First order traffic flow models: intersection modelling, network modelling, applications

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OUTLINE

1. Traffic modelling (objectives, state of art)
2. First order models, second order models
3. A fast review of the LWR model on a line
4. Local supply and demand, numerical schemes
5. Boundary conditions, intersection models
6. The LWR model on a network
7. Conclusions and next steps
Traffic flow modelling Objectives

Modelling tools are useful for traffic engineering tasks:

- Implementation of on-line control strategies
- Off-line evaluation of control strategies: ramp metering, speed control, collective route guidance via VMS, intersection control, externalities etc.
- Prediction and estimation of the traffic state
- New infrastructure construction etc.
Traffic flow modelling approaches

Two main types of modelling approaches:

1. Microscopic (evaluation, simulation)
   a. follow-the-leader,
   b. cellular automata,
   c. multi-agents

2. Macroscopic (evaluation, control):
   a. First order Modelling
   b. Second order Modelling
Macroscopic traffic description

- **Hydrodynamic analogy**
- **Continuum hypothesis:** traffic state can be described by functions of location $x$ and time $t$
- **Variables:**
  - Density $\rho(x,t)$ (or $K(x,t)$)
  - Flow $q(x,t)$
  - Velocity $v(x,t)$
Definitions

- Density

\[ N(x, \Delta x; t) \]

- Flow

\[ N(x; \Delta t, t) \]
Definitions (2)

- **Speed**

- **Conservation**

- Only for space-time scales > 100 meters x 5 seconds
Macroscopic traffic description

- Limiting factor of the continuum hypothesis:
  - Avogadro number $N = 6.025 \times 10^{23}$ (number of molecules per 22.4 liters of gas under normal conditions)
  - Maximum (jam) density on highways, as communicated by operators: 180 vh / km x lane
Macroscopic approaches: Basic equations

1- **Continuity Equation:**
\[ \partial_t \rho(x,t) + \partial_x q(x,t) = 0 \]

2- **Volume-Density-speed relationship:**
\[ q(x,t) = \rho(x,t) v(x,t) \]

3- **Fundamental Diagram** (equilibrium)
\[ v(x,t) = V_e(\rho(x,t)) \quad \text{where } V_e \text{ monotone decreasing function} \]

4- **Momentum equation**
\[ \frac{dv(x,t)}{dt} = \partial_t v(x,t) + v(x,t) \partial_x v(x,t) = G(\rho(x,t), v(x,t)) \]
Fundamental diagram, equilibrium

- No less than 25 FDs in the literature (TRB 165)
- Example of fundamental diagram

![Diagram of fundamental diagram](image)
1\textsuperscript{st} vs 2\textsuperscript{nd} order models

- 1\textsuperscript{st} order: assumed at equilibrium ($\rho, v$ on the fundamental diagram)

- 2\textsuperscript{nd} order: out of equilibrium ($\rho, v$ points are \textbf{not} on the fundamental diagram)

- Both are dynamic (term “equilibrium” is misleading)
Another example of FD

- Cf METACOR, STRADA
Another example still of FD

- Cf Newell, Daganzo
Momentum equation approaches

\[ \partial_t V + V \partial_x V = \frac{1}{\tau} (V_{eq}(\rho) - V) - \frac{1}{\rho} K^2(\rho) \partial_x \rho \]

- \( K(\rho) = \sqrt{-\frac{v}{2\tau} V_e'(\rho)} \) \quad \text{Payne model (1971):}
- \( K(\rho) = 0 \) \quad \text{Ross's model 1988}
- \( K(\rho) = \sqrt{-\rho V_e'(\rho) \exp \left( \frac{1}{a} (V_e(\rho) - v) \right) } \) \quad \text{Del Castillo's model 1993}
- \( K(\rho) = -\rho V_e'(\rho) \) \quad \text{Zhang's model 1998}
Momentum equation approaches (2)

\[ \partial_t V + V \partial_x V = \frac{1}{\tau} (V_{eq}(\rho) - V) - \frac{1}{\rho} K^2(\rho) \partial_x \rho \]

The conservative form of the system (including the CE):

\[ \partial_t U + \partial_x F(U) = S(U) \]

with 

\[ U = \begin{pmatrix} \rho \\ v \end{pmatrix} \text{ and } S(U) = \begin{pmatrix} 0 \\ \frac{1}{\tau} (V_{eq}(\rho) - v) \end{pmatrix} \]

\[ F(U) = \begin{pmatrix} v & \rho \\ \frac{1}{\rho} K^2(\rho) & v \end{pmatrix} \]

The dynamic of the system is given by the eigenvalues of \( F(U) \)
What does it mean?

- **Conservative form**: no mathematically “illegal” derivatives ⇔ no physically meaningless expressions

- **Eigenvalues describe the dynamics**: the characteristic speeds are the propagation speed of information, small perturbations in the flow (linearization)

\[
\partial_t V + V \partial_x V = \frac{1}{\tau} \left( V_{eq}(\rho) - V \right) - \frac{1}{\rho} K^2(\rho) \partial_x \rho \\
\partial_t U + \partial_x F(U) = S(U)
\]
Interpretation of the momentum equation

- Traffic acceleration is the result of
  - Relaxation term: traffic state tends towards equilibrium
  - Anticipation term: interaction of a vehicle with surrounding vehicles

\[
\partial_t V + V \partial_x V = \frac{1}{\tau} (V_{eq}(\rho) - V) - \frac{1}{\rho} K^2(\rho) \partial_x \rho
\]
Eigenvalues: interpretation

- Special waves:
  - Small perturbations of the traffic flow
  - Self-similar solutions

- The velocity of these special waves is equal to the eigenvalues of

\[
\partial_t U + \partial_x f(U) = 0
\]

\[
A(U) \overset{\text{def}}{=} \nabla f(U)
\]
Momentum equation approaches (3)

System eigenvalues:

\[\lambda_1(U) = v - K(\rho)\]
\[\lambda_2(U) = v + K(\rho)\]

The system is hyperbolic (except ROSS)
The anisotropic character of the traffic is not preserved (except ROSS)
due to:

\[\lambda_2(U) = v + K(\rho) > v\]

Daganzo, 1995 claims the « requiem for the second order traffic modelling »
Aw, Rascle 2000 « resurrection of the second order traffic model »
Why is the anisotropic character of traffic not respected?

- If some eigenvalue is > traffic speed, information travels faster than traffic.
- Information “catches up” drivers.
- Upstream lower density “attracts” vehicles backward ⇒ negative speeds.
The LWR model

- Introduced by Lighthill, Whitham (1955), Richards (1956)
- Traffic at equilibrium: speed-density points on the fundamental diagram
- A single conservation law (for density; the flux is the flow), a single eigenvalue
The LWR model in a nutshell

- The equations:

$$\frac{\partial K}{\partial t} + \frac{\partial Q}{\partial x} = 0$$  
conservation equation

$$Q = KV$$  
definition of $V$

$$V = V_e(K, x)$$  
behavioural equation

or:

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial x}Q_e(K, x) = 0$$
The LWR model (2)

Conventions

Equilibrium flow $Q_e$:

Equilibrium speed $V_e$:

Notations:

- $Q$: Flow
- $K$: Density
- $V$: Speed
- $V_e$: equilibrium speed
- $V_e$: equilibrium flow

$Q_e(K, x) \overset{def}{=} KV_e(K, x)$
Analytical solutions (reminder)

- Description of solutions:
  - using characteristics and shockwaves
  - characteristics with $> 0$ slope $\iff$ fluid traffic $K < K_{crit}$
  - characteristics with $< 0$ slope $\iff$ congested traffic $K > K_{crit}$
  - coordinates $(x, t)$

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Analytical solutions (2)

- Shock-waves:

Shock-wave velocity:

\[ v = \frac{Q_a - Q_b}{K_a - K_b} = \frac{[Q]}{[K]} \]

(Rankine-Hugoniot)

Concave fundamental diagram ⇒ Only *deceleration* shockwaves are allowed in entropy solutions.

Comment: Experimental fundamental diagrams need *not* be concave.
Analytical solutions (3)

- Rarefaction waves:
  - Characteristic speed (eigenvalue):
    \[ Q_e'(K) \]

- Comments:
  - Some intersection models require *boundedness of acceleration*. 
The LWR model: supply / demand

- the equilibrium supply $\Sigma_e$ and demand $\Delta_e$ functions (Lebacque, 1996)
The LWR model: the min formula

The local supply and demand:

\[ \Sigma(x,t) = \Sigma_e(K(x^+,t), x^+) \]
\[ \Delta(x,t) = \Delta_e(K(x^-,t), x^-) \]

The min formula

\[ Q(x,t) = \text{Min} \left[ \Sigma(x,t), \Delta(x,t) \right] \]
Local traffic demand

Local Traffic Demand Demand at a point \( x \) is the greatest possible outflow at that point:

\[
\Delta_e = Q_{\text{max}}(x^-) \quad K_{\text{crit}}(x^-) \quad K_{\text{max}}(x^-)
\]

\[
Q = Q_\text{c}(K) = Q
\]
Local traffic supply

Local Traffic Supply Supply at a point $x$ is the greatest possible inflow at that point.
Supply-demand boundary conditions

- Link supply: \( \Sigma(a,t) = \Sigma_e(K(a+,t),a) \)
- Link demand: \( \Delta(b,t) = \Delta_e(K(b-,t),b) \)
- Min formula:
  \[
  Q(a,t) = \text{Min} \left[ \Delta_u(t), \Sigma(a,t) \right]
  \]
  \[
  Q(b,t) = \text{Min} \left[ \Delta(b,t), \Sigma_d(t) \right]
  \]
Upstream boundary condition

- The upstream boundary condition determines the link inflow (the Min formula)

\[ Q(a,t) = \text{Min} [\Delta_u(t), \Sigma(a,t)] \]
Upstream boundary condition

- The upstream boundary condition determines the density at the link entry point.

- Symmetric rules apply at the link exit.

Let $q = \min[\delta, \Sigma_a]$ be the entry flow.

If $\Sigma_a < \delta$, $K_b = \Sigma_e^{-1}(q)$.

If $\Sigma_a \geq \delta$, $K_b = \Delta_e^{-1}(q)$.

Supply Regime

Demand Regime

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Upstream demand can be modified by boundary conditions

- If link supply $\Sigma_a(t)$ is less than upstream demand $\Delta_u(t)$, then
- The upstream demand is modified:
  \[ \Delta_u(t) \rightarrow \Delta_u(t+1) = Q_{max} \]
- This fact is fundamental for intersection modeling.
- Symmetric result for downstream supply.

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The BLN (Bardos-Leroux-Nédélec) boundary condition

- **Origin**: viscosity solutions of the LWR
- **Idea**: to impose a density-like $A$ at the boundary

Mathematical expression

\[ \{ \text{sgn} [K(x,t) - \kappa] - \text{sgn} [A(c,t) - \kappa] \} [Q_e(K(c,t),c) - Q_e(\kappa)].n(c) \geq 0 \]

\[ \forall c \in \partial D = \{a, b\} \quad \text{and} \quad \forall \kappa \geq 0 \]
BLN boundary conditions: the density at the boundary cannot be prescribed

\[
\begin{align*}
Q_e(K) &= \min_{\kappa \in [A,K]} Q_e(\kappa) \quad \text{if} \quad K \geq A \\
Q_e(K) &= \max_{\kappa \in [K,A]} Q_e(\kappa) \quad \text{if} \quad K \leq A
\end{align*}
\]

- With upstream boundary conditions:

  \[
  A \leq K_{\text{crit}} : \quad K \in \{A\} \cup [A^*, K_{\text{max}}]
  \]

  \[
  A \geq K_{\text{crit}} : \quad K \in [K_{\text{crit}}, K_{\text{max}}]
  \]

- \emph{A cannot} be prescribed
Graphical illustration (upstream boundary conditions)

\[
\begin{align*}
Q_e(K) &= \text{Min}_{\kappa \in [A,K]} Q_e(K) \quad \text{if} \quad K \geq A \\
Q_e(K) &= \text{Max}_{\kappa \in [K,A]} Q_e(K) \quad \text{if} \quad K \leq A
\end{align*}
\]

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The BLN and the supply / demand boundary conditions are equivalent

\[ A \leq K_{\text{crit}} : \quad K \in \{A\} \cup [A^*, K_{\text{max}}] \]
\[ A \geq K_{\text{crit}} : \quad K \in [K_{\text{crit}}, K_{\text{max}}] \]

- **Cases** \( K \neq A \):
  \[ \Leftrightarrow \quad \Delta_e(A) \leq \Sigma_e(K) \leq \Delta_e(A) \]
  - \( A \leq K_{\text{crit}} \) and \( K \in [A^*, K_{\text{max}}] \)
    \[ \Sigma_e(K) = Q_e(K) \leq Q_e(A) = \Delta_e(A) \]
  - \( A \geq K_{\text{crit}} \) and \( K \in [K_{\text{crit}}, K_{\text{max}}] \)
    \[ \Sigma_e(K) = Q_e(K) \leq Q_{\text{max}} = \Delta_e(A) \]
- \( A \) over-critical in all these cases
Point-wise intersections

- **Basic idea:** solve the (generalized) Riemann problem for the intersection
- **Result:** constraints on the through flows

\[
\begin{align*}
Q_i & \leq \Delta_{\epsilon}(K_{i0}) = \delta_i \\
& \text{and} \quad \begin{cases} 
K_i = \Sigma_{\epsilon}^{-1}(Q_i) & \text{if } Q_i < \delta_i \\
K_i = K_{i0} & \text{if } Q_i = \delta_i
\end{cases} \\
R_j & \leq \Sigma_{\epsilon}(K_{j0}) = \sigma_j \\
& \text{and} \quad \begin{cases} 
K_j = \Delta_{\epsilon}^{-1}(Q_j) & \text{if } R_j < \sigma_j \\
K_j = K_{j0} & \text{if } R_j = \sigma_j
\end{cases}
\end{align*}
\]
Riemann problem

- It is an archetype for many practical situations
- Initial conditions are piecewise constant
- Solutions are self-similar (waves)

- It is the key for developing numerical methods
Flow constraints imply that density changes propagate in the right direction.

\[
\begin{align*}
0 \leq Q_i & \leq \delta_i \quad \forall i \\
0 \leq R_j & \leq \sigma_j \quad \forall j \\
\sum_i Q_i & = \sum_j R_j
\end{align*}
\]
Necessity of intersection models

- Other flow constraints are possible (turning movement proportions, assignment coefficients...)
- Flow constraints do not suffice to determine the flow values
- A behavioral intersection model is necessary
  \[(Q, R) = f(\delta, \sigma)\]
- But not all models are consistent
Invariance principle 1

- **Upstream link** \(i\). If \(Q(t) < \delta_i(t)\) : supply regime at the link exit and \(\delta_i(t^+) = Q_{max}\)

- **Downstream link** \(j\). If \(R_j(t) < \sigma_j(t)\) : supply regime at the link exit and \(\sigma_j(t^+) = Q_{max}\)

- The intersection model \((Q, R) = f(\delta, \sigma)\) must be invariant by the transformation

\[
\begin{align*}
\delta_i &\rightarrow Q_{i, max} & \text{if } Q_i < \delta_i \\
\sigma_j &\rightarrow R_{j, max} & \text{if } R_j < \sigma_j
\end{align*}
\]
Invariance principle 2

- Another way of stating the invariance principle:
  - The intersection model must be compatible with self-similarity of solutions
  - Riemann problem at the node
The invariance principle: an example

- Distribution scheme:
- Numerical values:
  - $\sigma = 3000$ vh / h (2 lanes)
  - $\delta_1 = 2100$ vh / h (1 lane)
  - $\delta_2 = 1400$ vh / h (1 lane)
  - $Q_{k,max} = 2200$ vh / h

\[
\begin{align*}
Q_i &= \delta_i & \text{if } \sum_i \delta_i \leq \sigma \\
Q_i &= \frac{\delta_i}{\sum_k \delta_k} & \text{if } \sum_i \delta_i > \sigma
\end{align*}
\]
The invariance principle: an example (cont’d)

The flow values calculated by the distribution scheme imply a shift in the upstream demands

\[
Q_1 = \frac{2100}{2100 + 1400} \times 3000 = 1800 \text{ vh/h} < 2100 \text{ vh/h} = \delta_1(t)
\]

\[
Q_2 = \frac{1400}{2100 + 1400} \times 3000 = 1200 \text{ vh/h} < 1400 \text{ vh/h} = \delta_2(t)
\]

- The flow values calculated by the distribution scheme imply a shift in the upstream demands
  - \( \delta_1(t +) = 2200 \text{ vh/h} \)
  - \( \delta_2(t +) = 2200 \text{ vh/h} \)
The invariance principle: an example (cont’d)

- The new demand values determine **new flow values** $Q_1 = Q_2 = 1500$ vh/h
- Illustration for link [2]
- 3 traffic states
The invariance principle: an example (cont’d)

- States \((u), (i), (d)\)
- Velocity of the \((u) \rightarrow (i)\) shockwave < the velocity of the \((d) \rightarrow (i)\) rarefaction wave \(\Rightarrow\)
  - The state \((i)\) must vanish
- Velocity of the \((u) \rightarrow (d)\) shockwave > 0 \(\Rightarrow\)
  - state \((d)\) must vanish at \(t+\).
  - \(\Rightarrow\) inconsistency
An example of a node model satisfying the invariance principle: the optimization node model

\[
\text{Max } \sum_i \Phi_i(Q_i) + \sum_j \Psi_j(R_j)
\]

- \(Q_i \leq \delta_i \quad \forall i\)
- \(R_j \leq \sigma_j \quad \forall j\)
- \(\sum_i \gamma_{ij} Q_i - R_j = 0 \quad \forall j\)

- \(\gamma_{ij}\) : turning movement coefficients (deduced from the assignment coefficients)
- **Constraints:**
  - Node inflows less than upstream demands
  - Node outflows less than downstream supplies
  - Conservation of node out-flows
The Karush-Kuhn-Tucker optimality conditions yield \( s_j \) coefficient of the outflow (\( j \) conservation equation)

\[
Q_i = \min \left[ \delta_i, \Phi_i^{-1} \left( - \sum_j \gamma_{ij} s_j \right) \right] \\
R_j = \min \left[ \Psi_j^{-1} \left( s_j, \sigma_j \right) \right] \\
\sum_i \gamma_{ij} Q_i - R_j = 0 \quad \forall j
\]

The in- and out-flows are given by a Min-formula \( \Rightarrow \) The model satisfies the invariance principle

\[
Q_i = \min \left[ \delta_i, \varphi_i \right] \\
R_j = \min \left[ \psi_j, \sigma_j \right]
\]
Optimization node model (cont’d)

- Interpretation of the criterion:
  - $\Phi_i$: → partial supply of node (for link $(i)$)
  - $\Psi_j$: → partial demand of node (for link $(j)$)

- Coefficients $s_j$: “node state”

- Other models satisfy the invariance principle (dynamic pointwise, equilibrium)
A second example: dynamic node models

- They are characterized by inner state dynamics

\[ N = \sum_{ij} N_{ij} \]
\[ NO_j = \sum_i N_{ij} \]
\[ \Sigma_i(N) = \beta_i \Sigma(N) \quad \forall i \]

\[ \dot{N}_{ij} = \gamma_{ij} Q_i - \frac{N_{ij}}{NO_j} R_j \]
\[ Q_i = \text{Min} \left[ \delta_i, \beta_i \Sigma(N) \right] \]
\[ R_j = \text{Min} \left[ \frac{NO_j}{N} \Delta(N), \sigma_j \right] \]
Equilibrium node models

- They are derived from dynamic node models.
- **Assumption:** node time-scale $<<$ link time-scale

$\forall = \gamma - j QR_i j NN_j NOR_j iNQ j i j j i j j i j j i j i j j i j j i j i j i j j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i j i
Optimization vs Equilibrium node models

- Both models provide node supplies and demands
- Optimization and equilibrium node models are equivalent for
  - Merges
  - FIFO Diverges
Some numerical results

- The site: A4
- The sets of points are bounded by linear constraints \(\Rightarrow\) invariance with respect to aggregation
- Results for the merge
Some numerical results (cont’d): non stochastic scatter of data points

- Motorway: upstream vs downstream flow
Some numerical results (cont’d)

- On ramp flow vs motorway downstream flow
Some numerical results (cont’d)

- Onramp vs upstream motorway flow
Physical node models

- Internal state of node:
  - Stored vehicles
  - Node supply/demand functions
  - (Lebacque-Khoshyaran 1998-2002)

\[
\begin{align*}
N &= \sum_i Q_i - \sum_j Q_j \\
Q_i &= \min[\delta_i, \beta_i \Sigma_e(N)] \\
Q_j &= \min[\chi_j \Delta_e(N), \sigma_j]
\end{align*}
\]

Assume:
- fundamental diagram \(Q_e(N)\),
- and node traffic supply and demand functions \(\Sigma_e(N), \Delta_e(N)\).

\(\beta_i\): split coefficient for node supply
\(\chi_j\): composition coefficient of node traffic
Discretized node models

- Exchange zones
  - Generalize cells of Godunov scheme
  - Conflicts are described implicitly
  - Buisson, Lebacque, Lesort 1995-1996
  - Haj-Salem, Lebacque 2003: bounded acceleration
Exchange zones (cont’d)

- Features:
  - global variables $N$, $NI_i$, $NO_j$, $N_{ij}^d$ (movements, per final destination: non FIFO)
  - global supply and demand $\Sigma_e(N)$ and $\Delta_e(N)$
  - partial supplies and demands $\Sigma_i = \beta_i \Sigma_e(N)$,
    $\Delta_j = (NO_j/N) \Delta_e(N)$
Exchange zones (cont’d)

- Linear partial supply model
- Fits exp. data

(oversaturated merge, Oltra and Jardin 1998)
The SSMT node

- Models movements with overlapping cells
- SSMT (Lebacque 1984) and METACOR (Haj-Salem 1995)
- Model of priority conflicts (opposing movements)
Extension of the LWR model to networks

- **Links**: supply/demand boundary conditions for total flow
- **Composition** (assignment) coefficients are carried by traffic flow
- **Node models** connect the demand of their upstream nodes to the supplies of their downstream nodes
- Node models must satisfy the *invariance principle*
Multicommodity flow

- Flow is **disaggregated** per “commodity” (destination, path, driver category...) \( d \)

\[
K(x,t) = \sum_{d=1}^{D} K^d(x,t) \quad \forall x, t
\]

\[
Q(x,t) = \sum_{d=1}^{D} Q^d(x,t) \quad \forall x, t
\]

- **Conservation per commodity** (attribute)

\[
\frac{\partial K^d}{\partial t} + \frac{\partial Q^d}{\partial x} = 0 \quad \forall d = 1...D
\]

\[
Q^d = K^d V \quad \text{and} \quad K^d = \chi^d K, Q^d = \chi^d Q \quad \forall d
\]
Multicommodity flow: FIFO model

- **Principle**: all vehicles have the same speed, whichever their attribute.
- Composition stays constant along trajectories.

\[
\frac{\partial \chi^d}{\partial t} + V_e(K) \frac{\partial \chi^d}{\partial x} = 0 \quad \forall d = 1 \ldots D
\]
Network boundary data

- Supply, demand for the total flow
- Upstream composition for network inflow

\[
\begin{align*}
\chi_u^d (t) & \quad \Delta_u(t) \\
\sum^d (a,t) & \quad Q^d (a,t) \\
\end{align*}
\]

\[
\chi^d (a,t) = \chi_u^d (t) \\
Q(a,t) = \text{Min} [\Delta_u(t), \Sigma(a,t)] \\
Q^d (a,t) = \chi^d (a,t)Q(a,t)
\]
Multicommodity flow: intersections

- No change, except...
- Turning movements result from assignment coefficients (behavioral: VMS, user choice...)

$$\gamma_{ij}(t) = \sum_d \beta_{ij}^d(t) \chi^d(n, t; i)$$
Example: Godunov discretization scheme. Shockwave
Example: speed control does work

- The facts
No Control vs Speed Control

- Bounded acceleration node model for the exit of motorway
Data reconstruction works too

- **Problem:** reconstruct loop data (usually 10% out of order at any time)
- **Solution:** LWR model, fed by loop data
Data reconstruction

- Reconstructed vs actual loop data
- Volume (flow), Occupancy (density), Speed
- Haj-Salem, Lebacque 2002
Ramp metering

- **Principle**: limit access to motorway ⇔ limit conflicts and reacceleration

- Nominal capacity reduction ⇔ effective capacity increase (Braess-like paradox)

- Uses bounded acceleration node + ALINEA (linear feedback)
Ramp metering (ALINEA)

\[ \sigma(t) = q(t-1) + c(O^* - O_{out}(t)) \]

\[ q(t) = \text{Min} \{ \sigma(t), \delta(t) \} \]
Ramp metering 2

- Two strategies:
  - Ramp metering
  - Ramp metering + Speed control (highway)
Ramp metering 3 (simulation results)

- Ramp metering alone
- Gains downstream of node
Ramp metering 4 (simulation results)

- Ramp metering + Speed control
- Greater gains downstream of node
Ramp metering 4:

- **Persistence of gains** with respect to the distance traveled
- **Cause**: fluidity, regularity, no hysteresis
Conclusion & next steps

- Systematic approach to intersection modelling
- Network models (FIFO)
- Satisfactory empirical evidence

Next steps:

- Multicommodity, non FIFO flows on networks
- New integration methods: Hamilton-Jacobi, cumulative flows
- More physical intersection models
- Development hybrid models (Micro +macroscopic modeling)
- Development of the MAESTRAU kernel