# Intersections modelling with a class of "second order" models of traffic flow

Salissou Moutari[1]

Joint work with Michael Herty[2] and Michel Rascle[1]

[1] Laboratoire J. A. Dieudonné, University of Nice-Sophia Antipolis

[2] Fachbereich Mathematik, TU Kaiserslautern

# Outline

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- Introduction
- The "Aw-Rascle" Traffic model
- Junctions Modelling
- Numerical Examples

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# Introduction

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  - Preserve the mass flux and the pseudo momentum;
  - Maximize the total flux at the junction.
- Holden & Risebro (1995), Coclitte, Garavello & Piccoli (2005), Garavello & Piccoli (2005), Lebacque & Khoshyaran (2005), Haut & Bastin (2005), Siebel & Mauser (2005), Herty & Rascle (2006) . . .

# The “Aw-Rascle” macroscopic model

# The Model





The "Aw-Rascle" (AR) macroscopic model of traffic flow:

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$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho v w) = 0, \end{cases} \tag{1}$$

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$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho v w) = 0, \end{cases} \tag{1}$$

where,

- $\rho$: a dimensionless local density
  (the fraction of space occupied by cars),
- $v$: the macroscopic velocity of cars
- and $w$: a Lagrangian marker. E.g. $w = v + p(\rho)$, where $\rho \longmapsto p(\rho)$ is a known function such that

$$\rho p''(\rho) + 2p'(\rho) > 0. \tag{2}$$

# The model

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- $\lambda_2$ is linearly degenerate: 2-contact discontinuity.

# The A-R model on a network



## Basic notations:

# The A-R model on a network



**Basic notations:**

Road Network: Finite directed graph $G = (\mathcal{I}, \mathcal{N})$

with $|\mathcal{I}| = \mathbf{I}$ and $|\mathcal{N}| = \mathbf{N}$.

- Each arc $i = 1 \ldots \mathbf{I}$ corresponds to a road.
- Each vertex $n = 1 \ldots \mathbf{N}$ corresponds to junction.

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Each road $i$ is modelled by $I_i = [a_i, b_i]$.

# The A-R model on a network

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# The A-R model on a network



- We required the A-R system (1) to hold on each arc $i \in \mathcal{I}$ of the network.
- Weak solutions $U_i = (\rho_i, \rho_i v_i)$ of the network problem:

$$\sum_{i=1}^{\mathbf{I}} \int_0^{\infty} \int_{a_i}^{b_i} \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} \cdot \partial_t \phi_i + \begin{pmatrix} \rho_i v_i \\ \rho_i v_i w_i \end{pmatrix} \cdot \partial_x \phi_i dxdt$$

$$+ \int_{a_i}^{b_i} \begin{pmatrix} \rho_{i,0} \\ \rho_{i,0} w_{i,0} \end{pmatrix} \cdot \phi_i(x,0)dx = 0, \qquad (3)$$

for any set of smooth functions $\{\phi_i\}_{i \in \mathcal{I}} : [0, +\infty[ \times I_i \longrightarrow \mathbb{R}^2$ having compact support and also smooth across a junction $n$, i.e.

$$\phi_k(b_k) = \phi_j(a_j) \ \forall k \in \delta_n^- \text{ and } \forall j \in \delta_n^+. \qquad (4)$$

# The A-R model on a network



The Rankine–Hugoniot condition for piecewise smooth solutions

$$\sum_{k\in\delta^-}(\rho_k v_k)(b_k^-,t) = \sum_{j\in\delta^+}(\rho_j v_j)(a_j^+,t), \tag{5a}$$

$$\sum_{k\in\delta^-}(\rho_k v_k w_k)(b_k^-,t) = \sum_{j\in\delta^+}(\rho_j v_j w_j)(a_j^+,t). \tag{5b}$$

Conservation of mass and (pseudo)-“momentum”.

# Incoming half Riemann Problem





## Riemann Problem on an incoming road

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## Riemann Problem on an incoming road

- Classical: connect the left Riemann data $U_k^- = (\rho_k^-, v_k^-)$ through a 1-wave of negative speed to the state

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- $q_k$ is still unknown, depends on the demand(s) and supply(ies).

# Incoming half Riemann Problem



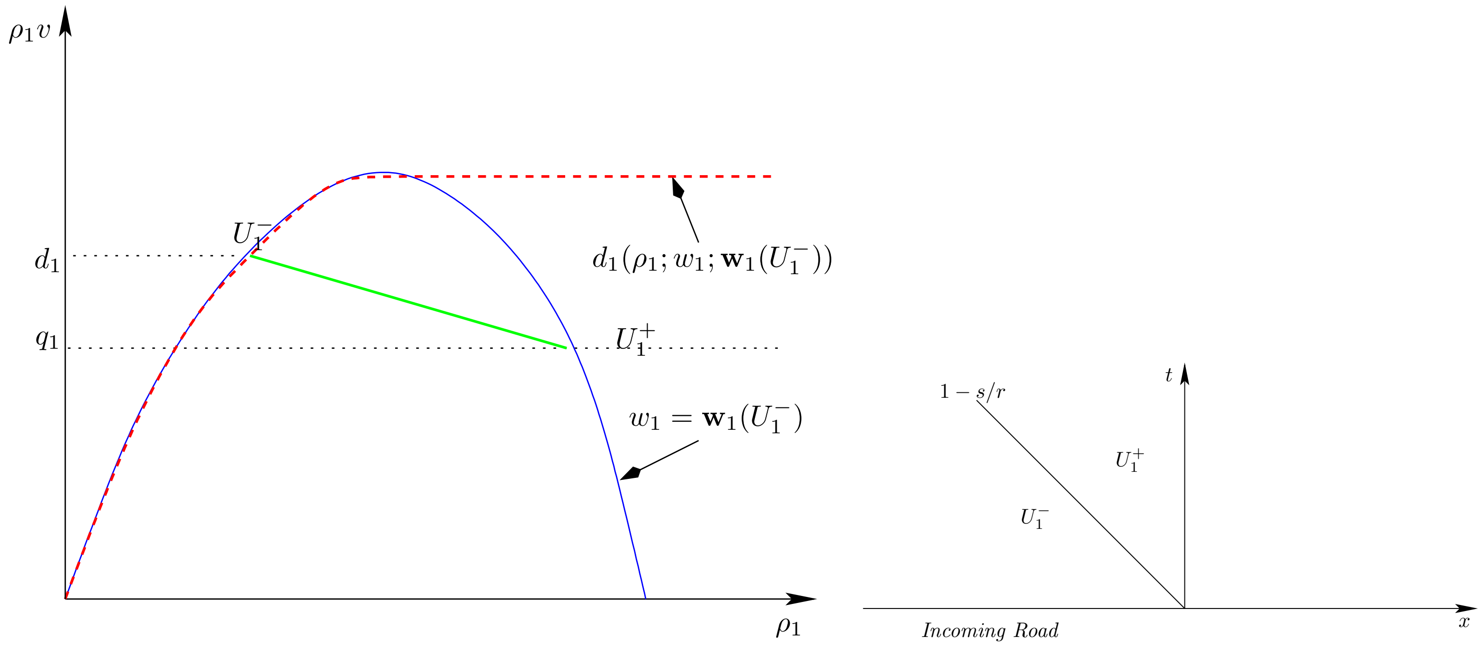


Figure 1: (Half-)Riemann Problem on an incoming road.

## Riemann Problem on an outgoing road

# Outgoing half Riemann Problem





## Riemann Problem on an outgoing road

- First, connect the right Riemann data $U_j^+ = (\rho_j^+, v_j^+)$ through a 2-contact discontinuity (of speed $v_j^+ > 0$) to the intermediate state

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- Then, connect $U_j^*$ (on the right) through a 1-wave of positive speed to the state

$$U_j^- = \{w_j = \bar{w}_j\} \cap \{\rho_3 v = q_j\}, \tag{8}$$

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- $q_j$ (Rankine-Hugoniot) and $\bar{w}_j$ still unknown.

# Outgoing half Riemann Problem



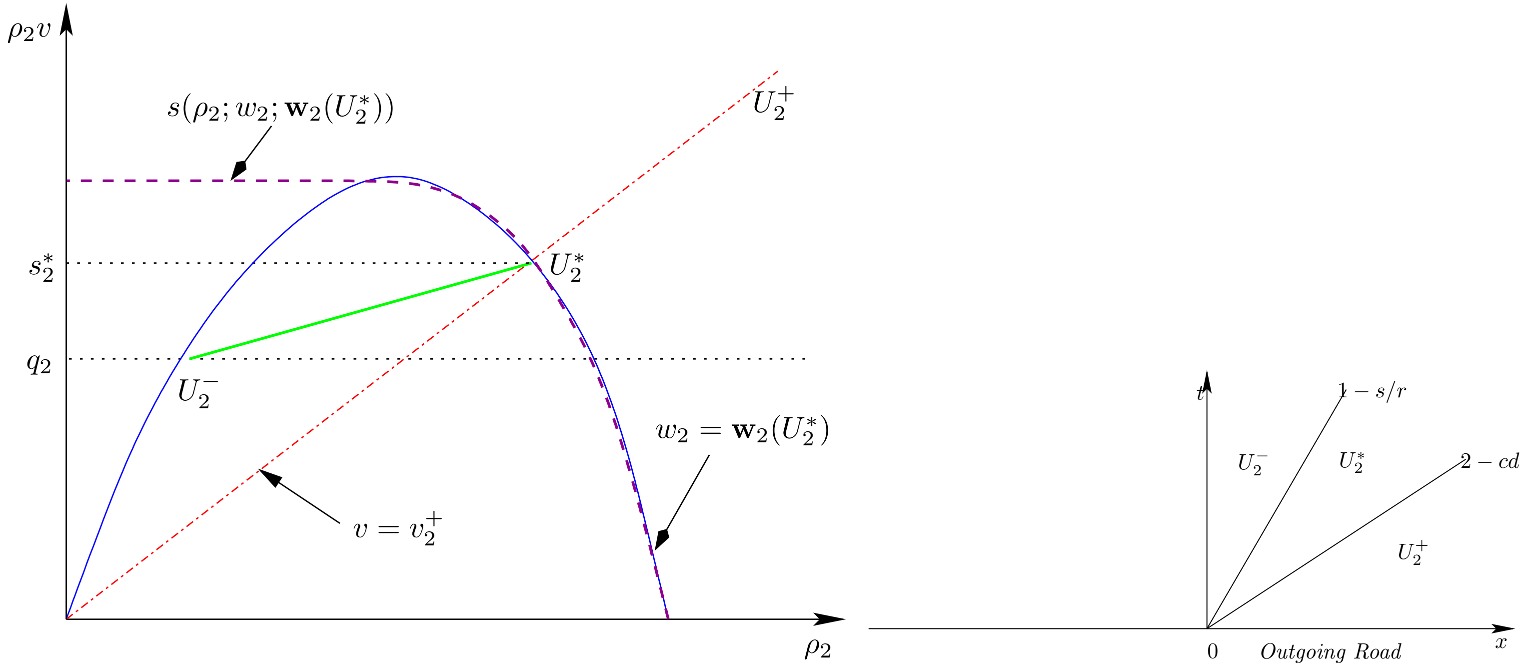


Figure 2: (Half-)Riemann Problem on an outgoing road.

# Homogenization





- A-R model in Lagrangian mass coordinates

$$\begin{cases} \partial_t \tau - \partial_X v = 0, \\ \partial_t w = 0, \end{cases} \quad (9)$$

where

$$w = v + P(\tau), \quad (10)$$

$\tau = F(v,w) = P^{-1}(w - v) = \frac{1}{\rho}$ and $P(\tau) = p\left(\frac{1}{\tau}\right)$.

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- $w$ and any $F(v, w)$, in particular $\tau$ can be homogenized cf. [P. Bagnerini and M. Rascle (2003)].

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- In particular,

$$\bar{w}_j := \sum_{k \in \delta^-} \beta_{jk} \mathbf{w}_k(U_{k,0}), \qquad (11)$$

$$\tau_j := \sum_{k \in \delta^-} \beta_{jk} P_j^{-1}(\mathbf{w}_k(U_{k,0}) - v), \qquad (12)$$

where $\beta_{jk}$, still unknown are the homogenization coefficients.

# Optimization Problem



## Notations

- $d_k :=$ demand on incoming road $k$;
- $s_j :=$ supply on outgoing road $j$;
- $q_k :=$ total flux on incoming road $k$;
- $q_j :=$ total flux on outgoing road $j$;
- $q_{jk} :=$ flux on road $j$ coming from road $k$.

# Optimization Problem



## Notations

- $\beta_{jk} :=$ proportion of flux on road $j$ coming from road $k$

$$\beta_{jk} = \frac{q_{jk}}{q_j} \tag{13}$$

$$\text{and} \qquad \sum_{k \in \delta^-} \beta_{jk} = 1, \qquad \forall j \in \delta^+. \tag{14}$$

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- $\alpha_{jk} :=$ proportion of flux on road $k$ willing to go to road $j$.

$$\text{So,} \quad \alpha_{jk} = \frac{q_{jk}}{q_k} \tag{15}$$

is assumed to be known $\forall\, k$, $j$ and

$$\sum_{j \in \delta^+} \alpha_{jk} = 1, \qquad \forall k \in \delta^-. \tag{16}$$

# Optimization Problem

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## Maximization Problem:

$$
\begin{cases}
\max \sum_{j \in \delta^+} q_j \\
0 \leq q_j \leq \frac{\alpha_{jk} d_k}{\beta_{jk}} \quad \forall k \in \delta^-, \forall j \in \delta^+ \\
0 \leq q_j \leq s_j(v_j, \beta_{jk}, k = 1..|\delta^-|) \quad \forall j \in \delta^+; \\
\beta_{jk} q_j = \alpha_{jk} q_k \quad \forall k \in \delta^-, \forall j \in \delta^+; \\
\sum_{k \in \delta^-} \beta_{jk} = 1 \quad \forall j \in \delta^+; \\
0 \leq \beta_{jk} \leq 1 \quad \forall k \in \delta^-, \forall j \in \delta^+.
\end{cases} \tag{17}
$$

# Junctions

# The case of a merging junction



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- $k = 1, 2$ (for incoming roads);

# The case of a merging junction

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We pose

- $k = 1, 2$ (for incoming roads);
- $j = 3$ (for the outgoing road);

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Let set

$$\beta_{31} = \beta_1 \tag{18}$$

and

$$\beta_{32} = (1 - \beta_1). \tag{19}$$

# The case of a merging junction



On the incoming roads:

$$\mathbf{w}_k(U) = v + p_k(\rho_k) = w_k \tag{20}$$

or

$$\mathbf{w}_k(U) = v + P_k(\tau_k) = w_k \qquad (k = 1, 2), \tag{21}$$

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On the outgoing road nearby the junction:

$$\mathbf{w}_3(U) = v + P_3(\tau_3) = \bar{w}, \tag{22}$$

with

$$\bar{w} = \beta_1 w_1 + (1 - \beta_1) w_2. \tag{23}$$

# The case of a merging junction





Homogenization of $\tau$ on the outgoing road:

$$\tau_3(X,t) = \sum_{k\in\delta^-} \beta_{3k} P_k^{-1}(w_k - v(X,t)). \tag{24}$$

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Hence $\tau_3 = \beta_1\tau_1 + (1-\beta_1)\tau_2$.

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Hence $\tau_3 = \beta_1 \tau_1 + (1-\beta_1)\tau_2$.

With $p_k(\rho) = \rho^\gamma$ (or $P_k(\tau) = 1/\tau^\gamma$) and $\gamma = 1$ for $i = 1,2,3$, we get

$$\tau_3 = \frac{\beta_1}{w_1 - v} + \frac{(1-\beta_1)}{w_2 - v}. \tag{25}$$

# The case of a merging junction





$$\tau_3 = \frac{\beta_1}{w_1 - v} + \frac{(1-\beta_1)}{w_2 - v} \Rightarrow \rho_3 v = \frac{(w_2 - v)(w_1 - v)v}{\beta_1(w_2 - w_1) + w_1 - v}. \quad (26)$$

The supply on the outgoing road is therefore

$$s_3(v_3, \beta_1) = \begin{cases} \frac{(w_2 - v_3)(w_1 - v_3)v_3}{\beta_1(w_2 - w_1) + w_1 - v_3} & if \quad v_3 \leq v_c; \\ \frac{(w_2 - v_c)(w_1 - v_c)v_c}{\beta_1(w_2 - w_1) + w_1 - v_c} & if \quad v_3 > v_c. \end{cases} \quad (27)$$

$v_c$ the velocity corresponding to the maximal flux on the outgoing road.

$$v_c : \frac{\partial(\rho_3 v)}{\partial v} = 0, \quad (28)$$

for any fixed $\beta_1$.

# The case of a merging junction



The maximization problem:

$$\mathcal{P} = \begin{cases} \max q_3 \\ 0 \leq \beta_1 q_3 \leq d_1; \\ 0 \leq (1-\beta_1) q_3 \leq d_2; \\ 0 \leq q_3 \leq s_3(v_3, \beta_1); \\ 0 \leq \beta_1 \leq 1; \end{cases} \iff \begin{cases} \max q_3 \\ 0 \leq q_3 \leq \frac{d_1}{\beta_1}; \\ 0 \leq q_3 \leq \frac{d_2}{(1-\beta_1)}; \\ 0 \leq q_3 \leq s_3(v_3, \beta_1); \\ 0 \leq \beta_1 \leq 1; \end{cases} \tag{29}$$

Solution of the maximization problem when $w_1 > w_2$.

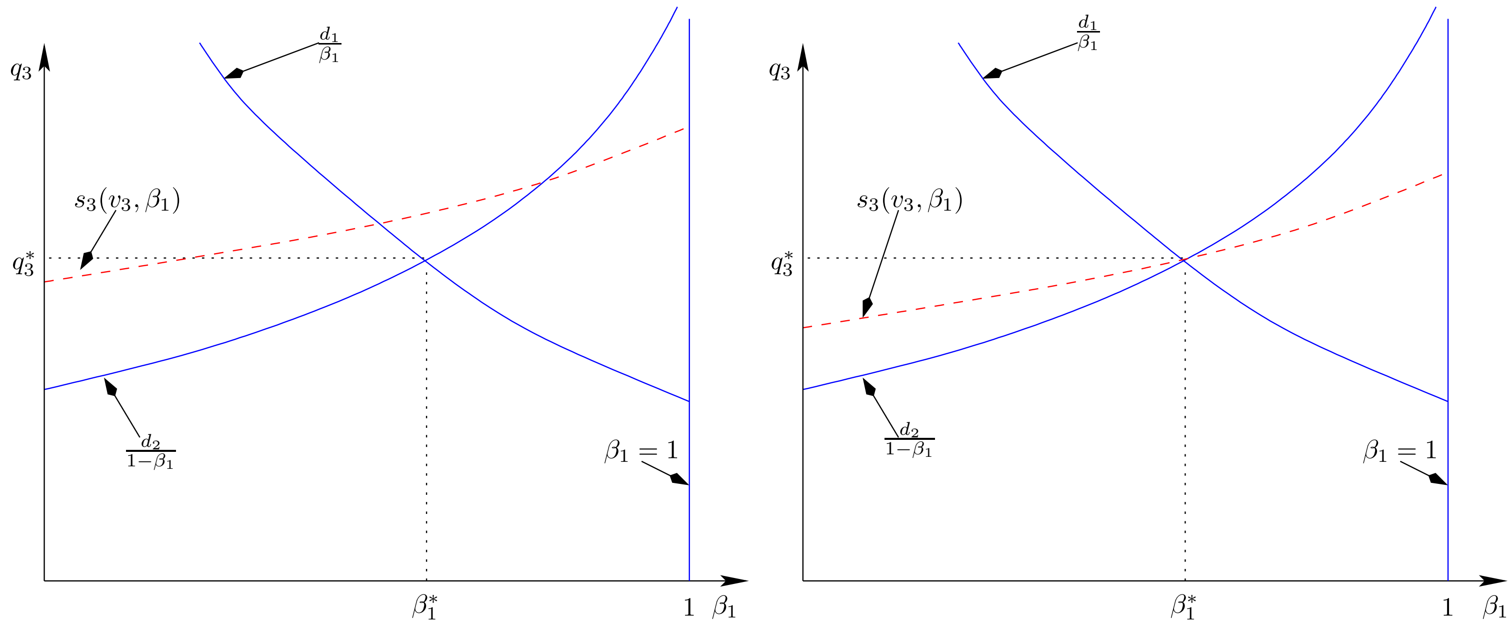


Figure 3: Examples of the optimal solution.

# The case of a merging junction



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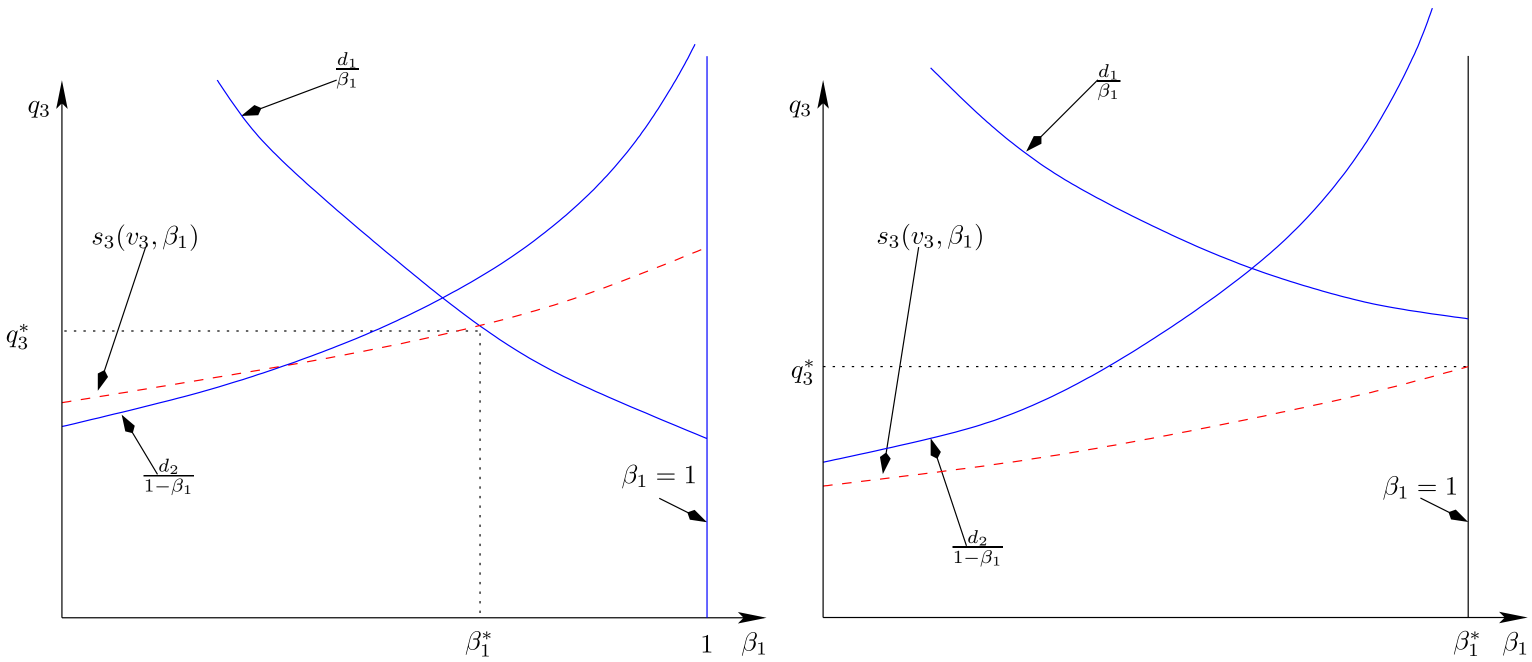


Figure 4: Examples of the optimal solution.

# The case of a merging junction



## Numerical Results

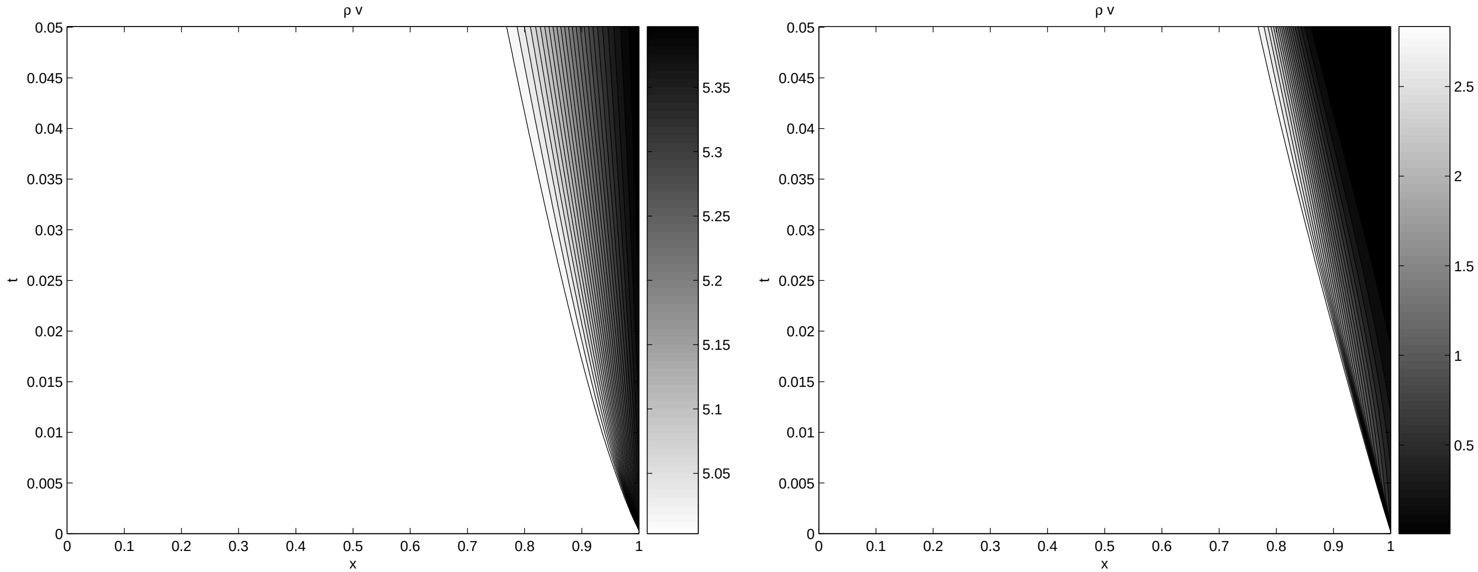


Figure 5: Plots of the level curves of the flux on the incoming roads: road 1 (left) and road 2 (right).

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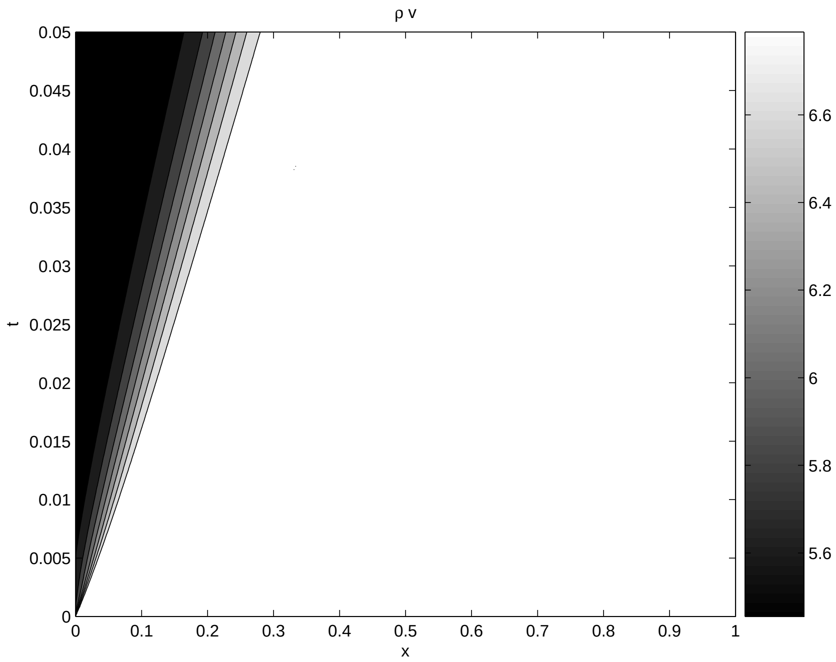


Figure 6: Plots of the level curves of the flux on the outgoing road 3.

# The case of a diverging junction



In this case, $k = 1$ for the incoming road and $j = 2, 3$ for the outgoing roads.
We have

$$\beta_{j1} = \frac{\alpha_{j1} q_1}{q_j}, \qquad j = 2, 3. \tag{30}$$

Since there is only one incoming road, $q_j = \alpha_{j1} q_1$, $j = 2, 3$ and therefore $\beta_{21} = \beta_{31} = 1$.
Obviously, here, **no** homogenization is needed, since there is a single incoming road: Therefore the maximization problem reduces to a linear program.

# Extension to a roundabout



Combination of merging and diverging junctions.

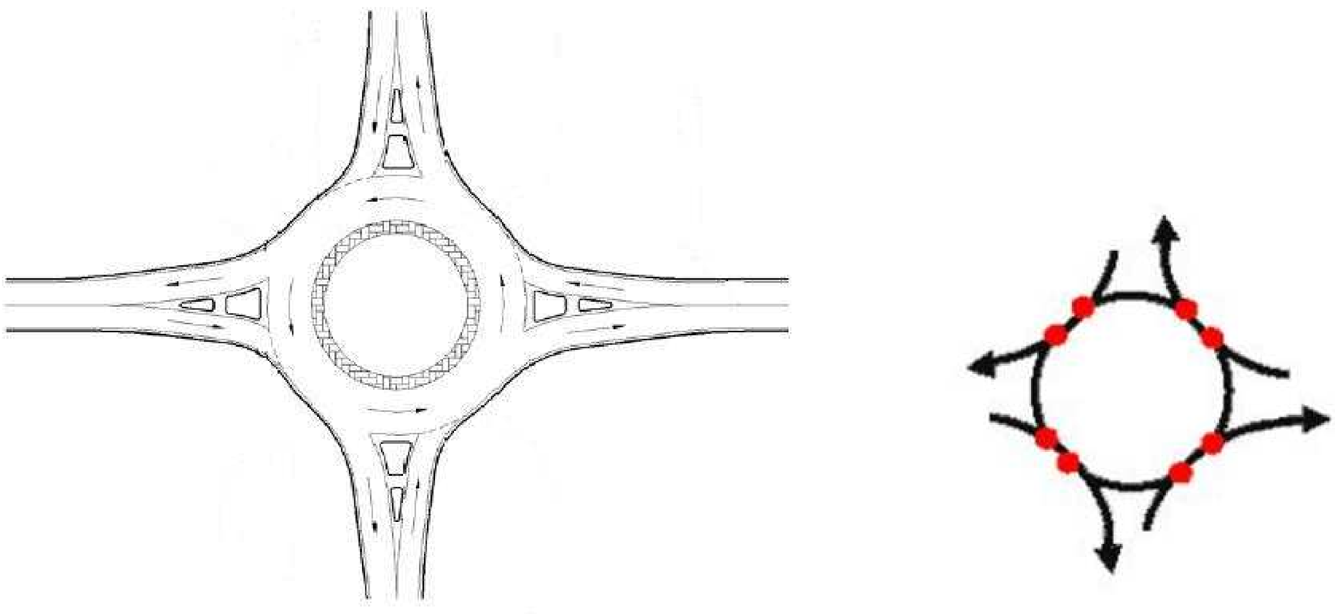

Figure 7: Roundabouts for a 4–4 junction.

# Conclusion and outlook





- Study of the boundary condition and solution to the Riemann problem through a junction.
- Further investigations:
  - Extension to a road network.

# Thank You!