Hybrid resolution of the LWR* model

Workshop on Mathematical Models of Traffic Flow

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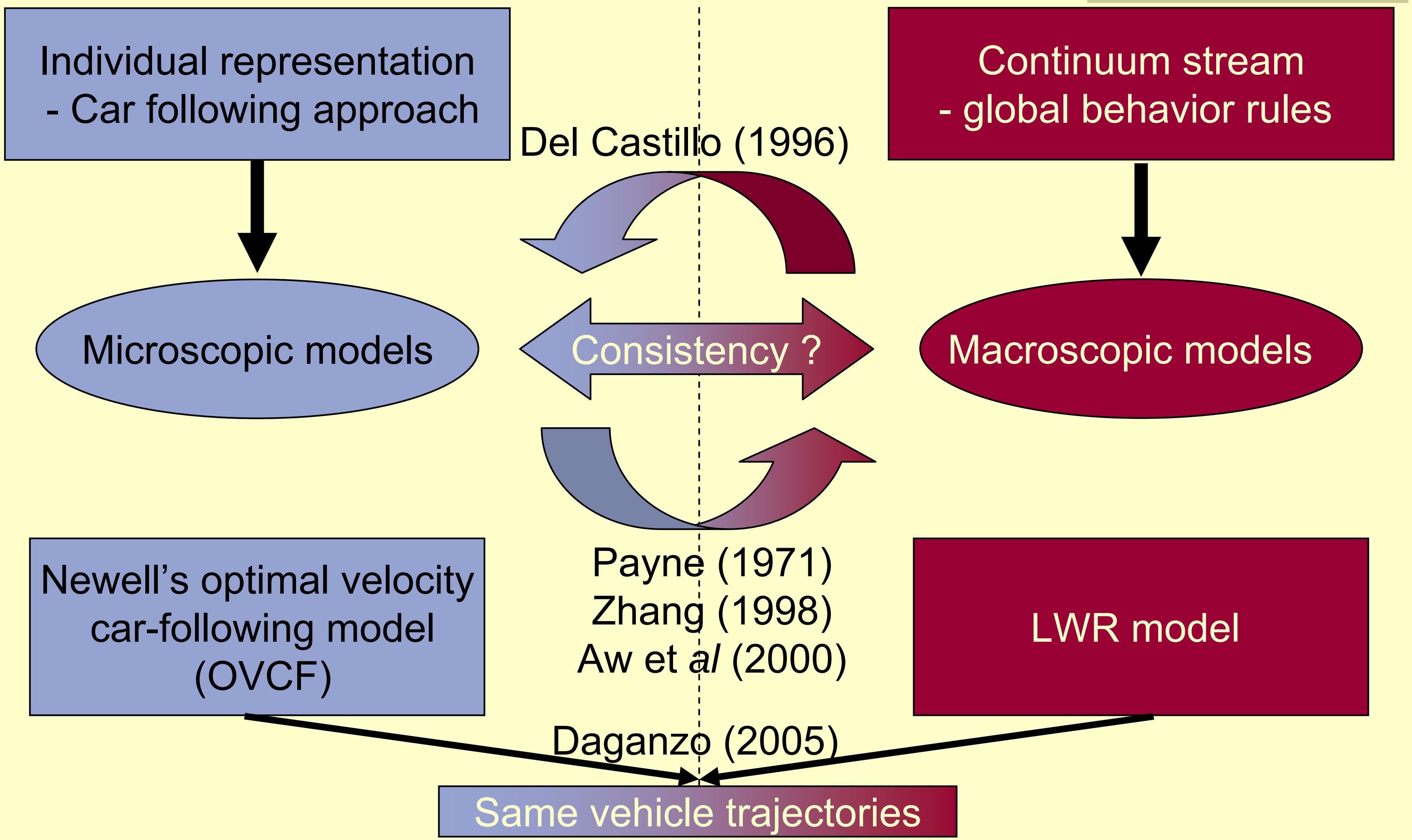
Background


Individual representation
- Car following approach
Continuum stream
- global behavior rules
Del Castillo (1996)
Microscopic models
Consistency ?
Macroscopic models
Payne (1971)
Zhang (1998)
Aw et *al* (2000)
Newell's optimal velocity car-following model (OVCF)
LWR model
Daganzo (2005)
Same vehicle trajectories

Outline

- Background
- Eulerian V.S Lagrangian resolution of the LWR model
- Coupling Lagrangian and Eulerian scheme
  - Lagrangian to Eulerian interface (L2E)
  - Eulerian to Lagrangian interface (E2L)

The LWR model

The conservation equation:

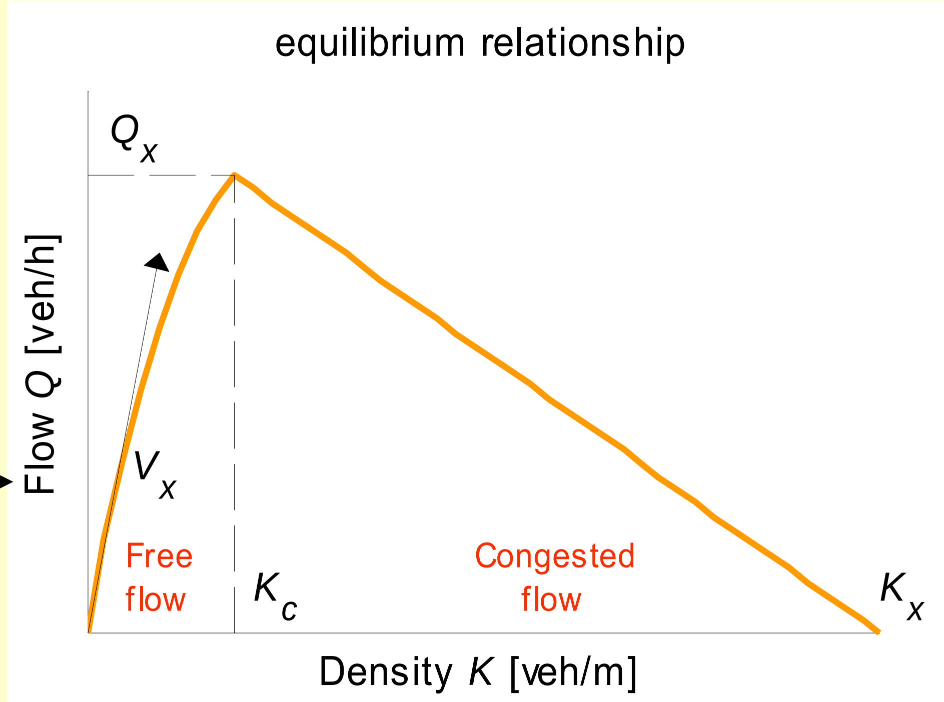
$$\partial_x Q(x,t) + \partial_t K(x,t) = 0$$

The speed definition:

$$V = Q / K$$

A fundamental diagram →

$$Q = Q_{eq}(K) \text{ or } V = V_{eq}(K)$$

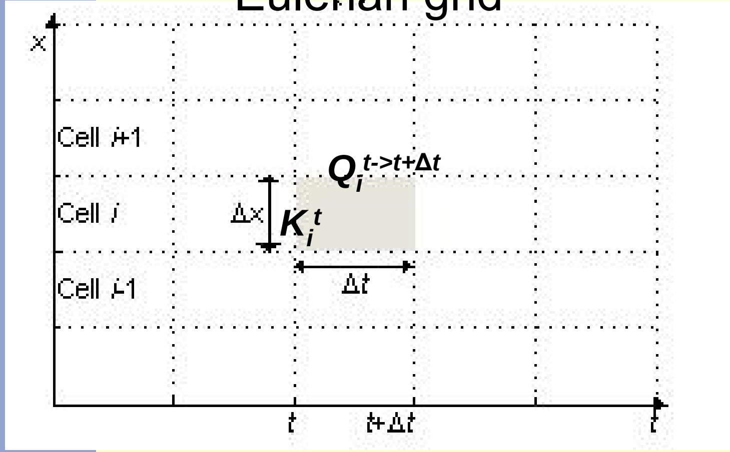


This can be synthesized under the following non-linear, hyperbolic equation:

$$\partial_t K(x,t) + \partial_x Q_{eq}\left(K\left(x,t\right)\right) = 0$$

Eulerian discretization

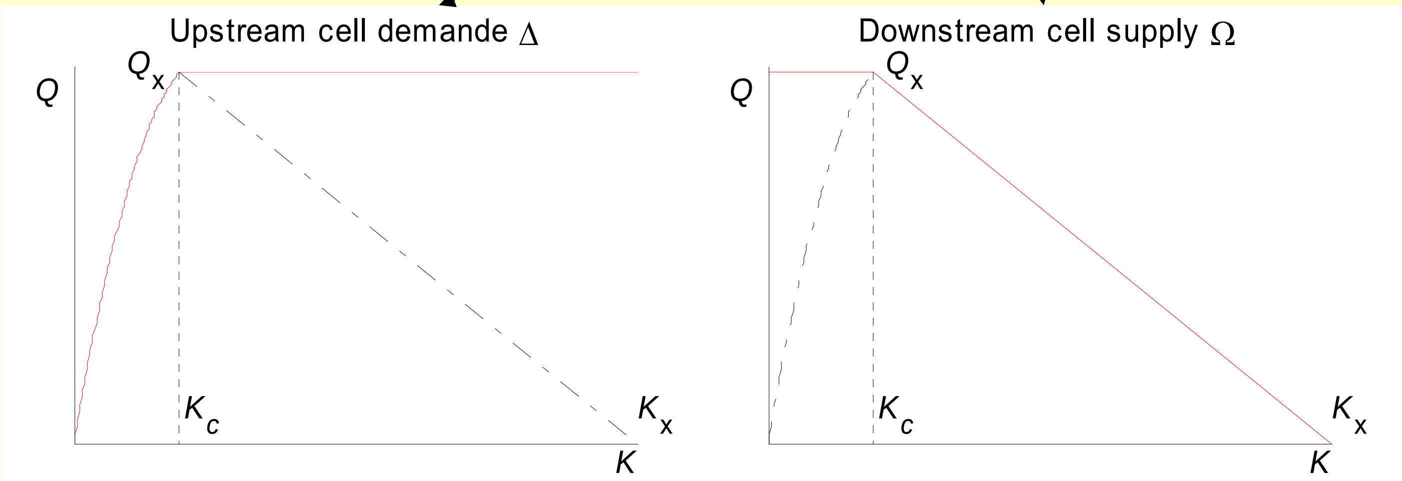
Eulerian grid



The Godunov scheme:

$$K_i^{t+\Delta t} = K_i^t + \left(Q_{i-1}^{t\to t+\Delta t} - Q_i^{t\to t+\Delta t}\right)\frac{\Delta t}{\Delta x}$$

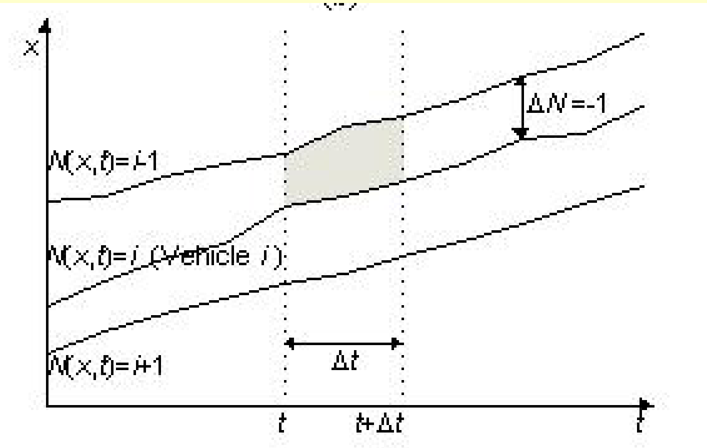
$$Q_i^{t\to t+\Delta t} = \min\left(\Delta_i\left(K_i^t\right), \Omega_{i+1}\left(K_{i+1}^t\right)\right)$$



CFL condition: $\Delta x / \Delta t \geq V_x$

Lagrangian formulation of the LWR model

Lagrangian grid



N(*x*,*t*): Newell's cumulative count curves (*N* curves)

N is constant along a vehicle trajectory

$$Q(x,t) = \partial_t N \quad \text{and} \quad K(x,t) = -\partial_x N$$

Eulerian coordinates (*x*,*t*) are changed into Lagrangian coordinates (*N*,*t*)

$$\partial_t K + \partial_x Q = 0 \quad \text{becomes} \quad \partial_t s + \partial_N V = 0 \quad \text{with} \quad s = 1/K$$

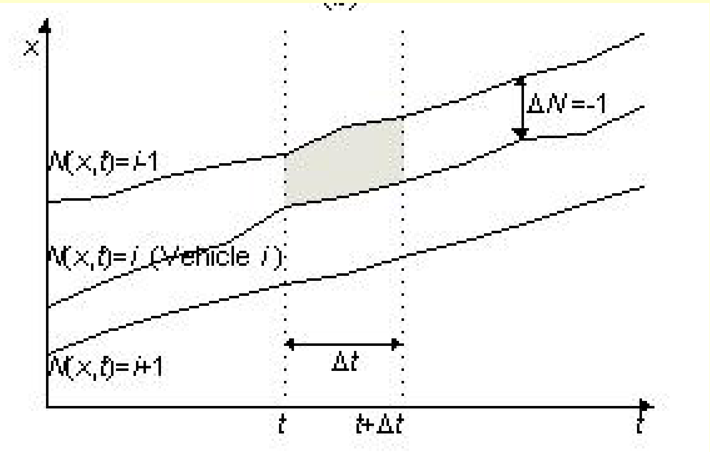
$$\text{Note that} \quad V = V_{eq}(K) = V_{eq}\left(1/s\right) = V_{eq}^{*}(s)$$

The LWR model, in (*N*,*t*) coordinates, corresponds to an hyperbolic equation in *s:*

$$\partial_t s + \partial_N V_{eq}^{*}(s) = 0$$

Lagrangian discretization

Lagrangian grid



Lagrangian grid where *i* follows *i*-1 => Δ*N*=-1

The Godunov scheme:

$$s_i^{t+\Delta t} = s_i^t + \frac{\Delta t}{\Delta N}\left(V_{eq}^*\left(s_i^t\right) - V_{eq}^*\left(s_{i-1}^t\right)\right)$$
$$= s_i^t + \Delta t\left(V_{eq}^*\left(s_{i-1}^t\right) - V_{eq}^*\left(s_i^t\right)\right)$$

If x_i^t is the position of vehicle *i* at time *t*, this scheme can be expressed as

$$x_i^{t+\Delta t} = x_i^t + V_{eq}^*\left(x_{i-1}^t - x_i^t\right)\Delta t$$

This is the discrete-time expression of the OVCF model

CFL condition: $$\frac{\Delta N}{\Delta t} \geq \max_s \left|V_{eq}^{*}{}'(s)\right| = -pK_x \quad \Rightarrow \quad \Delta t \leq -\frac{1}{pK_x}$$

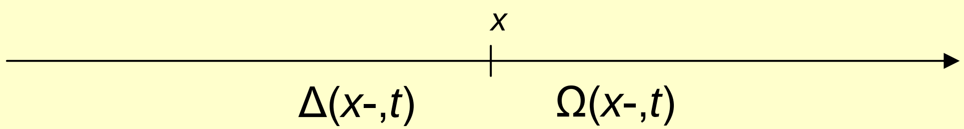
Remarks

- The OVCF model corresponds to the Lagrangian discretization of the LWR model
- Wagner (1987) has proved that the weak solutions of an hyperbolic problem are the same in Lagrangian and in Eulerian coordinates
- The LWR can so be discretized by two different ways:
  - The Eulerian discretization
  - The Lagrangian discretization

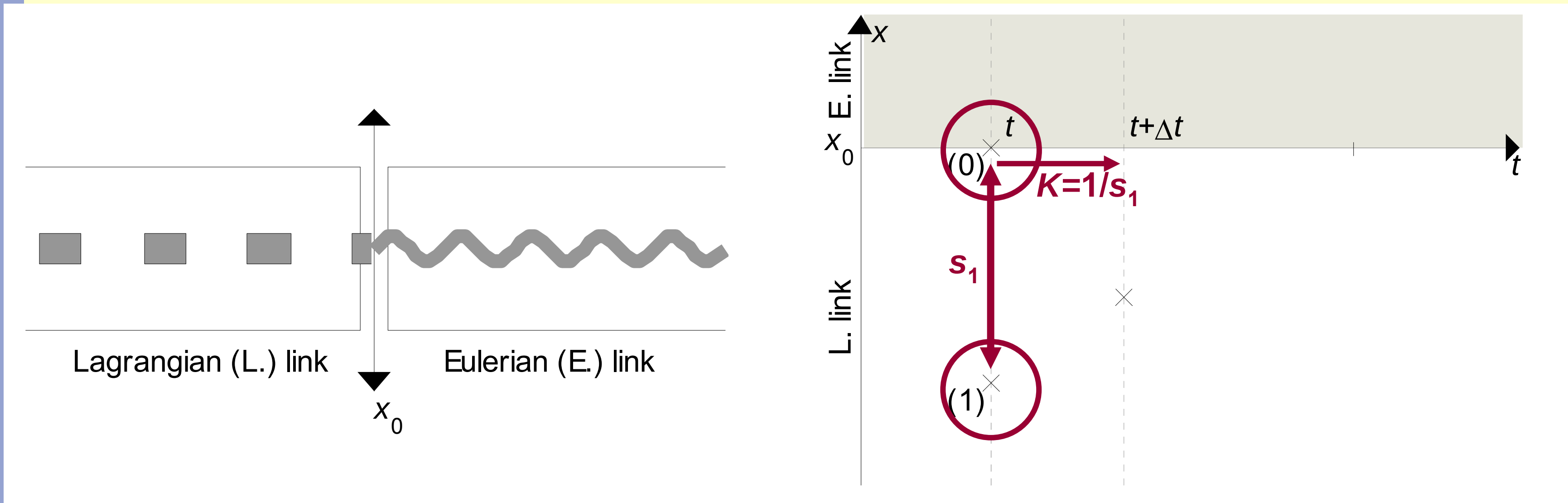
Coupling Lagrangian and Eulerian scheme

- Aims: coupling together the Lagrangian and Eulerian discretization of the LWR model
- Previous works use transition area and virtual vehicles
- Simpler interfaces will be proposed here:
  - Located at discrete points in space
  - Based on a generalized definition of the demand and supply

$$Q(x,t)=\min\left(\Delta\left(x^{-},t\right);\Omega\left(x^{+},t\right)\right)$$



Demand at the exit of a L. link



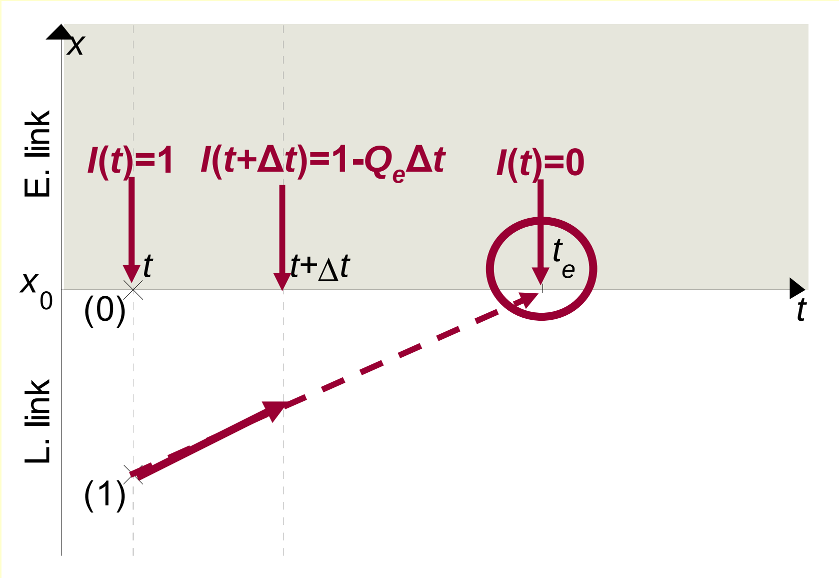
Demand at x_0^-

$$\Delta\left(x_0^-,t\right)=\begin{cases} K\left(x_0^-,t\right)V_{eq}\left(K\left(x_0^-,t\right)\right) & \text{if } K \le K_c \\ Q_x & \text{if } K > K_c \end{cases}$$

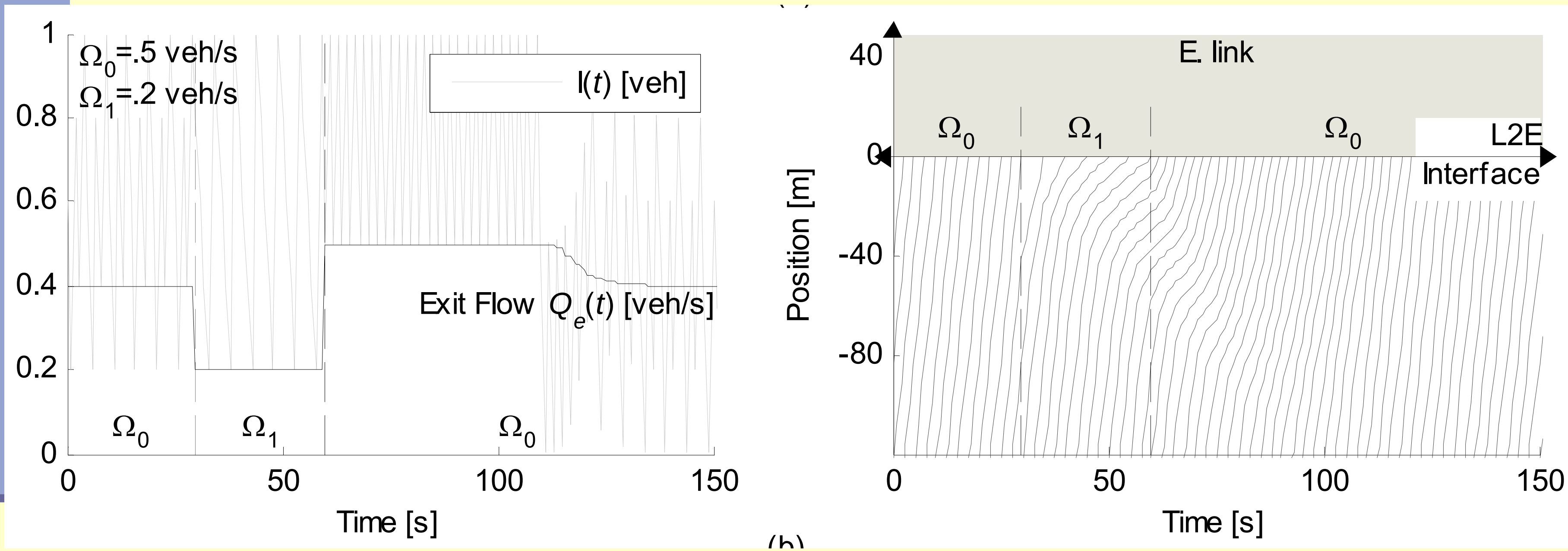
$$\Rightarrow \quad \Delta\left(x_0^-,t \to t+\Delta t\right)=\begin{cases} \dfrac{V_{eq}\left(1/s_1\right)}{s_1}=\dfrac{V_{eq}^*\left(s_1\right)}{s_1} & \text{if } s_1 \ge s_c \\ Q_x & \text{if } s_1 < s_c \end{cases}$$

L2E interface functioning

- Exit flow $Q_e\left(x_0, t \rightarrow t+\Delta t\right) = \min\left(\Delta\left(x_{0^-}, t \rightarrow t+\Delta t\right), \Omega\left(K_0^+\right)\right)$
- The cumulative flow at x_0 between two vehicles must be equal to 1
- The speed of the reference vehicle is adapted to respect this condition

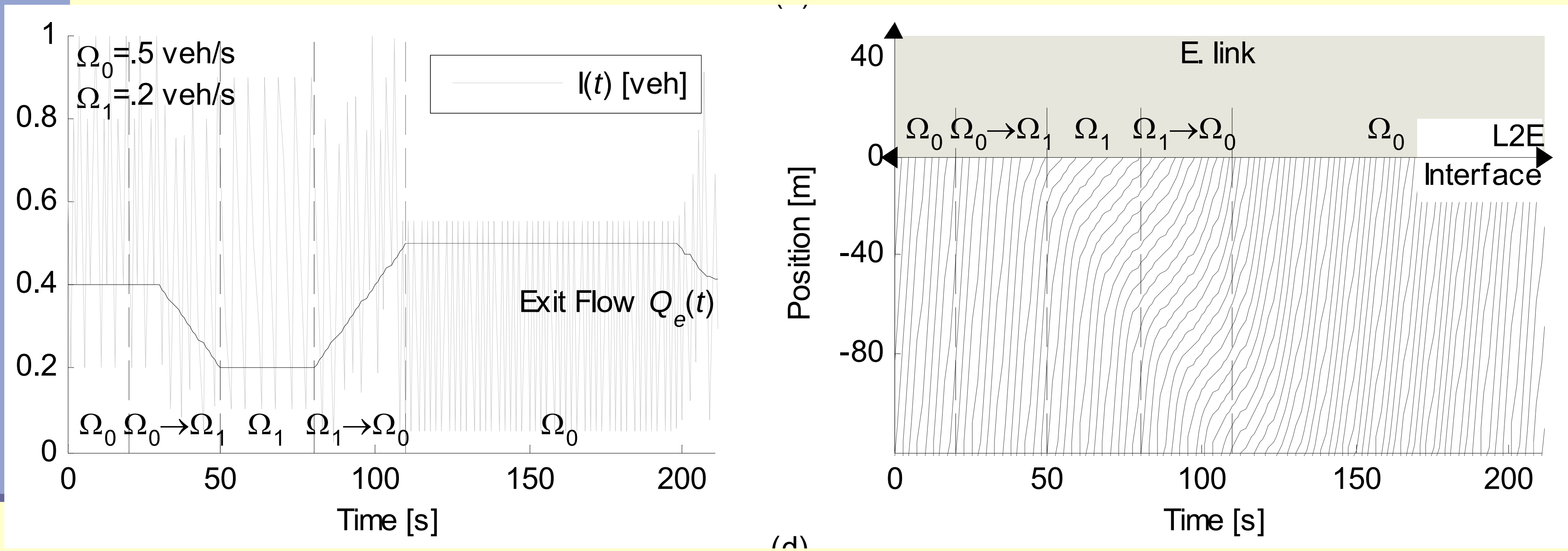


Simulation results (1)



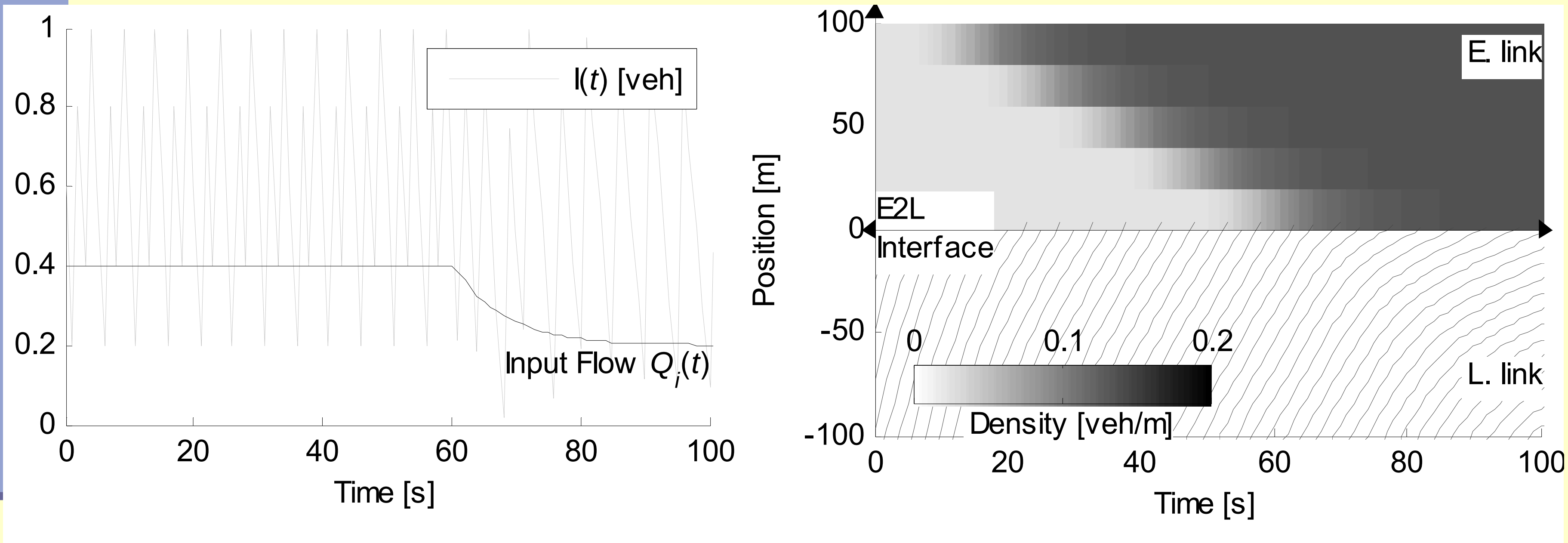
Discontinuous decrease and then increase of the downstream supply

Simulation results (2)



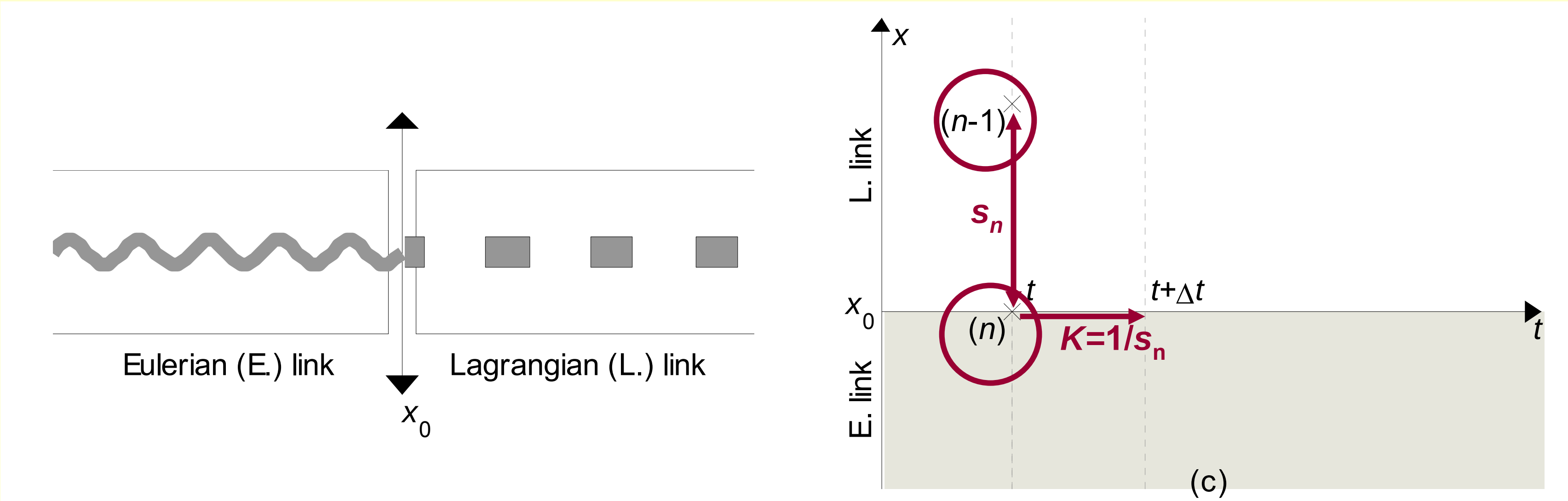
Continuous decrease and then increase of the downstream supply

Simulation results (3)



A shockwave coming from the E. link

Supply at the entrance of a L. link



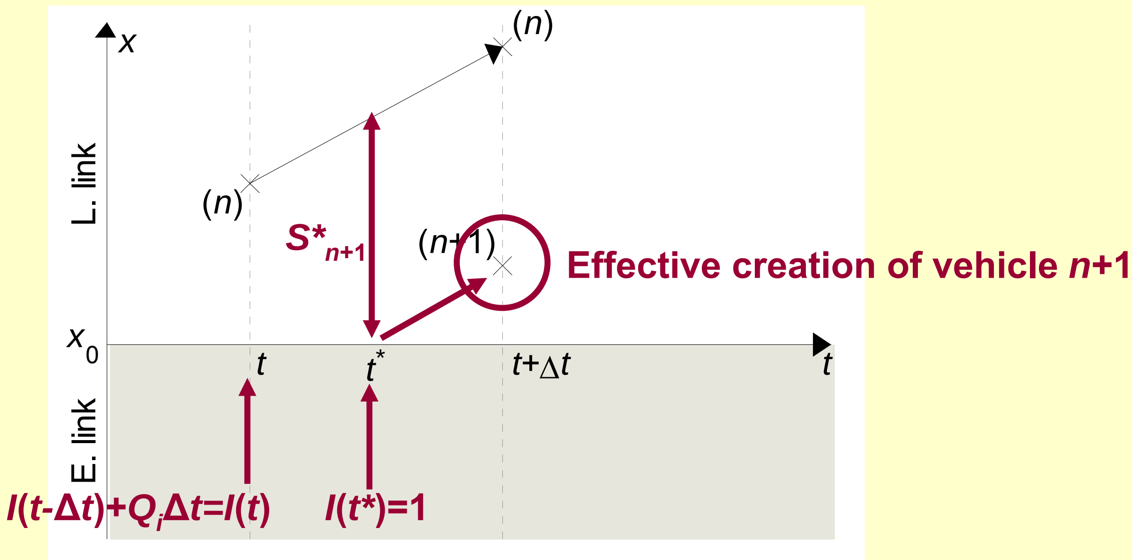
Supply at x_0^+

$$\Omega\left(x_0^+\right)=\begin{cases} Q_x & \text{if } K \le K_c \\ K\left(x_0^+,t\right)V_{eq}\left(K\left(x_0^+,t\right)\right) & \text{if } K > K_c \end{cases}$$

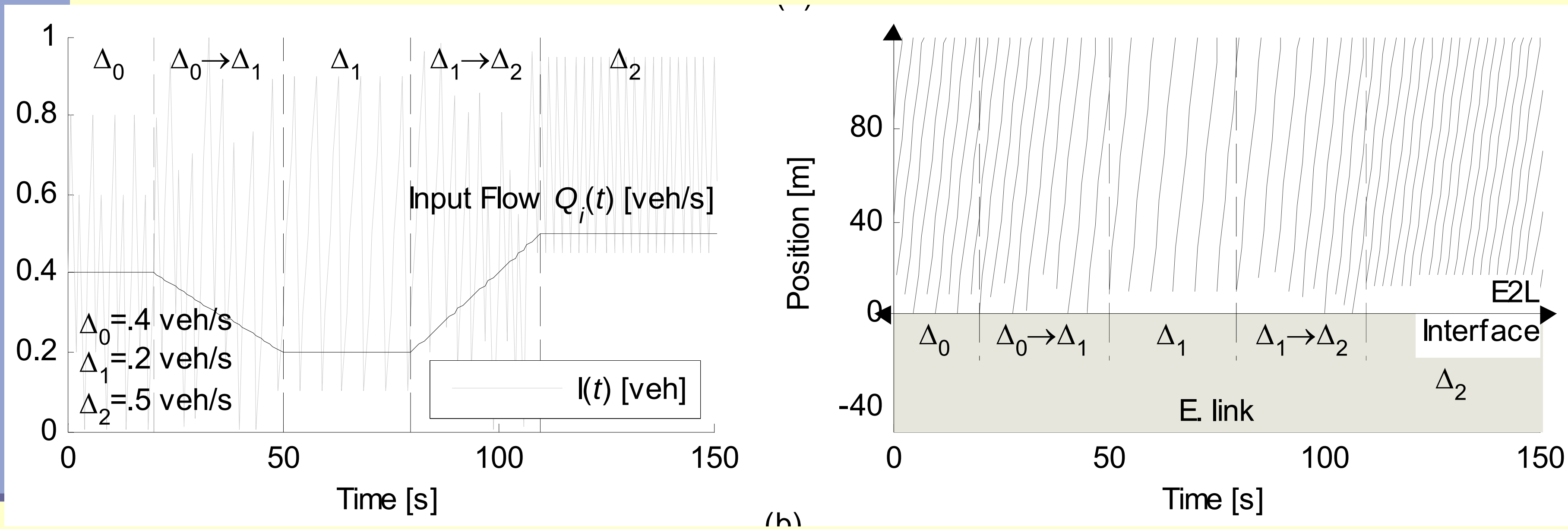
$$\Omega\left(x_0^+,t\rightarrow t+\Delta t\right)=\begin{cases} Q_x & \text{if } s_n \ge s_c \\ \dfrac{V_{eq}\left(1/s_n\right)}{s_n}=\dfrac{V_{eq}^*\left(s_n\right)}{s_n} & \text{if } s_n < s_c \end{cases}$$

E2L interface functioning

- Input flow $Q_i\left(x_0, t \to t+\Delta t\right) = \min\left(\Delta\left(K_0^-\right), \Omega\left(x_{0^+}, t \to t+\Delta t\right)\right)$
- A new vehicle is created when the cumulative flow is equal to 1 at x_0
- The speed of the created vehicle depends on the spacing with its leader

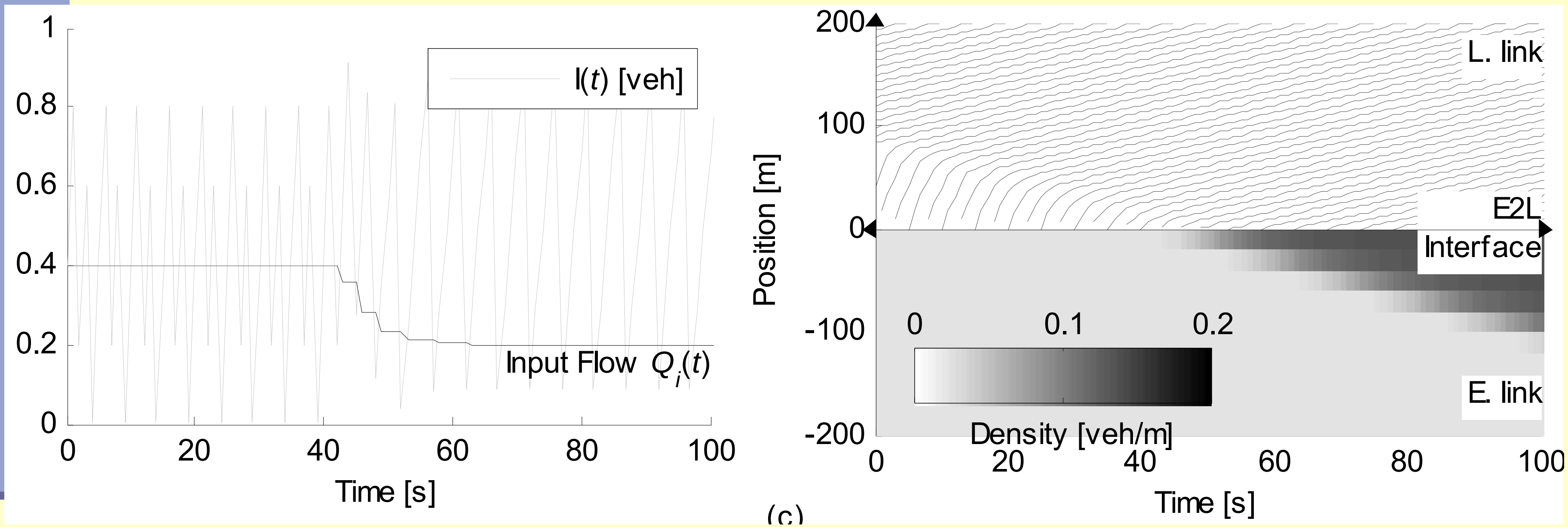


Simulation results (1)



Continuous decrease and then increase of the upstream demand

Simulation results (2)



A shockwave coming from the L. link

Conclusion

- The LWR model can be numerically solved:
  - Eulerian scheme
  - Lagrangian scheme following the *N* curves
- A hybrid model has been proposed coupling together both discretizations
- The proposed interfaces are fully compatible with major LWR extensions (bounded acceleration, intersections modeling, multi-lanes…)
- The proposed hybrid model could help the development of new extensions…

End of the presentation

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