Modelling of Crowd Motion

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Basic model

Theoretical framework

Numerical simulation

Macroscopic modelling

Some animations are downloadable at `www.math.u-psud.fr/~venel`

We consider N persons (rigid disks)

$$Q = \{\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_N) \in \mathbb{R}^{2N}\}$$

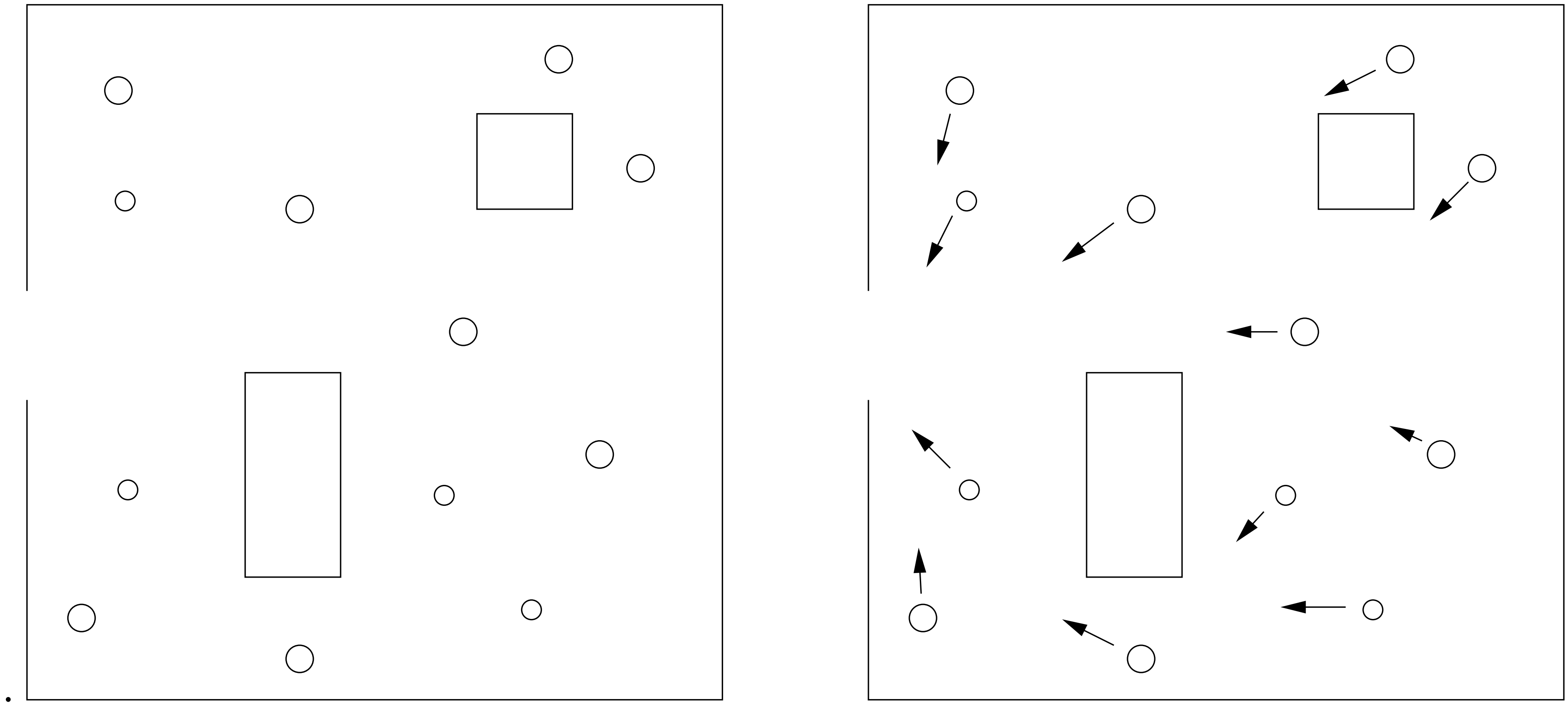
in a room, and a spontaneous velocity field

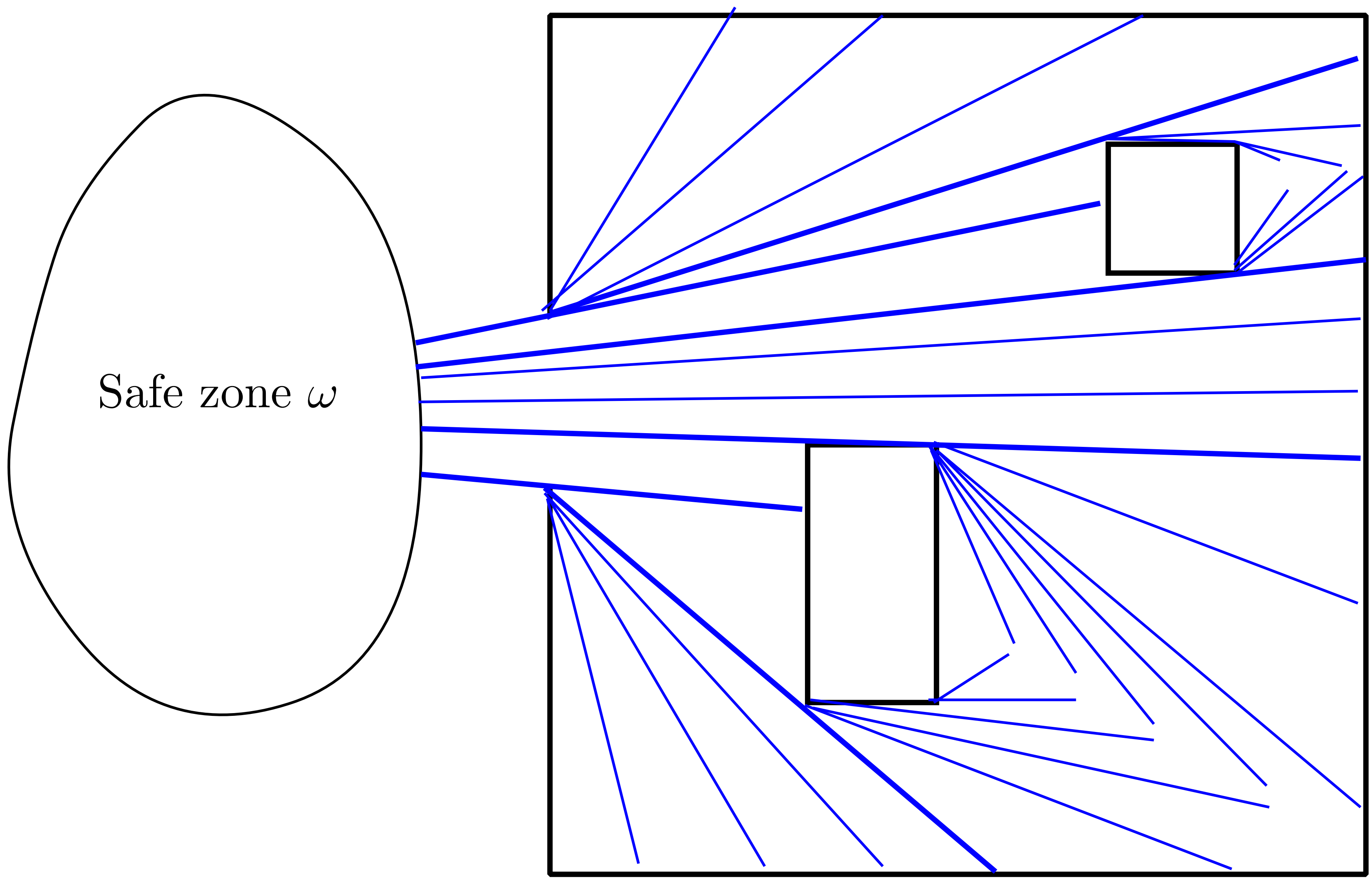
$$\mathbf{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_N)$$

Basic model : uniform, individualistic behaviour

$$\mathbf{U}(\mathbf{q}) = (\mathbf{U}_0(\mathbf{q}_1), \ldots, \mathbf{U}_0(\mathbf{q}_N))$$

$\mathbf{U}_0(\mathbf{q})$ points toward the optimal direction for a single person.




Safe zone ω

Set of feasible configurations

$$Q_0 = \{\mathbf{q} \in Q, D_{ij}(\mathbf{q}) = |\mathbf{q}_j - \mathbf{q}_i| - 2r \geq 0 \quad \forall i \neq j\}$$

(N.B. obstacle and walls are also included as constraints)

Set of feasible velocities (with $\mathbf{G}_{ij} = \nabla D_{ij}$)

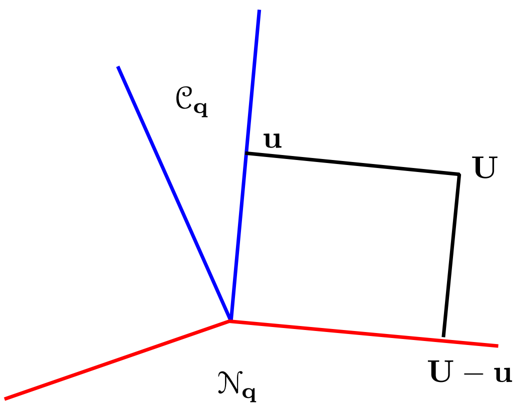
$$\mathcal{C}_\mathbf{q} = \{\mathbf{v}\,,\ \mathbf{G}_{ij} \cdot \mathbf{v} \geq 0 \quad \text{as soon as} \quad D_{ij}(\mathbf{q}) = 0\}\,,$$

Model : the actual velocity field is the feasible field which is the closest to $\mathbf{U}$.

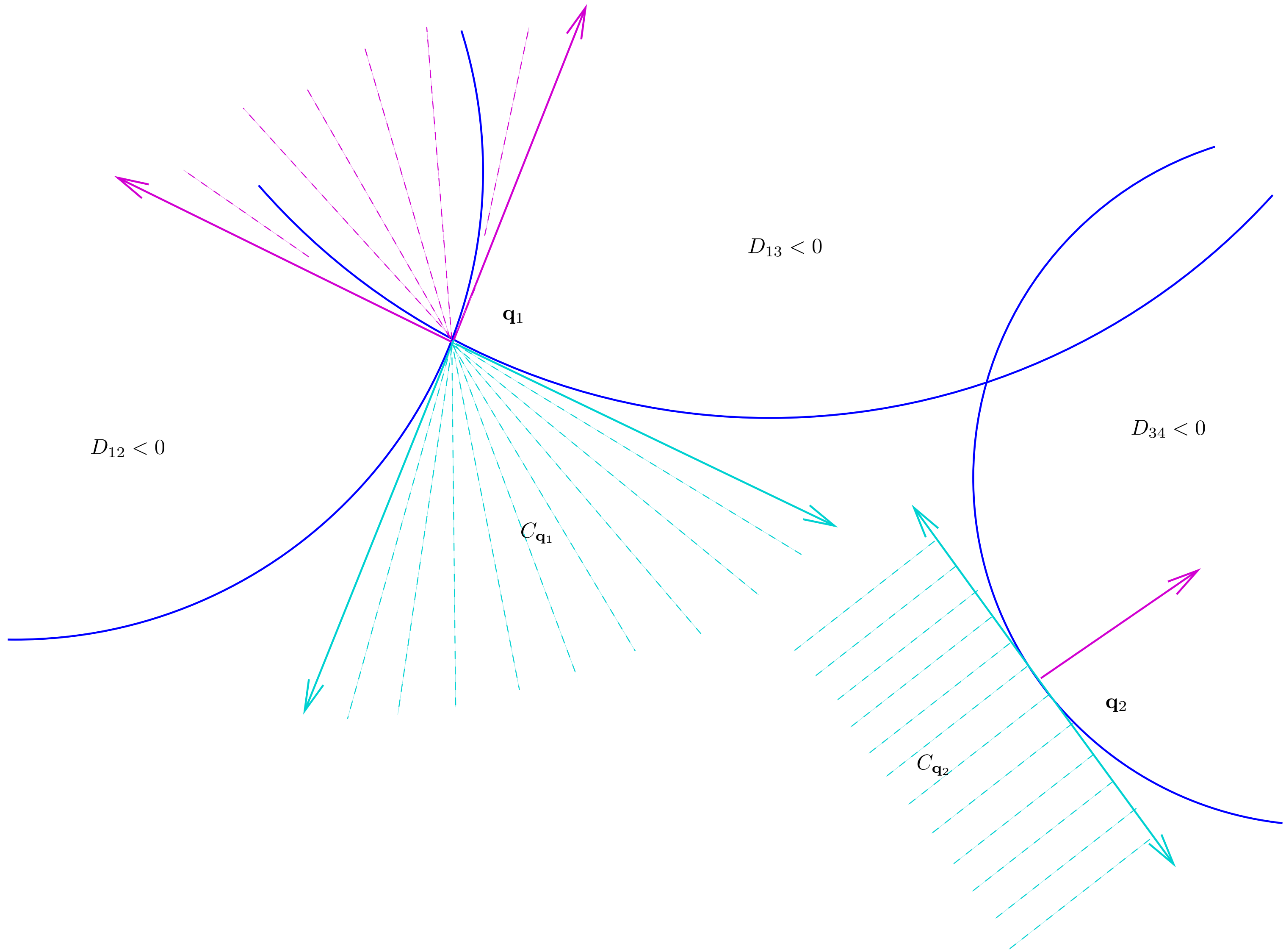
$$\frac{d\mathbf{q}}{dt} = P_{\mathcal{C}_\mathbf{q}} \mathbf{U}(\mathbf{q})$$

Reformulation of the problem

$$
\begin{aligned}
\mathcal{N}_{\mathbf{q}} &= \text{ outward normal cone to } Q_0 \\
&= \mathcal{C}_{\mathbf{q}}^{o} \text{ (polar cone to feasible directions)} \\
&= \{\mathbf{w}\,,\ (\mathbf{w},\mathbf{v}) \leq 0 \quad \forall \mathbf{v} \in \mathcal{C}_{\mathbf{q}}\} \\
&= \{-\sum \lambda_{ij}\mathbf{G}_{ij}\,,\ \lambda_{ij} \geq 0\,,\ D_{ij} = 0 \Longrightarrow \lambda_{ij} = 0\}
\end{aligned}
$$



$$I_d = P_{\mathcal{C}_{\mathbf{q}}} + P_{\mathcal{N}_{\mathbf{q}}} \text{ (Moreau)}$$



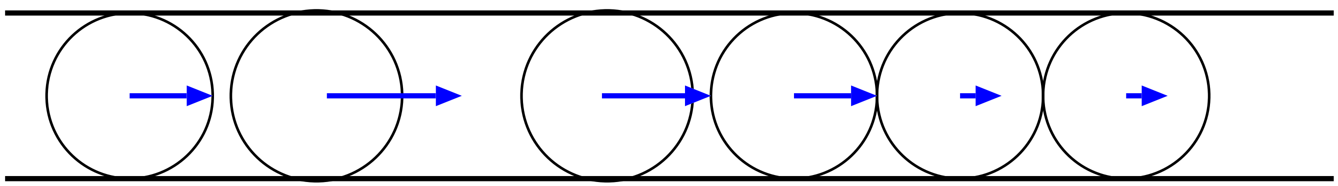
$D_{13} < 0$
$\mathbf{q}_1$
$D_{12} < 0$
$D_{34} < 0$
$C_{\mathbf{q}_1}$
$\mathbf{q}_2$
$C_{\mathbf{q}_2}$

$$\frac{d\mathbf{q}}{dt} = P_{\mathcal{C}_\mathbf{q}}\mathbf{U} = \mathbf{U} - P_{\mathcal{N}_\mathbf{q}}\mathbf{U}$$

As a consequence,

$$\frac{d\mathbf{q}}{dt} + \mathcal{N}_\mathbf{q} \ni \mathbf{U} \qquad \left(\Longleftrightarrow \left(\frac{d\mathbf{q}}{dt} - \mathbf{U}, \mathbf{w} \right) \geq 0 \quad \mathbf{w} \in \mathcal{C}_\mathbf{q} \right)$$

Particular situation : individuals in a corridor



$$Q_0 = \{\mathbf{q} = (q_1, \ldots, q_N)\,,\ q_{i+1} - q_i \geq 2r\} \text{ is convex, so that}$$

$$\mathbf{q} \longmapsto \mathcal{N}_\mathbf{q} = \partial I_{Q_0} \text{ is maximal monotone}$$

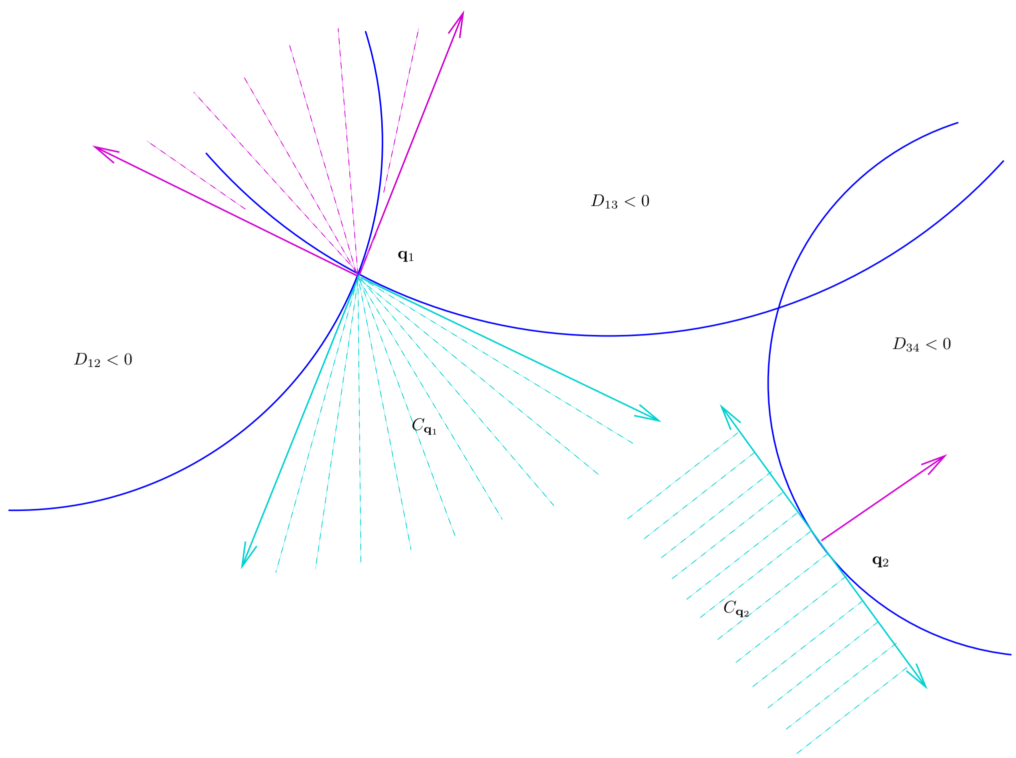
$\longrightarrow$ standard theory (see e.g. Brezis 1973) ensures well-posedness

General situation

Q_0 is not convex, $\mathbf{q} \longmapsto \mathcal{N}_\mathbf{q}$ is not maximal monotone, but

Q_0 is prox-regular (Moreau, Monteiro-Marquez, Thibault) as the intersection of complementary sets of smooth convex sets.

$\Longrightarrow$ the problem is well-posed (J. Venel, B.M., from Thibault 2006)



Numerical Simulation

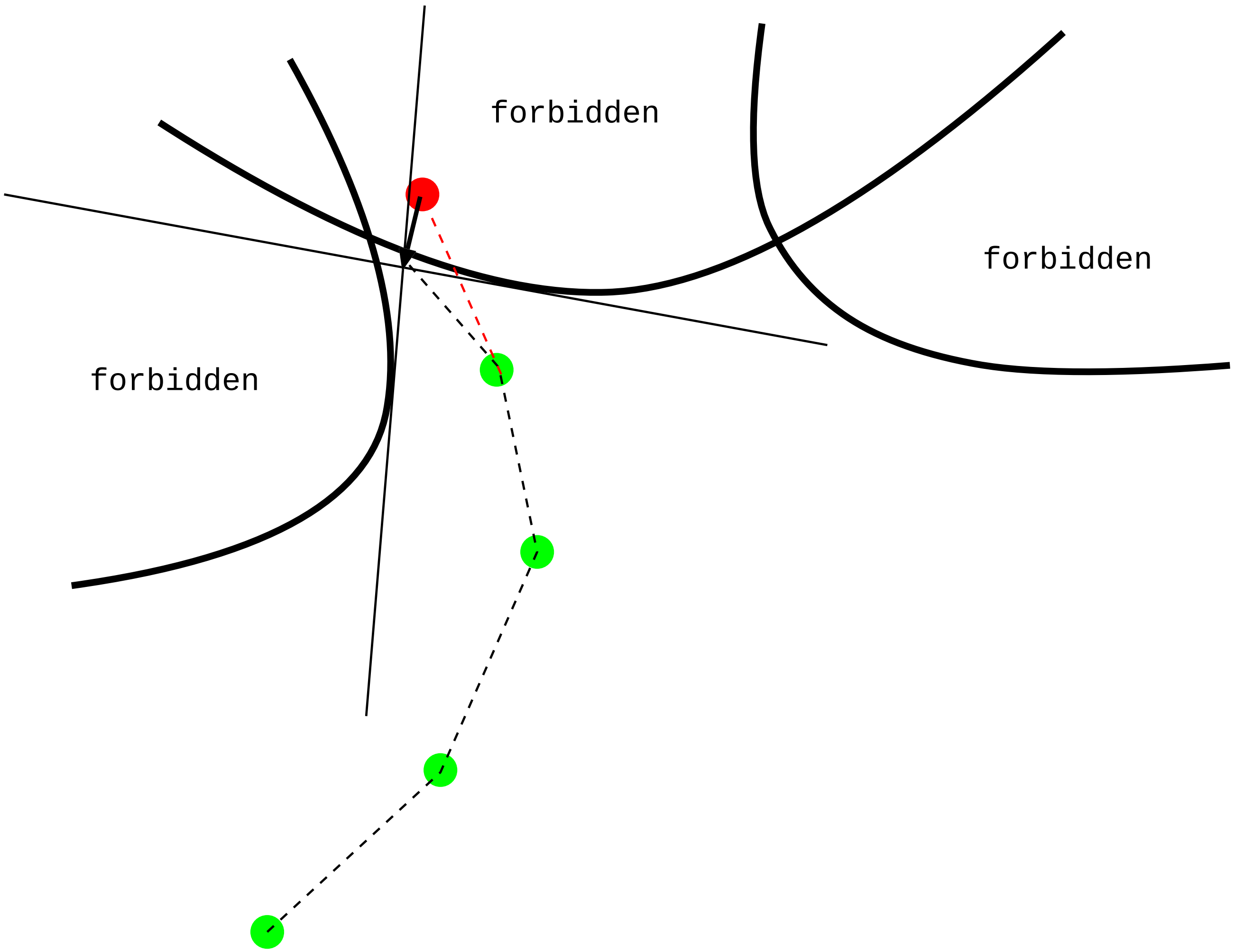
$\mathbf{q}^n$: configuration at time t^n.

$\mathbf{q}^{n+1} = \mathbf{q}^n + h\mathbf{u}^{n+1}$, where $\mathbf{u}^{n+1}$ minimizes

$$\frac{1}{2}\left|\mathbf{v} - \mathbf{U}\right|^2 \text{ over } \mathcal{C}^h_{\mathbf{q}^n}\,, \quad \text{with}$$

$$\mathcal{C}^h_{\mathbf{q}} = \{\mathbf{v}\,,\ \underbrace{D_{ij}(\mathbf{q}) + h\mathbf{G}_{ij}\cdot\mathbf{v}}_{\approx D_{ij}(\mathbf{q}+h\mathbf{v})} \geq 0\}$$

$$\text{By dualization } \longrightarrow \left\{\begin{array}{lcl} \mathbf{u} + B^\star\lambda & = & \mathbf{U} \\ B\mathbf{u} & \leq & 0. \end{array}\right.$$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INTERPRETATION OF THE λ_{ij}

Analogy with a “unilateral” Darcy problem

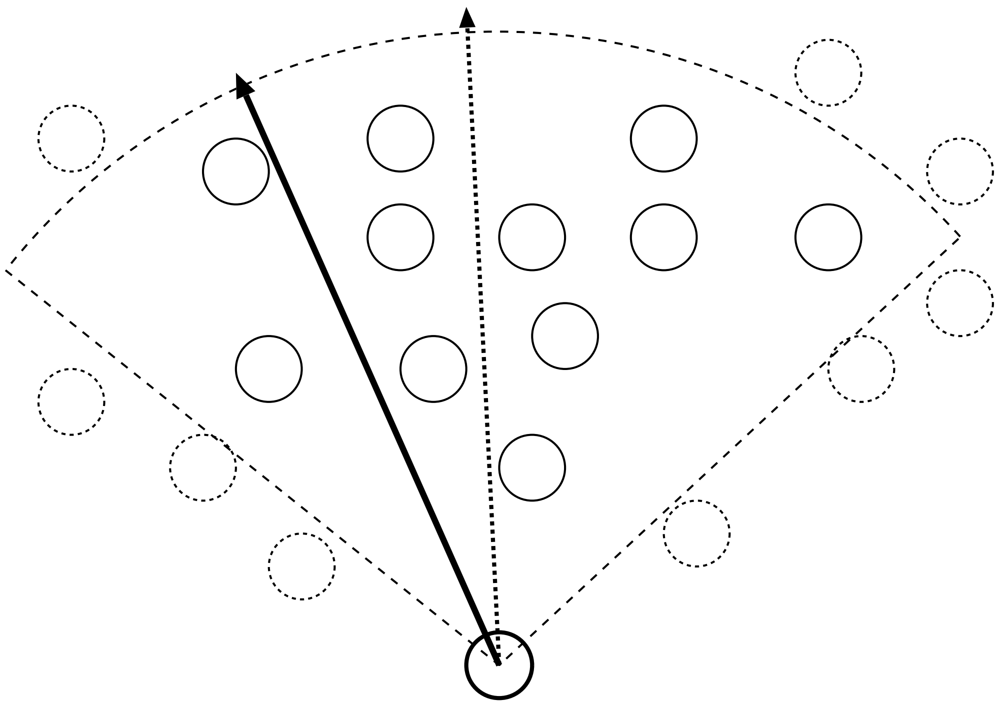
$$\begin{cases} \mathbf{u} + B^\star \lambda & = & \mathbf{U} \\ B\mathbf{u} & \leq & 0. \end{cases} \qquad \begin{cases} \mathbf{u} + \nabla p & = & \mathbf{F} \\ -\nabla \cdot \mathbf{u} & \leq & 0. \end{cases}$$

$$\mathbf{u} = \mathbf{U} + \text{ correction } = \mathbf{U} + \sum \lambda_{ij} \mathbf{G}_{ij}$$

λ_{ij} :pressure between i and j.

High values : risk of casualties

Differentiation among individuals, strategies



$$\longrightarrow \quad \mathbf{U}_i = \mathbf{U}_i \left(\mathbf{q}_i, \left(\mathbf{q}_j, \mathbf{U}_j\right)_{j \sim i} \right)$$

Macroscopic version

Domain Ω, delimited by Γ_{out} (exit) and Γ_W (walls)

ρ density of people, $\rho(x) \in [0,1]$, $\mathbf{U}$ spontaneous velocity field

$$\mathcal{C}_\rho = \text{Cl}\left\{\mathbf{v} \in H_{\text{div}}(\Omega)\,,\ \nabla\cdot\mathbf{v} \geq 0 \text{ a.e. on } [\rho = 1]\,,\ \mathbf{u}\cdot\mathbf{n} \leq \mathbf{0} \text{ on } \mathbf{\Gamma}_W\right\},$$

$$\begin{aligned} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) &= 0 \\ \mathbf{u} &= P_{\mathcal{C}_\rho}\mathbf{U}. \end{aligned}$$

Saddle point formulation of the instantaneous problem

$$\Lambda = \{\mathrm{q} \in H^1(\Omega)\,,\ \mathrm{q}_{|\Gamma_{\mathrm{out}}} = 0\,,\ \mathrm{q} = 0 \text{ a.e. on } [\rho < 1]\}$$

Λ^+ : non-negative fields in Λ.

There exists a unique $\mathrm{p} \in \Lambda^+$ such that

$$\begin{cases} \mathbf{u} + \nabla \mathrm{p} & = & \mathbf{U} \\ \displaystyle\int_\Omega \mathbf{u} \cdot \nabla \mathrm{q} & \leq & 0 \quad \forall \mathrm{q} \in \Lambda^+ \end{cases}$$

N.B. : Computations based on a bidual formulation

$$\begin{cases} -\triangle \mathrm{p} - \lambda & = & -\nabla \cdot \mathbf{U} \\ -\mathrm{p} & \leq & 0 \end{cases}$$