

# Mouvements of measures for optimal transportation problems

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Mathematical Models of Traffic Flow  
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What presented here is in the paper

*“An optimization problem  
for mass transportation  
with congested dynamics”*

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The goal of this research is to provide a **dynamical** formulation of mass transportation problems with possible **concentration** or **congestion** effects.

- **Concentration** effects for instance occur in several models of **branching** transportation (roots of trees, roads, delta of rivers, blood vessels, . . . )
- **Congestion** effects may be used to simulate traffic flows with **high density** and movement of crowds under **panic** effects.

The main tool is a good comprehension of lower semicontinuous functionals defined on the [space of measures](#).

A complete analysis (lower semicontinuity, relaxation, integral representation) for this kind of functionals was made in a series of paper by [Bouchitté-Buttazzo](#):

Nonlinear Anal., **15** (1990), 679–692

Ann.IHP Anal.NonLin., **9** (1992), 101–117

Ann.IHP Anal.NonLin., **10** (1993), 345–361

**Example 1 - Lebesgue** For  $L^p$  measures  $\mu = u dx$  define

$$F(\mu) = \int_{\Omega} |u|^p dx \quad p > 1.$$

**Example 2 - Dirac** For discrete measures  $\mu = \sum m_k \delta_{x_k}$  define

$$F(\mu) = \sum_k |m_k|^\alpha = \int_{\Omega} |\mu(x)|^\alpha d\#(x) \quad \alpha < 1.$$

**Example 3 - Mumford-Shah** For measures with no Cantor part  $\mu = u dx + \sum m_k \delta_{x_k}$  define

$$F(\mu) = \int_{\Omega} |u|^p dx + \int_{\Omega} |\mu(x)|^\alpha d\#(x) \quad p > 1, \alpha < 1.$$

A full **classification** of all weakly\* l.s.c. functionals on  $\mathcal{M}(\Omega)$  (translation invariant for simplicity) is the following

$$\begin{aligned}
 F(\mu) = & \int_{\Omega} f(\mu^a) dm(x) && \text{Lebesgue part} \\
 & + \int_{\Omega} f^{\infty}(\mu^c) && \text{Cantor part} \\
 & + \int_{\Omega} g(\mu(x)) d\#(x) && \text{Dirac part}
 \end{aligned}$$

where  $f$  is **convex**,  $f^{\infty}$  is its **recession** function,  $g$  is **subadditive**, and the **compatibility** condition  $f^{\infty} = g^0$  holds.

In **Example 1**  $f(z) = |z|^p$ ,  $g(z) \equiv +\infty$ ;

In **Example 2**  $f(z) \equiv +\infty$ ,  $g(z) = |z|^\alpha$ ;

In **Example 3**  $f(z) = |z|^p$ ,  $g(z) = |z|^\alpha$ .

Previous attempts have been made to model concentration/congestion effects:

- **Q. Xia** (2003) through the minimization of a suitable functional defined on **currents**;
- **V. Caselles, J. M. Morel, S. Solimini, ...** (2002) through a kind of analogy of fluid flow in **thin tubes**;
- **A. Brancolini, G. Buttazzo, F. Santambrogio** (2006) through **geodesic** curves in the space of measures.

The idea in this last paper was to study the evolution of densities as a curve in the space of probabilities  $\mathcal{P}(\Omega)$  endowed with the **Wasserstein** distance which minimize a kind of **length functional**:

$$\mathcal{L}(\mu) = \int_0^1 J(\mu(t)) |\mu'(t)|_W dt.$$

Here  $|\mu'|_W$  is the **metric derivative** in the Wasserstein space. In a general  $(X, d)$  space the definition of the metric derivative is

$$|x'(t)|_X = \lim_{\varepsilon \rightarrow 0} \frac{d(x(t + \varepsilon), x(t))}{\varepsilon}.$$

A coefficient  $J$  of **Lebesgue** type would provide a congestion functional, while  $J$  of **Dirac** type would give a model for describing concentrations.

In this presentation however we adopt a different point of view, introduced by **Brenier** to give a **dynamic formulation** of mass transportation problems.

$$\min \left\{ \int_0^1 \int_{\Omega} \rho |v|^2 dx dt : \rho_t + \operatorname{div}_x(\rho v) = 0 \right\}$$

under the initial/terminal conditions  $\rho|_{t=0} = \rho_0$  and  $\rho|_{t=1} = \rho_1$ .

Setting  $\rho v = E$  the continuity equation becomes **linear**:

$$\rho_t + \operatorname{div}_x E = 0$$

and the cost functional (representing the **kinetic energy**) becomes **convex**:

$$\int_0^1 \int_{\Omega} \frac{|E|^2}{\rho} dx dt.$$

To be precise, the correct meaning has to be given in terms of measures:

$$\int_0^1 \int_{\Omega} \left| \frac{dE}{d\rho} \right|^2 d\rho(x) dt.$$

Setting  $Q = [0, T] \times \Omega$ ,  $\sigma = (\rho, E)$ , and  $f = \delta_T(t) \otimes \rho_1(x) - \delta_0(t) \otimes \rho_0(x)$  the problem above can be written in the form

$$\min \left\{ \Psi(\sigma) : -\operatorname{div} \sigma = f \text{ in } Q, \sigma \cdot \nu = 0 \text{ on } \partial Q \right\}$$

where  $\Psi(\sigma)$  is a functional defined on  $\mathcal{M}(Q)$ .

**Theorem** *If  $\Psi$  is a weakly\* l.s.c. functional on  $\mathcal{M}(Q)$  and  $f \in \mathcal{M}(Q)$ , then the minimum problem*

$$\min \left\{ \Psi(\sigma) : -\operatorname{div} \sigma = f \text{ in } Q, \sigma \cdot \nu = 0 \text{ on } \partial Q \right\}$$

*has a solution, provided  $\int_Q df = 0$  and  $\Psi$  is **coercive**, i.e.  $\Psi(\sigma) \geq c|\sigma| - c_1$ .*

The functionals  $\Psi$  we have in mind are of the form

$$\Psi(\sigma) = \int_0^T J(\sigma(t)) dt$$

and again  $J$  of **Lebesgue** type would provide **congestion** models, while  $J$  of **Dirac** type would provide **concentration** models.

From now on we limit ourselves to the case of congestion, where the function  $J$  is **convex**. Similar arguments for the other **non-convex** cases have not yet been developed.

Dual formulation:

$$\sup \left\{ \langle f, \phi \rangle - \Psi^*(D\phi) : \phi \in C^1(Q) \right\}.$$

Primal-dual relation:

$$\Psi(\sigma_{opt}) + \Psi^*(D\phi_{opt}) = \langle \sigma_{opt}, D\phi_{opt} \rangle.$$

The point is that the maximizer in the dual formulation is not of class  $C^1$  in general. A relaxation formula is then needed for  $\Psi^*$  to extend it to its **natural space**.

The natural spaces for functionals like  $\Psi^*$  are the **Sobolev** spaces  $W_\mu^{1,p}$  with respect to a measure  $\mu$ , defined by relaxation of the energies

$$\int |Du|^p d\mu.$$

All the usual properties known for the standard Sobolev spaces continue to hold, provided the gradient is replaced by the **tangential gradient**  $D_\mu u$  suitably defined.

We do not enter in the details of this rather delicate theory, referring to **Bouchitté-Buttazzo-Seppecher** (Calc.Var. 1997).

The numerical approximation has been performed following the scheme used in **Benamou-Brenier**, which is an **augmented Lagrangian** algorithm. This consists in solving, instead of

$$\min \left\{ \Psi(\sigma) : -\operatorname{div} \sigma = f \text{ in } Q, \sigma \cdot \nu = 0 \text{ on } \partial Q \right\}$$

the min-max problem

$$\min_{\sigma} \max_{\varphi \in \mathcal{C}(Q)} L(\sigma, \varphi)$$

where  $L(\sigma, \varphi)$  is the **Lagrangian**:

$$L(\sigma, \varphi) = \Psi(\sigma) - \langle D\varphi, \sigma \rangle + \langle \varphi, f \rangle.$$

Using the **primal-dual relation** this is in turn equivalent to solve the max-min problem

$$\max_{\sigma} \min_{\sigma^*, \varphi} L_r(\sigma, \sigma^*, \varphi)$$

where  $L_r$  is the **augmented Lagrangian**

$$L_r(\sigma, \sigma^*, \varphi) := \Psi^*(\sigma^*) + \langle D\varphi - \sigma^*, \sigma \rangle - \langle \varphi, f \rangle + \frac{r}{2} \int |D\varphi - \sigma^*|^2 dy$$

for  $r > 0$  fixed.

This is the iterative process we used (algorithm ALG2, **Fortin-Glowinski**):

- let  $(\sigma_n, \sigma_{n-1}^*, \varphi_{n-1})$  be given;
- Step A: find  $\varphi_n$  solving (freeFEM3D by Del Pino-Pironneau)

$$\min \left\{ L_r(\sigma_n, \sigma_{n-1}^*, \varphi) : \varphi \in \mathcal{C}^1(Q) \right\}$$

- Step B: find  $\sigma_n^*$  solving

$$\min \left\{ L_r(\sigma_n, \sigma^*, \varphi_n) : \sigma^* \in \mathcal{C}^1(Q, \mathbf{R}^{d+1}) \right\}$$

- Step C: set  $\sigma_{n+1} = \sigma_n + r(D\varphi_n - \sigma_n^*)$ ;
- go back to Step A.

The following animations deal with a domain  $\Omega$  not convex (a kind of [subway gate](#)) and with the cases:

- $J(\rho, E) = \frac{|E|^2}{\rho}$  in which the transportation simply follows the Wasserstein geodesics.
- $J(\rho, E) = \frac{|E|^2}{\rho} + c\rho^2$  in which the Wasserstein transportation is perturbed by the addition of a diffusion term ([panic effect](#)).
- $J(\rho, E) = \frac{|E|^2}{\rho} + \chi_{\{\rho \leq M\}}$  in which there is the additional constraint that two different individual [do not want to stay too close](#).