(1)}^{O_1} \times \ldots \times J_{(\ell)}^{O_\ell}$.

Since $\#O_k \leq m \leq 20$, the orbit O_k has 1, 7 or 15 elements ([D-M], thm. 5.2.A). If $\#O_k = 1$, $\mathfrak{A}_7$ acts on the Jacobian $J_{(k)}$; by the lemma this action is faithful, contradicting the beginning of the proof. Thus $\#O_k = 7$ or 15 for each k, which contradicts the equality $\sum \#O_k \ \dim(J_{(k)}) = 20$. □

Remarks. The same kind of argument gives the non-rationality of the threefold $\sum X_i^2 = \sum X_i^3 = 0$ in $\mathbb{P}^5$, using the action of $\mathfrak{S}_6$. It also gives a simple proof of the non-rationality of the Klein cubic threefold, defined by $\sum_{i \in \mathbb{Z}/5} X_i^2 X_{i+1} = 0$ in $\mathbb{P}^4$ (and, by the same token, of the general cubic threefold). The automorphism group of the Klein cubic is $\mathrm{PSL}_2(\mathbb{F}_{11})$, of order 660, while its intermediate Jacobian has dimension 5. It is easily seen as above that a 5-dimensional principally polarized abelian variety with an action of $\mathrm{PSL}_2(\mathbb{F}_{11})$ cannot be a Jacobian or a product of Jacobians (see also [Z] for a somewhat analogous, though more sophisticated, proof).

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