Non-rationality of the symmetric sextic Fano threefold

Arnaud Beauville

Pour Gerard Van der Geer, en l’honneur de son 60ème anniversaire

Abstract. We prove that the symmetric sextic Fano threefold, defined by the equations \[ \sum X_i = \sum X_i^2 = \sum X_i^3 = 0 \] in \( \mathbb{P}^6 \), is not rational. In view of the work of Prokhorov [P], our result implies that the alternating group \( \mathfrak{A}_7 \) admits only one embedding into the Cremona group \( \text{Cr}_3 \) up to conjugacy.

Résumé. Nous prouvons que le solide de Fano d’équations \( \sum X_i = \sum X_i^2 = \sum X_i^3 = 0 \) dans \( \mathbb{P}^6 \) n’est pas rationnel. Grâce aux résultats de Prokhorov [P], cela entraîne que le groupe alterné \( \mathfrak{A}_7 \) admet un seul plongement (à conjugaison près) dans le groupe de Cremona à 3 variables.

2010 Mathematics Subject Classification. 14E08, 14M20, 14E07, 14J45.

Keywords. Rationality questions, unirational varieties, Cremona group.

Introduction

The symmetric sextic Fano threefold is the subvariety \( X \) of \( \mathbb{P}^6 \) defined by the equations

\[ \sum X_i = \sum X_i^2 = \sum X_i^3 = 0. \]

It is a smooth complete intersection of a quadric and a cubic in \( \mathbb{P}^5 \), with an action of \( S_7 \). We will prove that it is not rational.

Any smooth complete intersection of a quadric and a cubic in \( \mathbb{P}^5 \) is unirational [E]. It is known that a general such intersection is not rational: this is proved in [B] (thm. 5.6) using the intermediate Jacobian, and in [Pu] using the group of birational automorphisms. But neither of these methods allows to prove the non-rationality of any particular such threefold. Our method gives the above explicit (and very simple) counter-example to the Lüroth problem.

Our motivation comes from the recent paper of Prokhorov [P], which classifies the simple finite subgroups of the Cremona group \( \text{Cr}_3 = \text{Bir}(\mathbb{P}^3) \). In view of this work our result implies that the alternating group \( \mathfrak{A}_7 \) admits only one embedding into \( \text{Cr}_3 \) up to conjugacy.

Our proof uses the Clemens-Griñths criterion ([C-G], Cor. 3.26): if \( X \) is rational, its intermediate Jacobian \( JX \) is the Jacobian of a curve, or a product of Jacobians. The presence of the automorphism group \( S_7 \), together with the celebrated bound \( \# \text{Aut}(C) \leq 84(g-1) \) for a curve \( C \) of genus \( g \), immediately implies
that $JX$ is not isomorphic to the Jacobian of a curve. To rule out products of Jacobians we need some more information, which is provided by a simple analysis of the representation of $S_7$ on the tangent space $T_0(JX)$.

**Proof of the result**

**Theorem.** The intermediate Jacobian $JX$ is not isomorphic to a Jacobian or a product of Jacobians. As a consequence, $X$ is not rational.

The second assertion follows from the first by the Clemens-Gri	hths criterion mentioned in the introduction. Since the Jacobians and their products form a closed subvariety of the moduli space of principally polarized abelian varieties, this gives an easy proof of the fact that a general intersection of a quadric and a cubic in $\mathbb{P}^5$ is not rational.

As mentioned in the introduction, the classification in [P] together with the theorem implies:

**Corollary.** Up to conjugacy, there is only one embedding of $\mathfrak{A}_7$ into the Cremona group $\mathfrak{Cr}_3$, given by an embedding $\mathfrak{A}_7 \subset \text{PGL}_4(\mathbb{C})$.

(The embedding $\mathfrak{A}_7 \subset \text{PGL}_4(\mathbb{C})$ is the composition of the standard representation $\mathfrak{A}_7 \to \text{SO}_6(\mathbb{C})$ and the double covering $\text{SO}_6(\mathbb{C}) \to \text{PGL}_4(\mathbb{C})$.)

The intermediate Jacobian $JX$ has dimension 20. The group $S_7$ acts on $JX$ and therefore on the tangent space $T_0(JX)$; we will first determine this action.

**Lemma.** As a $S_7$-module $T_0(JX)$ is the sum of two irreducible representations, of dimensions 6 and 14.

**Proof.** Let $V$ be the standard (6-dimensional) representation of $S_7$, and put $P := P(V)$; we will view $X$ as a subvariety of $P$, stable under $S_7$.

By definition $T_0(JX)$ is $H^2(X, \omega_X^1)$. Every $S_7$-module is isomorphic to its dual, so we can identify $T_0(JX)$ with $H^1(X, TX(-1))$ by Serre duality. The exact sequence

$$0 \to TX \to TP_{|X} \to O_X(2) \oplus O_X(3) \to 0$$

twisted by $O_X(-1)$, gives a cohomology exact sequence

$$0 \to H^0(X, TP(-1)_{|X}) \to H^0(X, O_X(1)) \oplus H^0(X, O_X(2)) \to H^1(X, TX(-1)) \to H^1(X, TP(-1)_{|X}) \, .$$

From the Euler exact sequence $0 \to O_X \to O_X(1) \otimes_{\mathbb{C}} V \to TP_{|X} \to 0$ we deduce $H^1(X, TP(-1)_{|X}) = 0$ and an isomorphism $V \isom H^0(X, TP(-1)_{|X})$. Thus we find an exact sequence

$$0 \to V \to H^0(X, O_X(1)) \oplus H^0(X, O_X(2)) \to T_0(JX) \to 0 \, .$$
which is equivariant with respect to the action of $S_7$. As representations of $S_7$, $H^0(X, O_X(1))$ is isomorphic to $V$ and $H^0(X, O_X(2))$ to $S^2V/\mathbb{C}q$, where $q$ corresponds to the quadric containing $X$. On the other hand $S^2V = \mathbb{C} \oplus V \oplus V_{(5,2)}$, where $V_{(5,2)}$ is the irreducible representation of $S_7$ corresponding to the partition $(5,2)$ of 7 ([F-H], exercise 4.19). Thus we get $T_0(JX) \cong V \oplus V_{(5,2)}$. Since $\dim T_0(JX) = 20$ and $\dim(V) = 6$ we find $\dim V_{(5,2)} = 14$.

Proof of the theorem. We first observe that $A_7$ cannot act non-trivially on the Jacobian $JC$ of a curve of genus $g \leq 20$. Indeed by the Torelli theorem we have $\text{Aut}(JC) \cong \text{Aut}(C)$ if $C$ is hyperelliptic and $\text{Aut}(JC) \cong \text{Aut}(C) \times \mathbb{Z}/2$ otherwise. Since $A_7$ is simple we find $\# \text{Aut}(C) \leq 2520$. On the other hand we have $\# \text{Aut}(C) \leq 84(g-1) \leq 1596$, a contradiction.

Now assume that $JX$ is a product $J_1 \times \ldots \times J_m$ of Jacobians. Such a decomposition is unique up to the order of the factors: it corresponds to the decomposition of the Theta divisor into irreducible components, see [C-G], Cor. 3.23. Thus the group $A_7$ acts on $[1, m]$ by permuting the factors. Let $O_1, \ldots, O_\ell$ be the orbits of this action. For $1 \leq k \leq \ell$ we put $J(k) := J_{m_k}$ with $m_k = \min O_k$; then for each $i$ in $O_k$ $J_i$ is isomorphic to $J(k)$, so our decomposition can be written $JX \cong J_{O_1}(1) \times \ldots \times J_{O_\ell}(\ell)$.

Since $\#O_k \leq m \leq 20$, the orbit $O_k$ has 1, 7 or 15 elements ([D-M], thm. 5.2.A). If $\#O_k = 1$, $A_7$ acts on the Jacobian $J(k)$; by the lemma this action is faithful, contradicting the beginning of the proof. Thus $\#O_k = 7$ or 15 for each $k$, which contradicts the equality $\sum \#O_k \dim(J(k)) = 20$.

Remarks. The same kind of argument gives the non-rationality of the threefold $\sum X_i^2 = \sum X_i^3 = 0$ in $\mathbb{P}^5$, using the action of $S_6$. It also gives a simple proof of the non-rationality of the Klein cubic threelfold, defined by $\sum_{i \in \mathbb{Z}/5} X_i^3 X_{i+1} = 0$ in $\mathbb{P}^4$ (and, by the same token, of the general cubic threelfold). The automorphism group of the Klein cubic is $\text{PSL}_2(\mathbb{F}_{11})$, of order 660, while its intermediate Jacobian has dimension 5. It is easily seen as above that a 5-dimensional principally polarized abelian variety with an action of $\text{PSL}_2(\mathbb{F}_{11})$ cannot be a Jacobian or a product of Jacobians (see also [Z] for a somewhat analogous, though more sophisticated, proof).

References


Arnaud Beauville, Laboratoire J.-A. Dieudonné, UMR 6621 du CNRS, Université de Nice, Parc Valrose, F-06108 Nice cedex 2, France
E-mail: arnaud.beauville@unice.fr