Lyapunov stability analysis of networks of conservation laws

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Outline

2x2 hyperbolic systems of conservation laws
Steady-state and characteristic form
Networks of conservation laws
Boundary conditions
Exponential stability
Lyapunov stability analysis
Real-life application
Final comments
Open channels, St Venant equations.

\[ \partial_t h + \partial_x (hv) = 0 \]
\[ \partial_t v + \partial_x (gh + \frac{1}{2} v^2) = gS - Cv^2/h \]

\( h = \) water level, \( v = \) water velocity,
\( g = \) gravity, \( S = \) canal slope, \( C = \) friction coefficient

Road traffic, Aw-Rascle equations.

\[ \partial_t \rho + \partial_x (\rho v) = 0 \]
\[ \partial_t (v + p(\rho)) + v \partial_x (v + p(\rho)) = (V(\rho) - v)/\tau \]

\( \rho = \) traffic density, \( v = \) traffic velocity,
\( p(\rho) = \) ”traffic pressure”, \( V(\rho) = \) preferential velocity,
\( \sigma = \) constant
2x2 hyperbolic systems

\[ \partial_t Y + \partial_x f(Y) = g(Y) \]

Space \( x \in [0, L] \)

Time \( t \in [0, +\infty) \)

State \( Y(t, x) \triangleq \begin{pmatrix} y_1(t, x) \\ y_2(t, x) \end{pmatrix} \)

\[ \partial_t Y + A(Y) \partial_x Y = g(Y) \]

\( A(Y) \) has 2 distinct real eigenvalues
Steady state \[ \partial_t Y + A(Y) \partial_x Y = g(Y) \]

A steady-state is a constant solution \( Y(t, x) \equiv \bar{Y} \) which satisfies the equation \( g(\bar{Y}) = 0 \) and (obviously) the state equation \( \partial_t \bar{Y} + A(\bar{Y}) \partial_x \bar{Y} = g(\bar{Y}) \)

Road traffic
\begin{align*}
\rho &= \text{density} \\
v &= \text{velocity}
\end{align*}

Steady state \( \bar{v} = V(\bar{\rho}) \)

Open channels
\begin{align*}
h &= \text{water depth} \\
v &= \text{velocity}
\end{align*}

Steady state \( \bar{v} = \sqrt{\frac{gS}{C}} \bar{h} \)
Characteristics form

- Hyperbolic system: $\partial_t Y + A(Y)\partial_x Y = g(Y)$

- Change of coordinates:
  \[
  \xi(Y) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \partial_t \xi + \begin{pmatrix} c_1(\xi) & 0 \\ 0 & c_2(\xi) \end{pmatrix} \partial_x \xi = h(\xi)
  \]
  (Riemann)

  with $c_1(\xi) \neq c_2(\xi)$
  eigenvalues of $A(Y)$

The change of coordinates $\xi(Y)$ is defined up to a constant. It can therefore be selected such that $\xi(\bar{Y}) = 0 \Rightarrow h(0) = 0.$
Generalisation to networks of 2x2 hyperbolic systems

(e.g. hydraulic networks (irrigation, waterways) or road traffic networks)

- directed graph
- $n$ arcs
- one system of two conservation laws attached to each arc

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \begin{pmatrix} c_i(\xi) & 0 \\ 0 & c_{n+i}(\xi) \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} &= h \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} \\
&= (i = 1, \ldots, n)
\end{align*}
\]
System
\[ \partial_t \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \begin{pmatrix} c_i(\xi) & 0 \\ 0 & c_{n+i}(\xi) \end{pmatrix} \partial_x \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = h \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} \]

\[ \partial_t \xi + C(\xi) \partial_x \xi = h(\xi) \]

Boundary conditions
\[ \xi(t, 0) = G(\xi(t, L)) \]

Problem
\[ G \text{ such that boundary conditions are dissipative?} \]

i.e. steady-state \( \xi \equiv 0 \) is exponentially stable
Boundary conditions = Physical constraints

\[ \xi(t, 0) = G(\xi(t, L)) \quad \iff \quad F(Y(t, 0), Y(t, L)) = 0 \]

Road traffic
\[ \rho = \text{density} \]
\[ v = \text{velocity} \]

Open channels
\[ h = \text{water depth} \]
\[ v = \text{velocity} \]

Flow conservation at a junction
\[ \rho_3(t, 0)v_3(t, 0) = \rho_1(t, L)v_1(t, L) + \rho_2(t, L)v_2(t, L) \]

Modelling of hydraulic gates
\[ h_2(t, 0)v_2(t, 0) = \alpha(h_1(t, L) - u)^{3/2} \]
Boundary conditions = boundary feedback control

Road traffic

- $\rho = \text{density}$
- $v = \text{velocity}$
- $q = \rho v = \text{flux}$

Open channels

- $h = \text{water depth}$
- $v = \text{velocity}$

Feedback implementation of ramp metering

Feedback control of water depth in navigable waterways

$G$ function of the control tuning parameters: How to design the control laws to make the boundary conditions dissipative?
Definition of exponential stability

∃ 𝜖, 𝜇, ν > 0 such that, classical solutions on [0, L] satisfy

\[ \| \xi(t, 0) \| \leq \varepsilon \implies \| \xi(t, x) \|_{H^2} \leq \gamma e^{-\nu t} \| \xi(t, 0) \|_{H^2} \]

\[ \partial_t \xi + C(\xi) \partial_x \xi = h(\xi) \]
\[ \xi(t, 0) = G(\xi(t, L)) \]
\[ t \in [0, +\infty) \quad x \in [0, L] \]

G such that boundary conditions are dissipative, i.e. \( \xi = 0 \) is exp. stable?
Sufficient exponential stability conditions

Slemrod (1983)  
Greenberg and Li (1984)  
Qin, Yu and Li (1985)  
etc...

\[ \rho \left( \left| G'(0) \right| \right) < 1 \]

A weaker condition (2007)

\[ \rho_1 \left( G'(0) \right) \triangleq \inf_{\Delta} \left\{ \left\| \Delta G'(0) \Delta^{-1} \right\| \right\} < 1 \]

\[ \partial_t \xi + C(\xi) \partial_x \xi = 0 \]

\[ \xi(t, 0) = G(\xi(t, L)) \]

\[ t \in [0, +\infty) \quad x \in [0, L] \]

strictly positive diagonal matrices
Lyapunov stability analysis (first trial)

\[ \partial_t \xi + C(\xi) \partial_x \xi = 0 \]

\[ \xi(t, 0) = G(\xi(t, L)) \]

\[ t \in [0, +\infty) \quad x \in [0, L] \]

notations:

\[ \Lambda \triangleq C(0) \]

\[ K \triangleq G'(0) \]

Stability condition \( \rho_1(K) < 1 \Rightarrow \exists D \text{ (diag)} \text{ s.t. } \|DKD^{-1}\| < 1 \)

Lyapunov function candidate

\[ V = \int_0^L E(\xi) \, dx \quad \text{with} \quad E(\xi) \triangleq \xi^T D^2 \Lambda^{-1} \xi \]

Entropy

\[ \partial_t E(\xi) + \partial_x F(\xi) = 0 \]

Entropy Flux

\[ \frac{dV}{dt} = - F(\xi)_L^0 \]
Lyapunov stability analysis (first trial continued)

\[
\frac{dV}{dt} = - \left[ F(\xi) \right]_0^L \quad \text{depends only on B.C.} \quad \xi(t, 0) = G(\xi(t, L))
\]

\[
\frac{dV}{dt} = |D\xi(t, L)|^2 \left( \|DKD^{-1}\|^2 - 1 \right) + \text{H.O.T.}
\]

\[
\text{stab. cond.} < 1
\]

But time derivative of Lyapunov function only semi-negative definite ...
A strict Lyapunov function

\[ V = \int_0^L (\xi^T D^2 \Lambda^{-1} \xi) e^{-\mu x} \, dx \]

\[
\frac{dV}{dt} \leq -\mu V - [F(\xi)]_0^L + \beta \int_0^L |\xi|^2 |\partial_x \xi| \, dx
\]

O.K. for exponential stability ≤ 0 see above

Problem!
Extended model

\[ \begin{align*}
\partial_t \xi + C(\xi) \partial_x \xi &= 0 \\
\partial_t \zeta + C(\xi) \partial_x \zeta + [C'(\xi) \zeta] \zeta &= 0 \\
\partial_t \eta + C(\xi) \partial_x \eta + [C'(\xi) \eta] \zeta + 2[C'(\xi) \zeta] \eta + [(C^{''}(\xi) \eta) \eta] \zeta &= 0
\end{align*} \]

Extended Lyapunov function

\[ V = \int_0^L \left[ (\xi^T D^2 \Lambda^{-1} \xi) + (\zeta^T \Lambda D^2 \zeta) + (\eta^T \Lambda^2 D^2 \Lambda \eta) \right] e^{-\mu x} dx \]

\[ \Rightarrow \quad \frac{dV}{dt} \leq -\lambda V \quad \Rightarrow \quad \text{exponential stability in } H_2 \text{ norm} \]

\[ \|\xi\|_{H_2} = \left( \int_0^L (|\xi|^2 + |\zeta|^2 + |\eta|^2) dx \right)^{1/2} \]
Theorem

If $\rho_1(G'(0)) < 1$, the equilibrium $\xi \equiv 0$ of the quasi-linear hyperbolic system
\[
\partial_t \xi + C(\xi) \partial_x \xi = 0 \quad \xi(t, 0) = G(\xi(t, L))
\]
is exponentially stable.

The same Lyapunov function may be used for:

If $\rho_1(G'(0)) < 1$ and $h'(0) < \kappa$ sufficiently small, the equilibrium $\xi \equiv 0$ of the quasi-linear hyperbolic system
\[
\partial_t \xi + C(\xi) \partial_x \xi = h(\xi) \quad \xi(t, 0) = G(\xi(t, L))
\]
is exponentially stable.

(References
Single 2x2 system : IEEE-TAC 2007
Network of scalar systems : accepted NHM
General result : submitted paper)
Positive and negative characteristic velocities

System

\[
\partial_t \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \begin{pmatrix} c_i(\xi) & 0 \\ 0 & c_{n+i}(\xi) \end{pmatrix} \partial_x \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = h \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix}
\]

\[c_{n+i}(0) < 0 < c_i(0)\]

e.g. St Venant equations (fluvial flow)

Notations

\[\xi = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}, \quad \xi_+ = (\xi_1, \ldots, \xi_n)^T, \quad \xi_- = (\xi_{n+1}, \ldots, \xi_{2n})^T\]

Boundary conditions

\[
\begin{pmatrix} \xi_+(t, 0) \\ \xi_-(t, L) \end{pmatrix} = G \begin{pmatrix} \xi_+(t, L) \\ \xi_-(t, 0) \end{pmatrix}
\]

Same stability condition

\[\rho_1(G'(0)) < 1\]
Real-life application: level and flow control in navigable waterways. Sambre river (Belgium).
Navigable waterway = cascade of pools
Control design and stability analysis

Riemann coordinates

\[
\begin{pmatrix}
\xi_i \\
\xi_{n+i}
\end{pmatrix}
= 
\begin{pmatrix}
v_i + 2\sqrt{gh_i} - \bar{v} - 2\sqrt{gh} \\
v_i - 2\sqrt{gh_i} - \bar{v} + 2\sqrt{gh}
\end{pmatrix}
\]

Control design :
Nonlinear control laws such that

\[
\xi_{n+i}(L) = k_{Li}\xi_i(L) + k_{oi}\xi_{n+i+1}(0) \quad \Rightarrow \quad \rho_1(G'(0)) = \max_i(|k_{oi}k_{Li}|) < 1
\]
Flow Rates $Q$

Open loop

Closed loop

Water levels $H - \bar{H}$

Open loop

Closed loop
Some final comments

- Lyapunov stability analysis with dynamic boundary control (PI-type) + experimental validation on a laboratory pilot plant. (paper accepted in Automatica).

- Riemann coordinates may also be useful for the design of feedforward controllers (cancellation of disturbances) (see IEEE CDC 2005) and for the design of exponentially converging observers (paper in preparation).

- Application to flow control in navigable water-ways. (Automatica 2003)