The ARZ model: Supply demand analysis of the inhomogeneous Riemann problem, boundary conditions

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INRETS-GRETIA

OUTLINE

1. 2nd order macroscopic models
2. The ARZ model description
3. Resolution in the homogeneous case
4. Resolution in the inhomogeneous case
5. Boundary conditions
6. Conclusions and next steps

Traffic flow modelling approaches

Two main types of modelling approaches:

1. Microscopic
2. Macroscopic:
   a. First order Modelling
   b. Second order Modelling

Macroscopic traffic description

- Continuum hypothesis: traffic state can be described by functions of location x and time t
- Variables:
  - Density $\rho(x,t)$
  - Flow $q(x,t)$
  - Velocity $v(x,t)$

Macroscopic approaches: Basic equations

1- Vehicle conservation Equation:

$$\partial_t \rho(x,t) + \partial_x q(x,t) = 0$$

2- Volume-Density-speed relationship:

$$q(x,t) = \rho(x,t)\, v(x,t)$$

3- Fundamental Diagram (equilibrium)

$$v(x,t) = Q_e(\rho(x,t))$$ where V_e monotone decreasing function

First order (1–3)

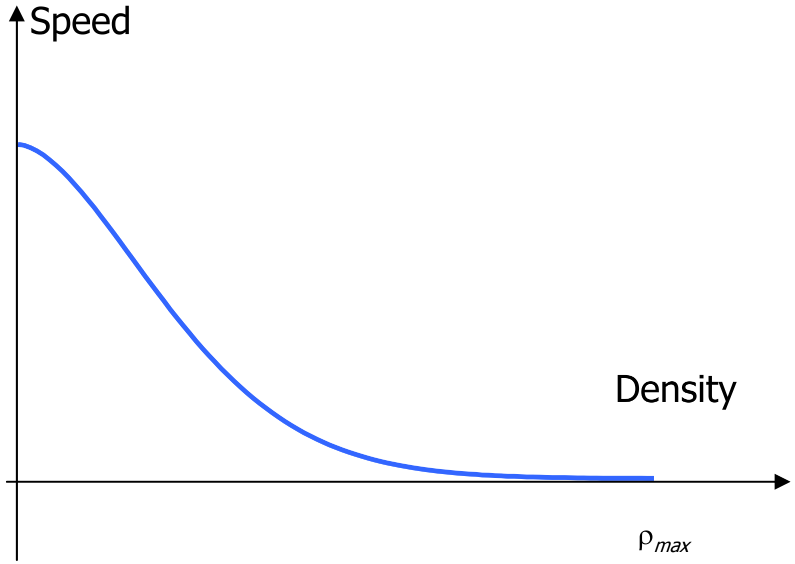
4- *Momentum equation*

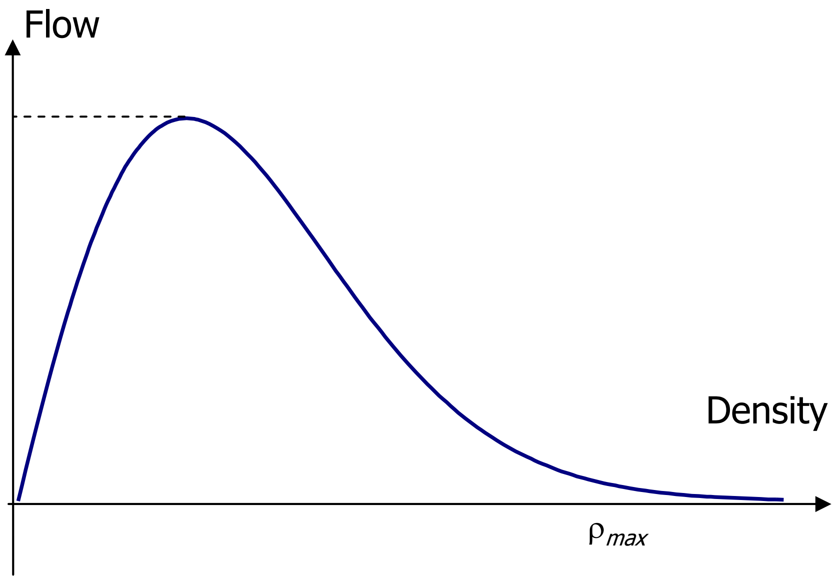
$$dv(x,t)/dt = \partial_t v(x,t) + v(x,t)\partial_x v(x,t) = G(\rho(x,t), v(x,t))$$

Second order (1–4)

Fundamental diagram, equilibrium

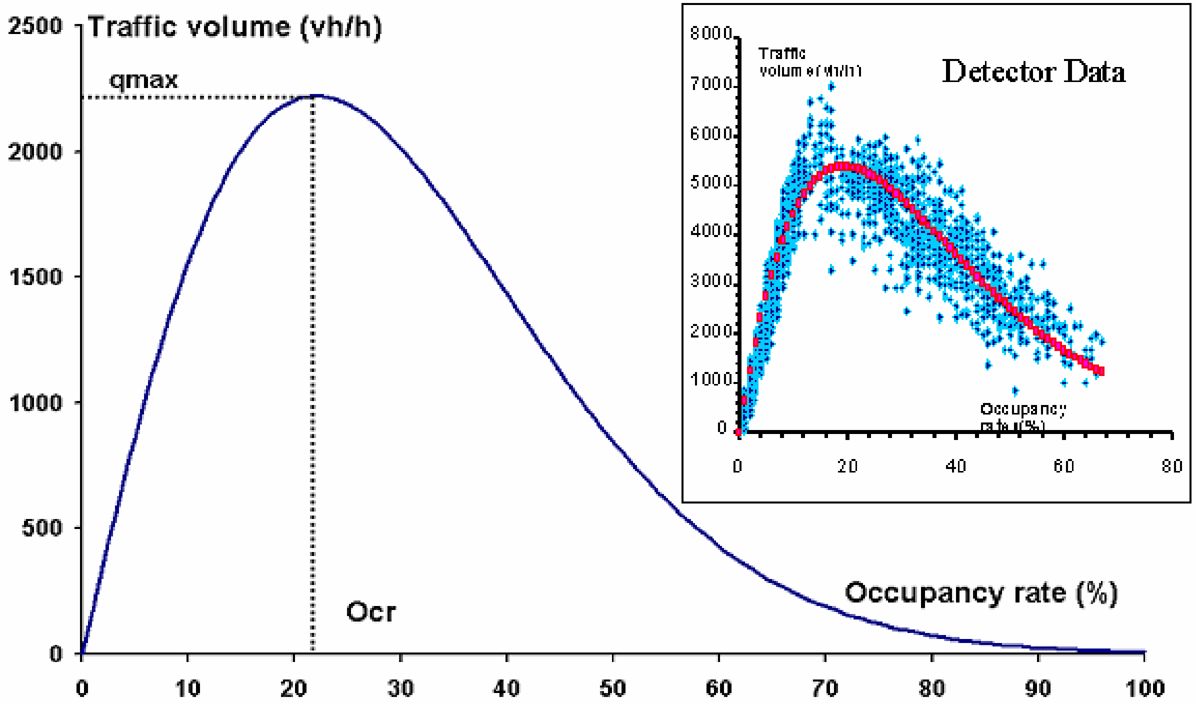
- Example of fundamental diagram





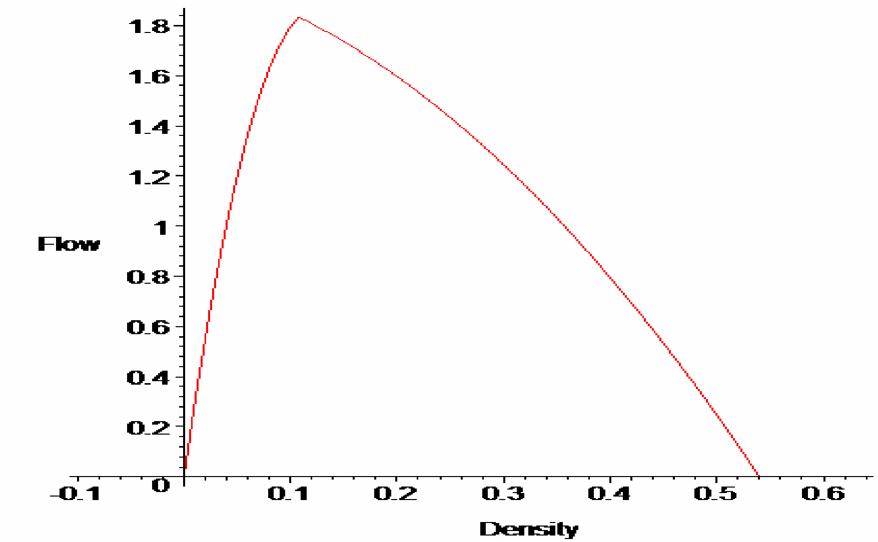
1st vs 2nd order models

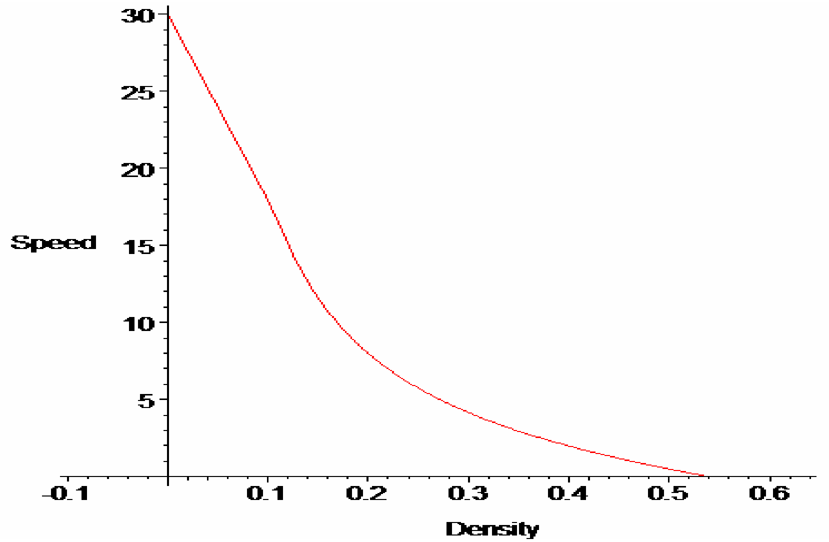
- 2nd order: out of equilibrium (ρ,*v* points are not on the fundamental diagram)
- *I* (relative speed) *y* (relative flow) both measure the "distance to equilibrium"



Another example of FD

- Cf METACOR, STRADA





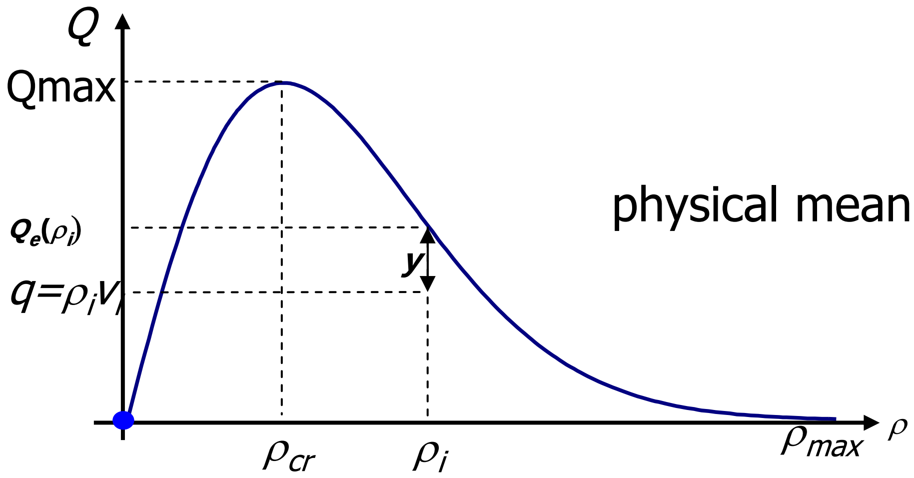
The ARZ model: conservation of density and relative flow

$$\partial_t U + \partial_x f(U) = 0$$ with $U = \begin{pmatrix} \rho \\ y \end{pmatrix}$ and $f(U) = \begin{pmatrix} \rho v \\ p \end{pmatrix}$

The relative speed $I \overset{def}{=} v - V_e(\rho)$

The relative flow $y = \rho I = q - Q_e(\rho)$

The flux of relative flow (relative pressure) $p = \rho v (v - V_e(\rho)) = yv$



physical meaning of y

Eigenvalues: interpretation

- Special waves:
  - Small perturbations of the traffic flow
  - Self-similar solutions

$$\partial_t U + \partial_x f(U) = 0$$

- The velocity of these special waves is equal to the eigenvalues of

$$A(U) \overset{def}{=} \nabla f(U)$$

ARZ system eigenvalues

- The eigenvalues of $A(U) \overset{def}{=} \nabla f(U)$ are:

$$\lambda_1(U) = \frac{y}{\rho} + Q_e'(\rho) = v + \rho V_e'(\rho) < v$$

$$\lambda_2(U) = \frac{y}{\rho} + V_e'(\rho) = v$$

- eigenvectors

$$r^1(U) = \begin{pmatrix} -\rho \\ -y \end{pmatrix} \text{ and } r^2(U) = \begin{pmatrix} \rho \\ y - \rho^2 V_e'(\rho) \end{pmatrix}$$

Associated waves to the eigenvalues

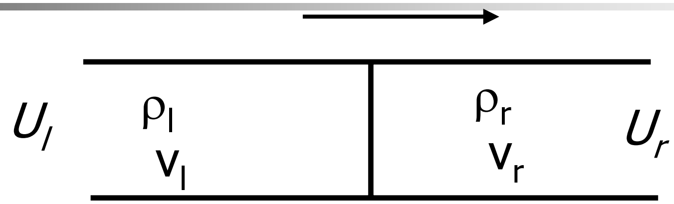
- $\lambda_1(U)$ is truly nonlinear: $$\nabla\lambda_1(U).r^1(U)\neq 0$$

  The associated waves correspond to rarefaction and/or shock waves
  Classical (LWR) dynamics: acceleration /deceleration waves, I conserved

- $\lambda_2(U)$ is linearly degenerate: $$\nabla\lambda_2(U).r^2(U)=0$$

  The associated waves correspond to contact discontinuities.
  speed is conserved, propagation of I discontinuities

Elementary waves



U_l is connected to downstream state U_r through a 1-Wave if and only if:

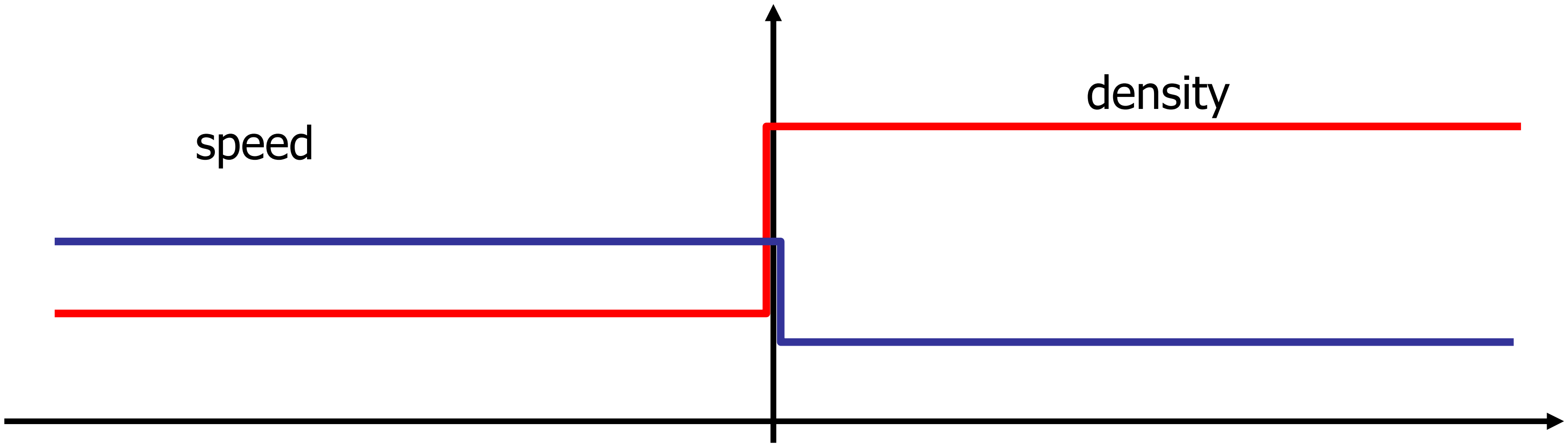
$$\frac{y_l}{\rho_l} - \frac{y_r}{\rho_r} = 0 \Leftrightarrow I_l = I_r$$

- if $\rho_l > \rho_r$, (and $v_l < v_r$) : rarefaction wave (acceleration wave)
- if $\rho_l < \rho_r$, (and $v_l > v_r$) : shock wave (deceleration wave).
- The upstream state is linked to a downstream state through a contact discontinuity if and only if:

$$\frac{y_l + Q_e(\rho_l)}{\rho_l} - \frac{y_r + Q_e(\rho_r)}{\rho_r} = 0 \quad \Leftrightarrow \quad v_l - v_r = 0$$

Riemann problem

- It is an archetype for many practical situations



- It is the key for developing numerical methods

Riemann Problem resolution

- Includes: Shock, rarefaction waves and contact discontinuities which connect:

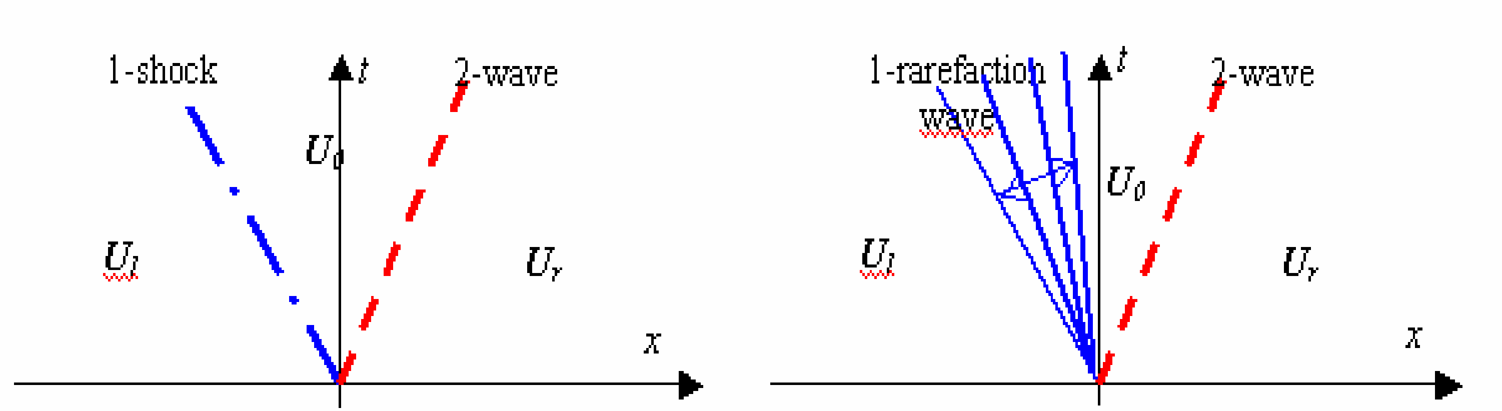
  The upstream state (U_l) to downstream state (U_r) through an auto similar solution
- The general solution associated to the eigenvalues:
  - 1-wave connecting the upstream state U_l to an intermediate state U_0 *(to be determined)*
  - 2-wave connecting the intermediate state U_0 to the downstream state U_r.
- The general solution is self-similar

Analytical Riemann problem solutions

1-waves can be either shock or rarefaction waves -> two main types of solutions:

Type 1: 1-shock connecting U_l to the intermediate state U_0, followed by a 2-contact discontinuity wave connecting U_0 to U_r

Type 2: 1-rarefaction connecting U_l to U_0 followed by a 2- contact discontinuity wave connecting U_0 to U_r .

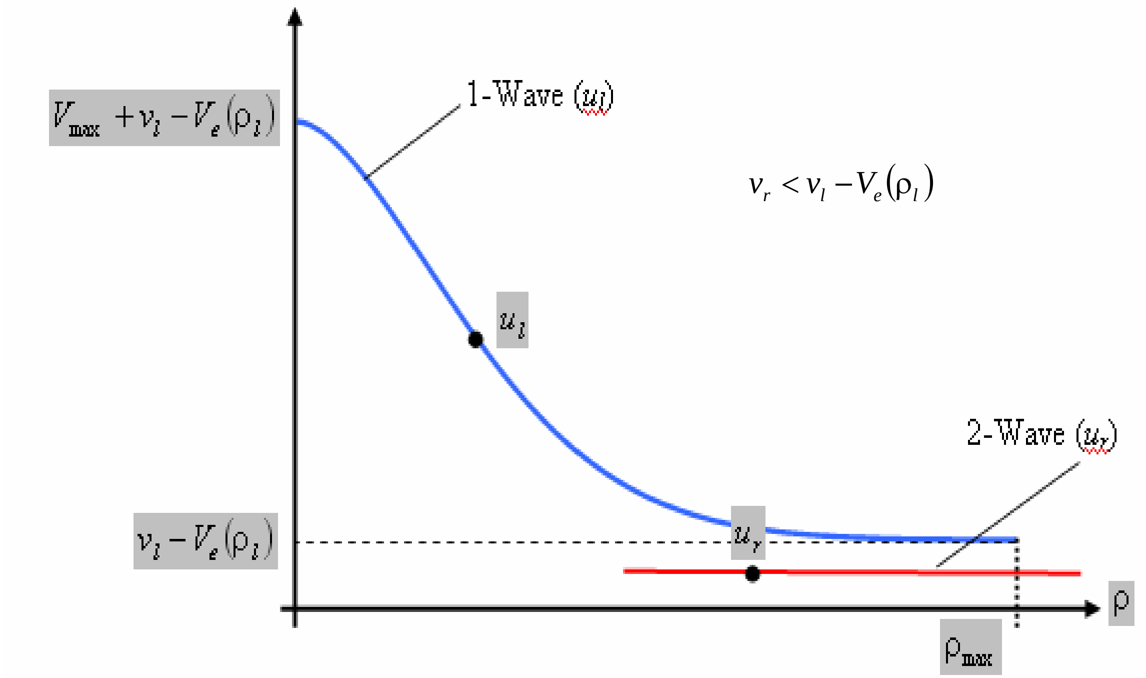


How to solve the R. Problem ? Extensions of the FD

In some cases, the original ARZ model does not admit any solution to the Riemann problem. Other pathological behaviour: non-physical solutions

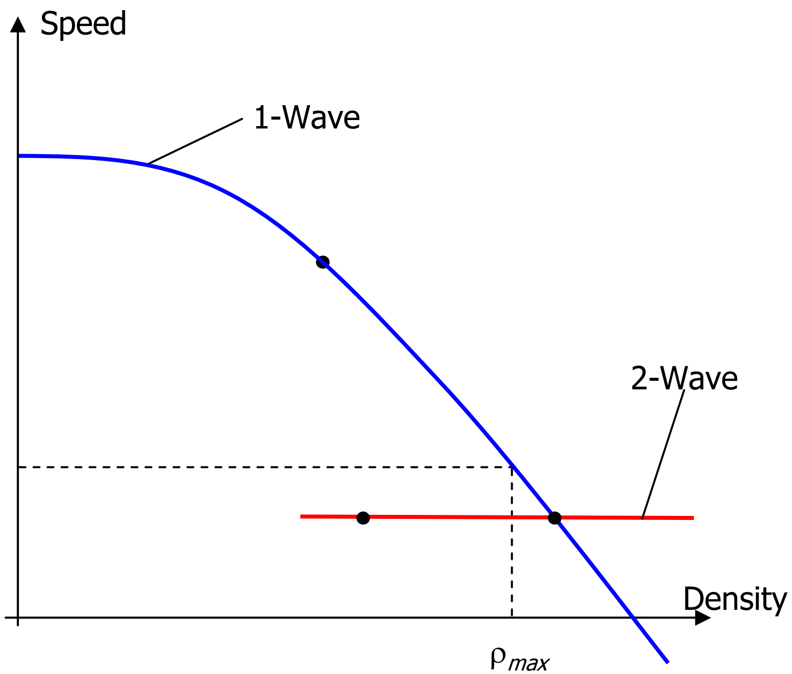
If $v_r < v_l - V_e(\rho)$, the Riemann problem does not admit a solution

Case of no solution to the Riemann problem.



Example of non-physical solutions: density higher than jam density

- Fundamental diagram: Rascle

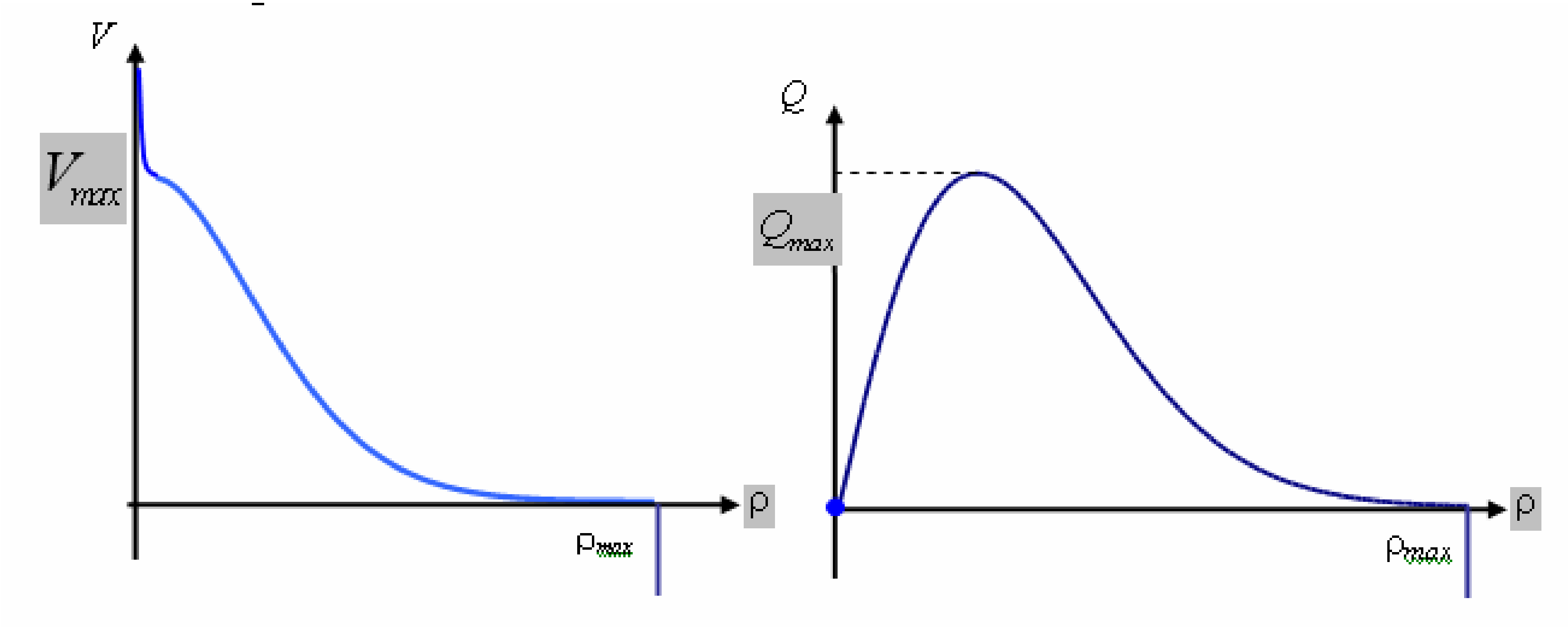


$$V_e(\rho) = V_{max}\left(1 - \left(\frac{\rho}{\rho_{max}}\right)^{\gamma}\right)$$

Extension of the fundamental diagram

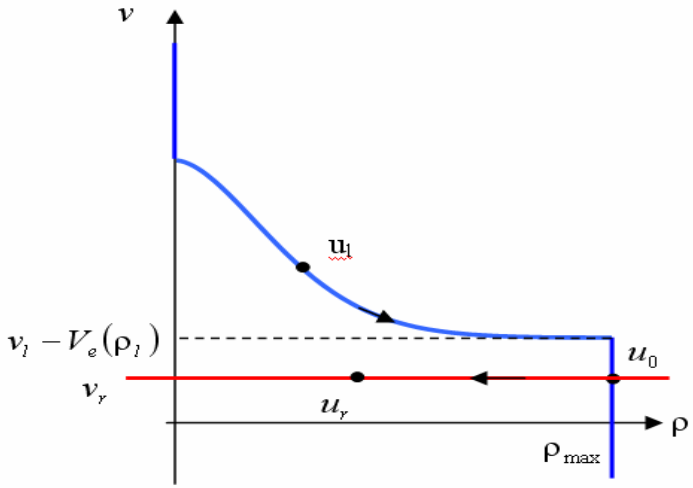
- The extension

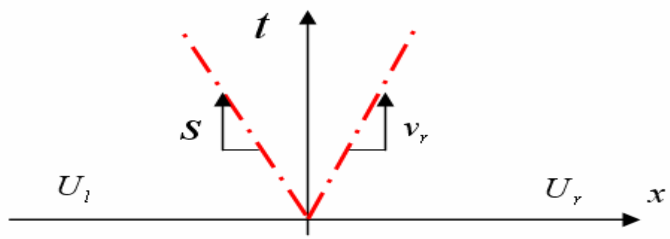
$$Q_e(\rho)=0 \quad \text{if} \quad \rho=0$$
$$Q_e(\rho)=]-\infty,0] \quad \text{if} \quad \rho=\rho_{max}$$



Example of analytical solution of the R. problem: Homogeneous case

Initial condition: $v_r < v_l$



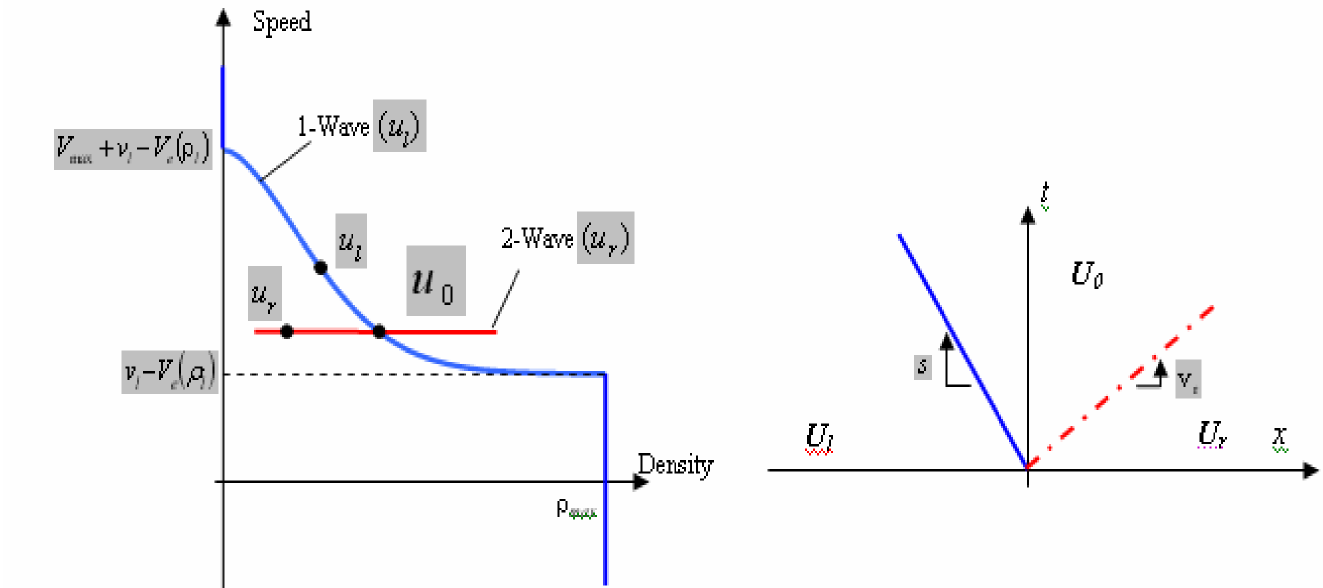


The Riemann problem solutions include:

- Shock wave ($U_l \rightarrow U_0$)
- Rarefaction wave ($U_0 \rightarrow U_r$)

Example of analytical solution of the R. problem: Homogeneous case (2)

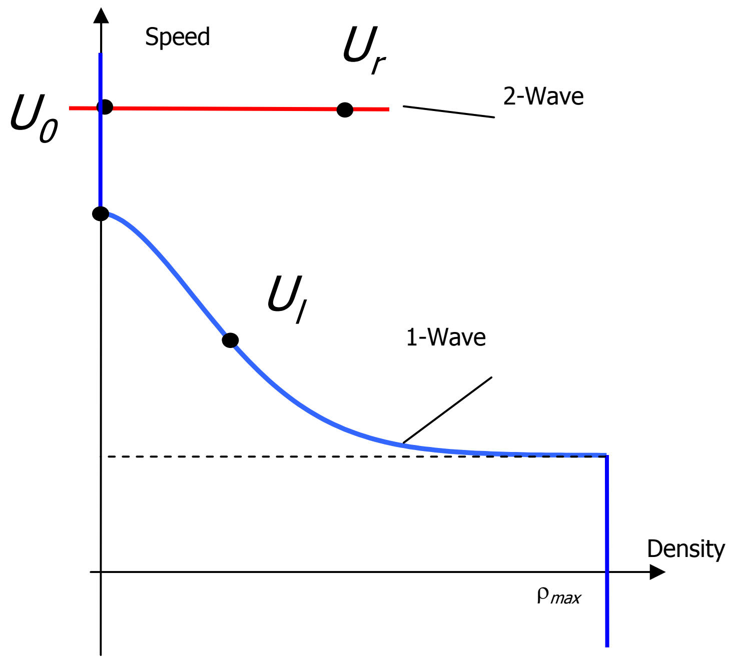
Analytical solution to the Riemann problem with ($v_r < v_l$) as initial condition

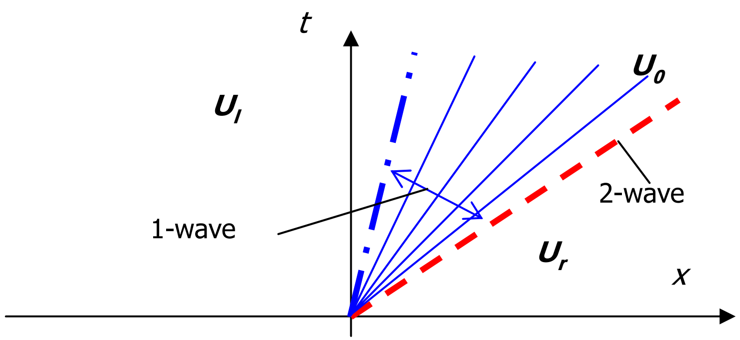


The solution of the Riemann problem includes: a shock wave which links the upstream state U_l to an intermediate state U_0. The latter state is connected to the downstream state U_r by a contact discontinuity.

Example of analytical solution of the R. problem: Homogeneous case (3)

- Example: very fast downstream traffic

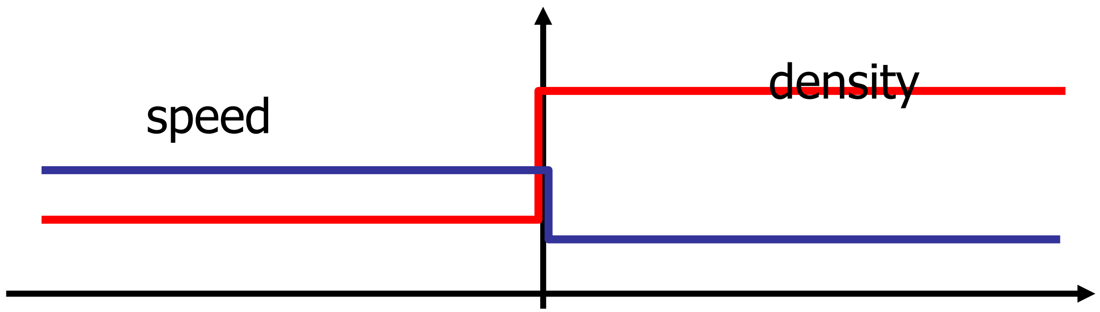




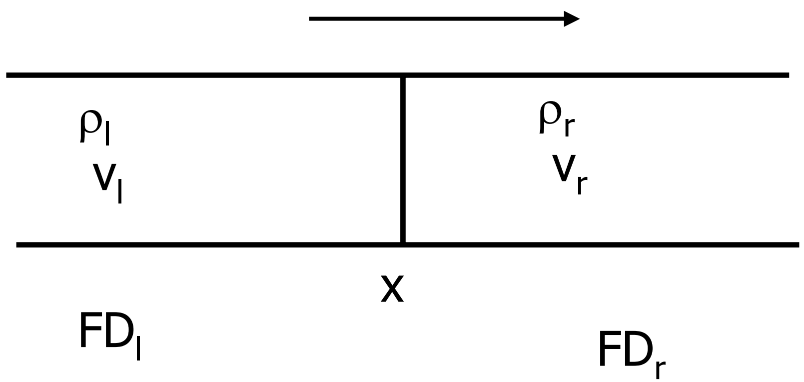
Inhomogeneous Riemann problem: what is it?

- Initial conditions:

$$U(x,0)=\begin{cases}U_l & if\ \ x<0\\ U_r & if\ \ x>0\end{cases}$$



- FD_l # FD_r



Inhomogeneous Riemann problem: why solve it?

- It is an archetype for:
  - Boundary conditions (network entry/exit points)
  - Intersections
  - Variable motorway sections (Number of lanes, speed limits…)
- It is the key to
  - Network modelling
  - Fast numerical solutions

Riemann Problem solutions

- The classical approach: Shock, rarefaction waves and contact discontinuities which connect:

  The upstream state (U_l) to downstream state (U_r) through an auto similar solution

- Difficulty: The general solution requires up to **3 waves**, not 2 as in the homogeneous case:
  - Two 1-wave connecting the upstream state U_l to an intermediate state U_0 *(to be determined)*
  - A 2-wave connecting the intermediate state U_0 to the downstream state U_r.

Riemann Problem solutions (2)

- Intuitive explanation: the discontinuity of the FD adds a standing wave at the origin
- Consequence: tractable but too many possible cases
- Necessity of a more concise and explanatory approach

The key to the solution: Dynamics of the relative speed *I*

- definition *of I* : $I \overset{def}{=} v - V_e(\rho)$

- *I* is constant along vehicle trajectories (lines).

$$\partial_t v + \left(v - \rho V_e^{'}(\rho)\right)\partial_x v = -\frac{A}{T}\left(v - V_e(\rho)\right) \overset{if\ A=0}{\Longrightarrow}$$

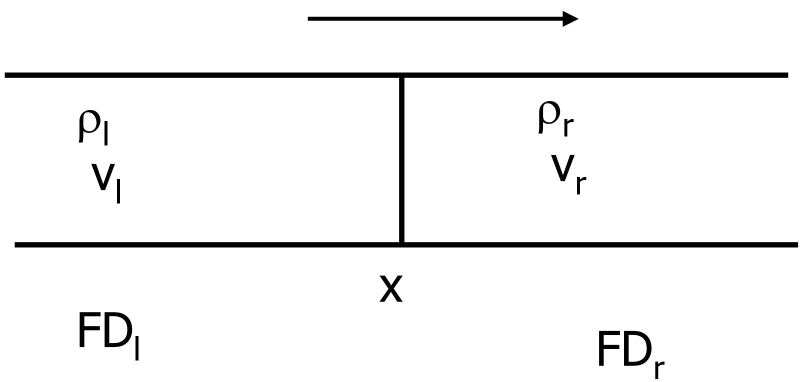
$$\dot{I} = \partial_t I + v\partial_x I = 0$$

- *I* is conserved through 1-waves

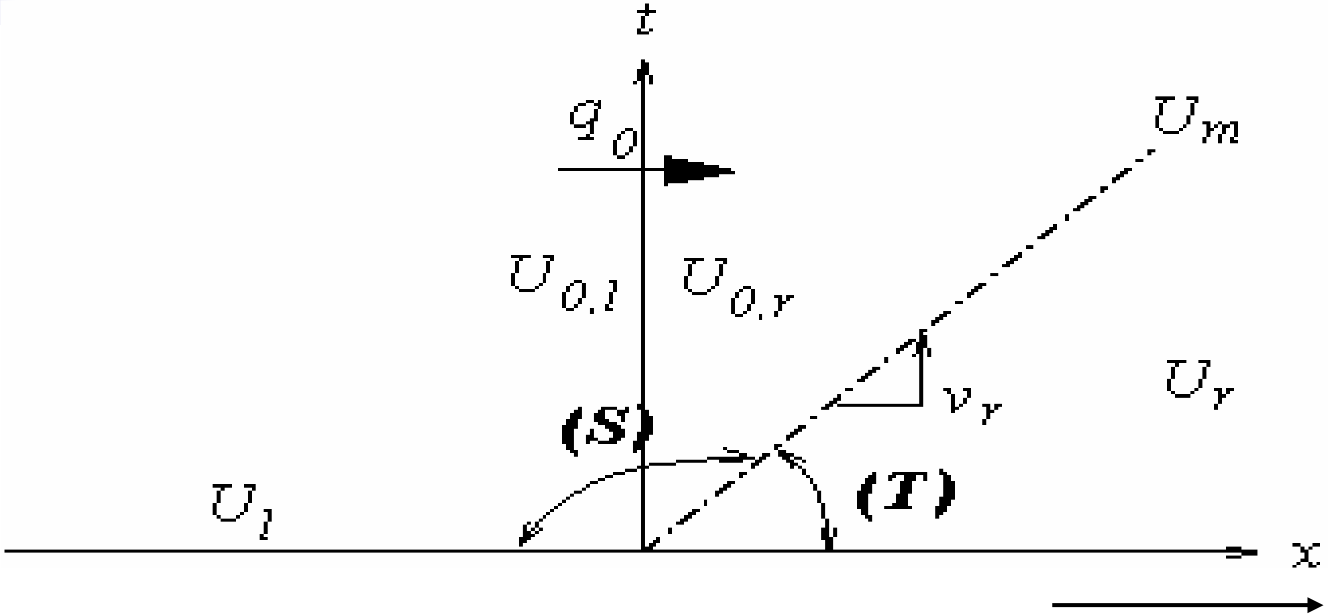
- *I* is conserved through stationary discontinuities (Rankine Hugoniot Conditions)

Conservation of I through stationary discontinuities

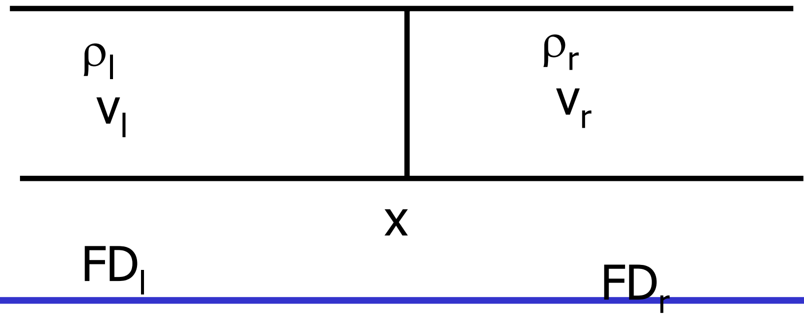
- Rankine-Hugoniot (conservation of traffic flow):
  - Flow q is conserved
  - Relative pressure $p = q\,I$ is conserved
  - $\Rightarrow I$ is conserved



Consequence: I_l is "carried over" on the right side



- $I = I_l$ in all of sector (***S***)
- $I = I_r$ in all of sector (***T***)



In sector (*S*), the ARZ model is equivalent to a LWR model with shifted Fundamental Diagram

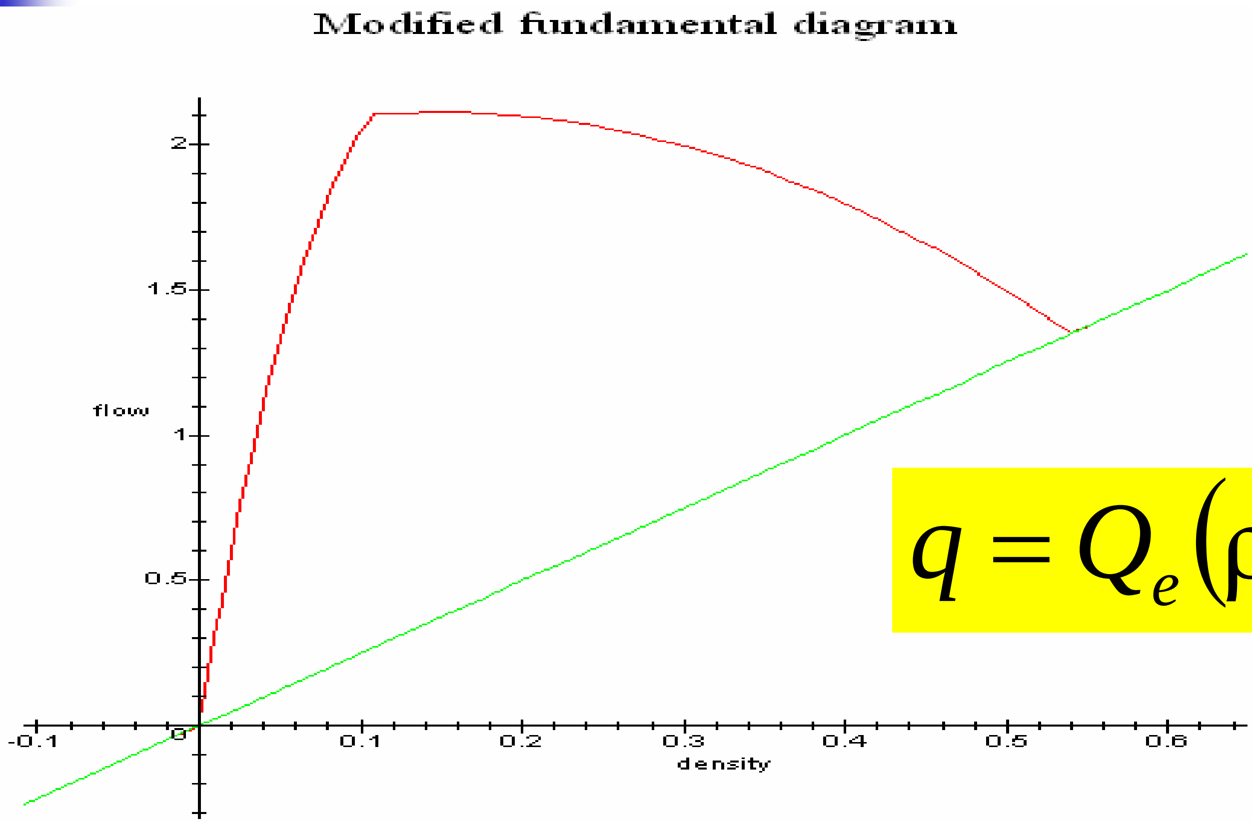
- In sector (*S*): $I = I_l \quad \Rightarrow$

$$v = V_e(\rho, x) + I_l$$
$$q = Q_e(\rho, x) + \rho I_l$$

- This is a shifted fundamental diagram (relationship between speed/flow and density)
- FD + conservation of vehicles ⇔ LWR model

Shifted LWR:

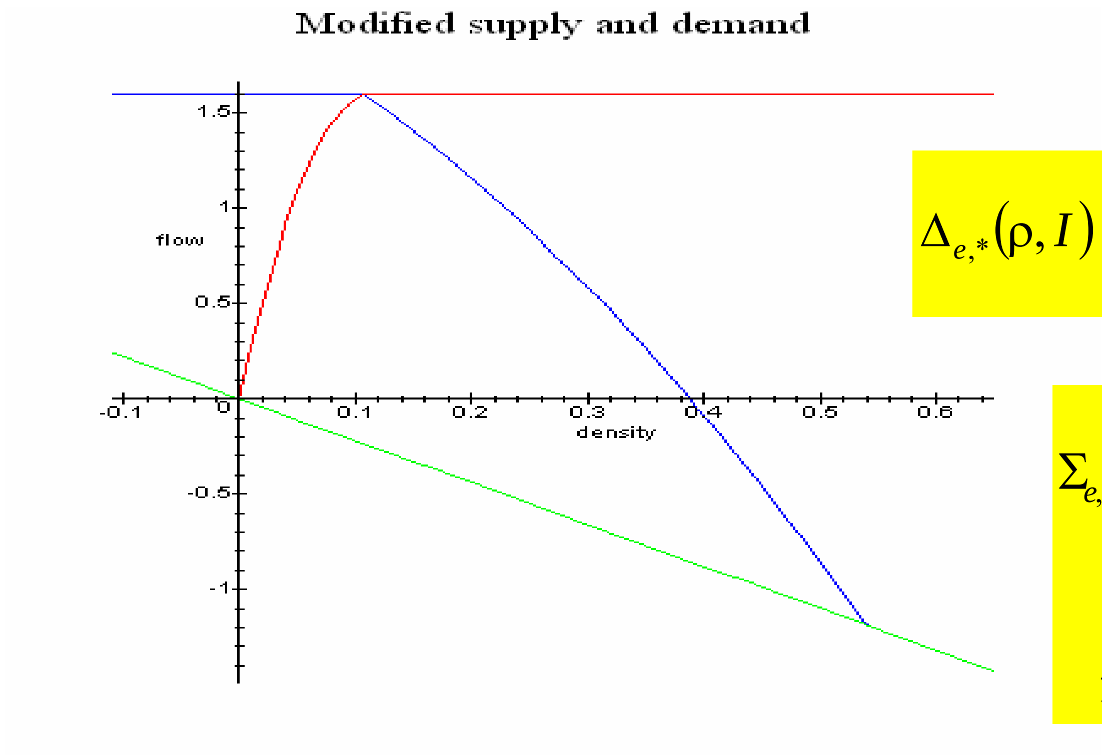
$$\partial_t \rho + \partial_x q = 0 \quad \text{and} \quad q = Q_e(\rho, x) + \rho I_l$$
$$\Rightarrow$$
$$\partial_t \rho + \partial_x \left(Q_e(\rho, x) + \rho I_l \right) = 0$$



$$q = Q_e(\rho, x) + \rho I_l$$

Solution to the Riemann problem: classical LWR supply/demand analysis

- Supply and demand for the modified FD



$$\Delta_{e,*}(\rho, I) = \begin{cases} Q_{e,*}(\rho, I) + \rho I & \text{if} \quad \rho \le \rho_{crit,*}(I) \\ q_{max,*}(I) & \text{if} \quad \rho \ge \rho_{crit,*}(I) \end{cases}$$

$$\Sigma_{e,*}(\rho, I) = \begin{cases} q_{max,*}(I) & \text{if} \quad \rho \le \rho_{crit,*}(I) \\ Q_{e,*}(\rho, I) + \rho I & \text{if} \quad \rho \ge \rho_{crit,*}(I) \end{cases}$$

$\text{for } * = l, r$

Riemann problem solution

- Define upstream demand and downstream supply (both depend on I_l):

$$\Delta_l \overset{def}{=} \Delta_{e,l}(\rho_l, I_l), \quad \Sigma_r \overset{def}{=} \Sigma_{e,r}(\rho_m, I_l)$$

- Through flow q_0 = Min [upstream demand, downstream supply]

$$q_0 = \mathrm{Min}[\Delta_l, \Sigma_r]$$

- Through relative pressure $p_0 = q_0 \times I_l$

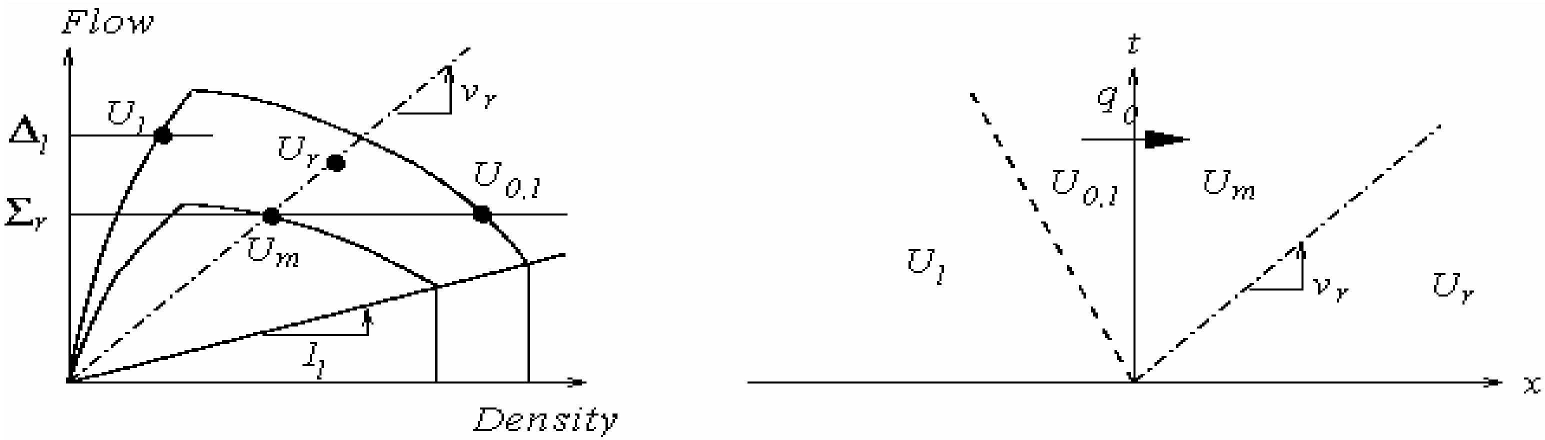
- The state U_m is given by

$$v_m = v_r$$
$$I_m = I_l \quad i.e. \quad q_m = Q_e(\rho_m) + I_l \rho_m$$

Example: Analytical solution of an inhomogeneous Riemann problem

- Case: insufficient downstream supply
- Steps:
  - Construct the shifted FD s
  - Construct U_m:
  - Construct U_{0l} (defines the supply on the l.h.s
  - Construct the shock $U_l \rightarrow U_{0l}$

$$v_m = v_r$$
$$I_m = I_l$$
$$i.e. \quad q_m = Q_e(\rho_m) + I_l \rho_m$$



Example: Upstream boundary conditions for a link

$\Delta_u(t), I_u(t)$

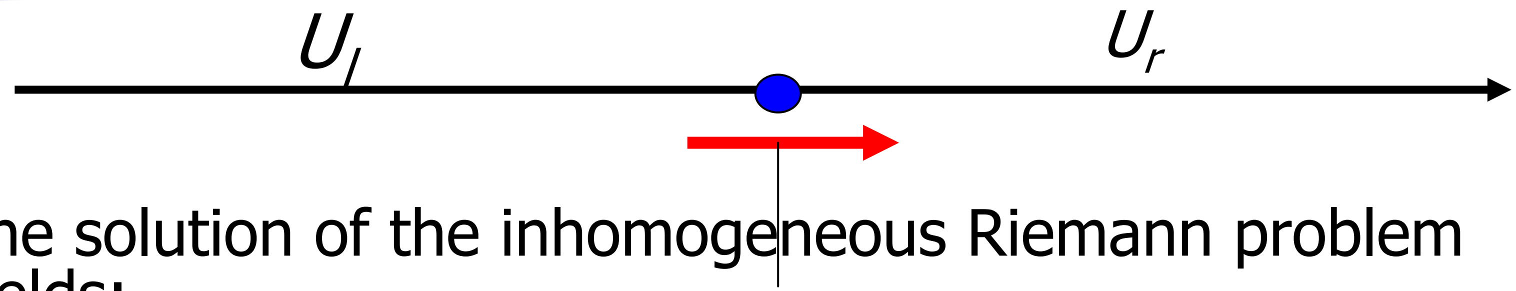
- Upstream boundary data: demand Δ_u , relative speed I_u
- Upstream supply $\Sigma(a, I_u)$

$$\Sigma(a, I_u) \stackrel{def}{=} \Sigma_e(\rho(a,t), I_u)$$

- Link inflow
- Link relative flow inflow

$$q(a,t) = \mathrm{Min}[\Sigma(\mathrm{a}, \mathrm{I_u}), \Delta_u(t)]$$
$$p(a,t) = q(a,t)\,\mathrm{I_u}$$

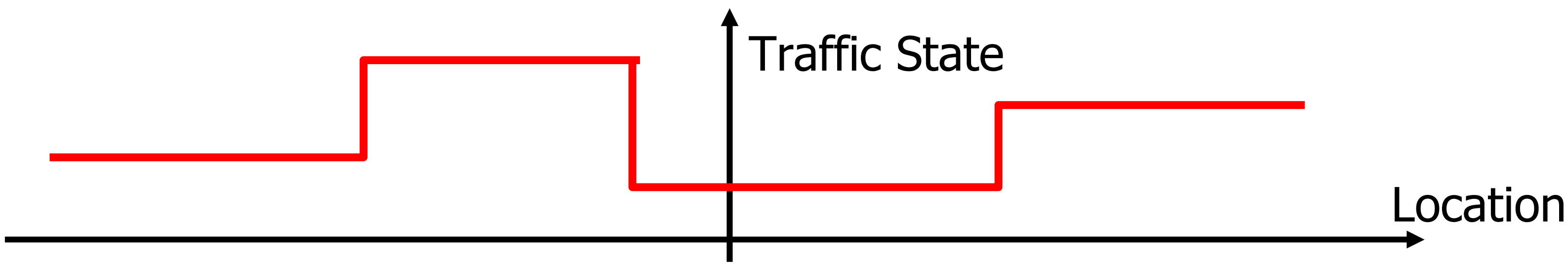
Example: Godunov scheme for the ARZ model



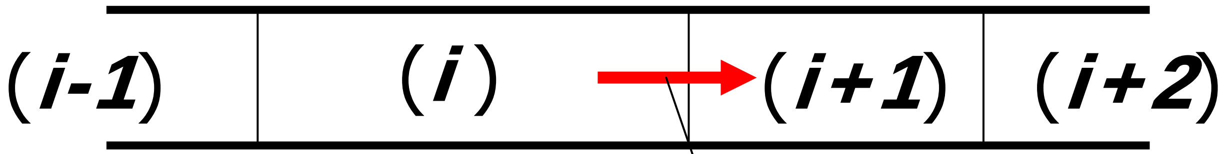
- The solution of the inhomogeneous Riemann problem yields:

$$\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} = F_{l,r}(U_l, U_r)$$

- The Godunov scheme relies on a piecewise constant approximation of the traffic state



Example: Godunov scheme for the ARZ model (2)



- At each interface point, we solve a local Riemann problem

$$\begin{pmatrix} q_i^t \\ p_i^t \end{pmatrix} = F_{i,i+1}\left(U_i^t, U_{i+1}^t\right)$$

- Then conservation of density and relative flow is applied

$$\rho_i^t = \frac{\Delta t}{\Delta x_i}\left(q_{i-1}^t - q_i^t\right)$$

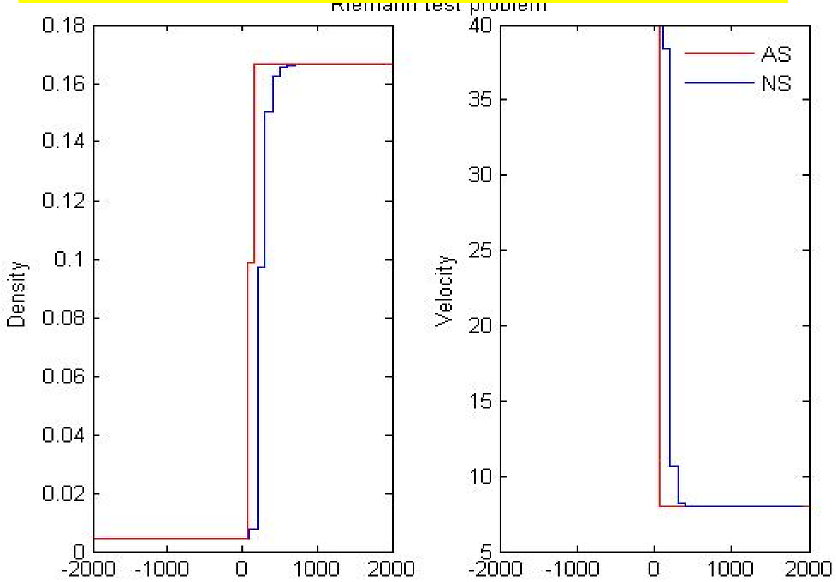
$$y_i^t = \frac{\Delta t}{\Delta x_i}\left(p_{i-1}^t - p_i^t\right)$$

Simulation results :

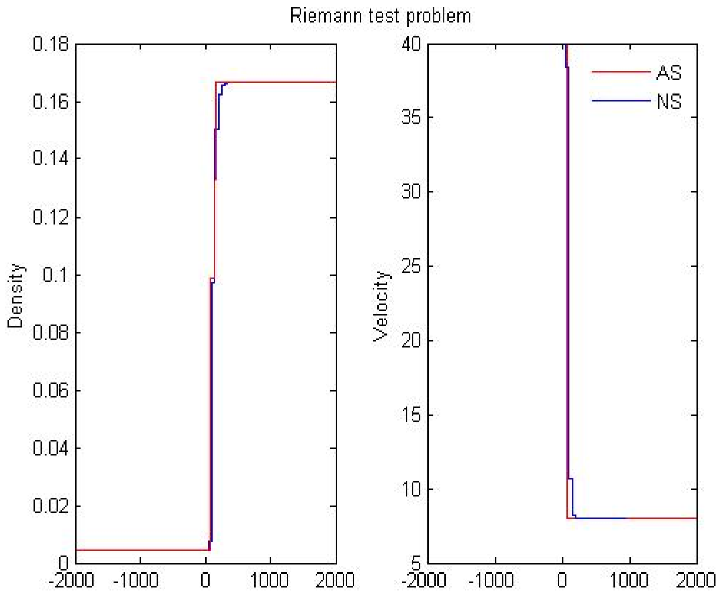
Validation of the numerical scheme (GODUNOV)

The scheme is tested against known analytical solutions ⇒ validation

$\Delta x = 100$ meters *and* $\Delta t = 2$ second

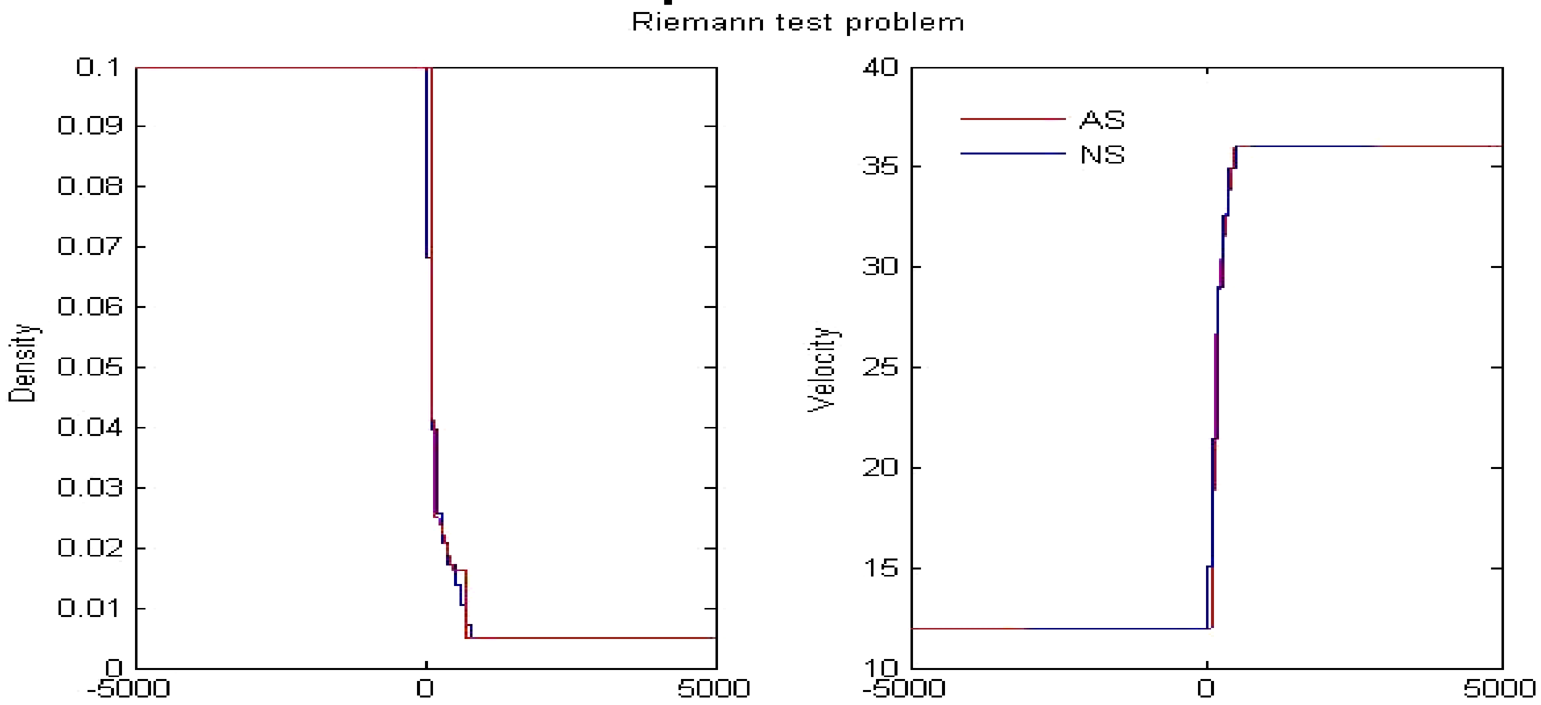


$\Delta x = 50$ meters *and* $\Delta t = 1$ second



Simulation results (2)

- Some more examples



Simulation results (3)

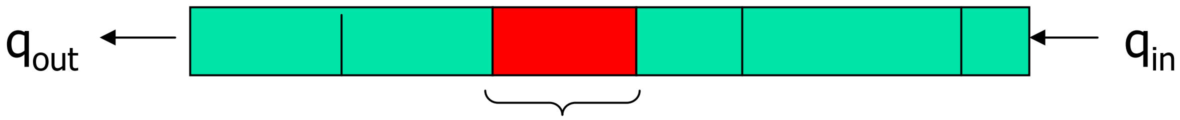
Convergence of the second order model to the LWR

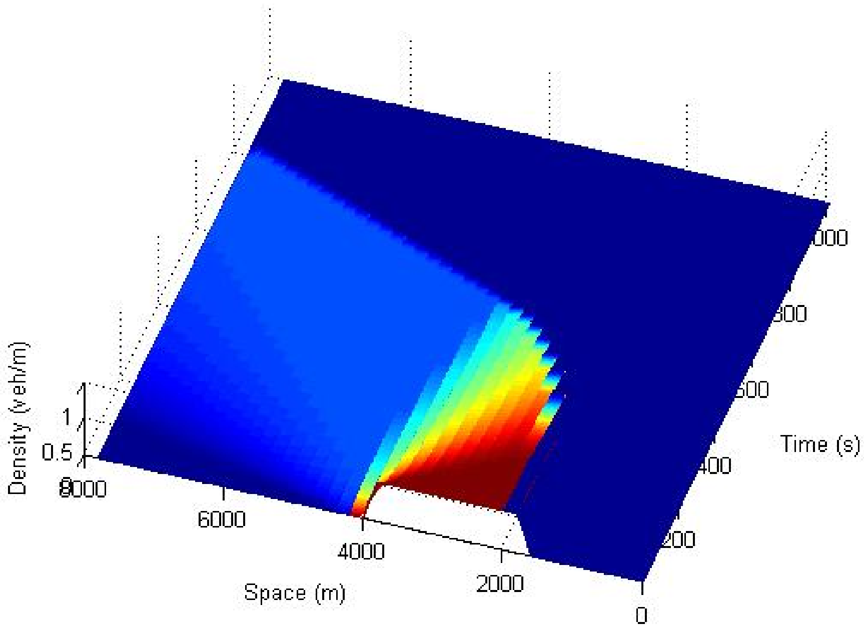
$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0 \\ \partial_t y + \partial_x p = 0 \end{cases} \overset{v=V_e(\rho)}{\Leftrightarrow} \quad \partial_t \rho + \partial_x (\rho V_e(\rho)) = \partial_t \rho + \partial_x Q_e(\rho) = 0$$

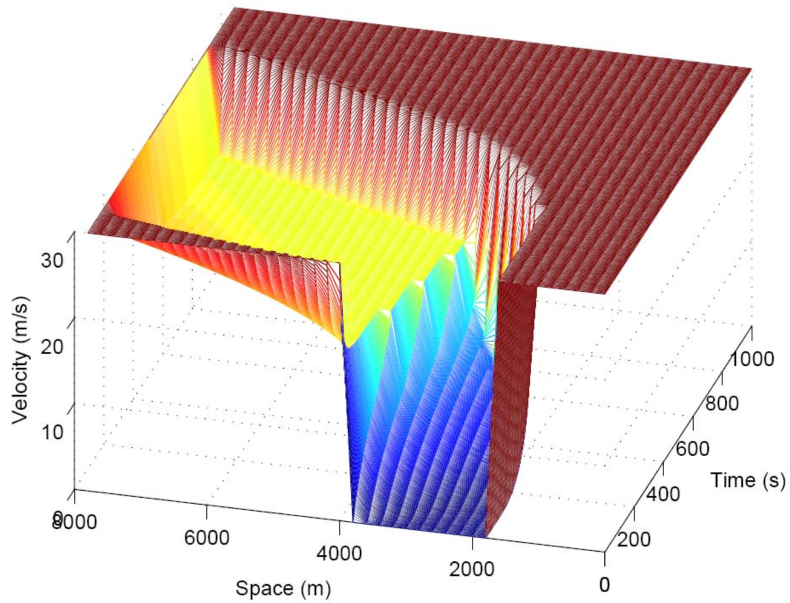
The ARZ model is consistent with LWR model. (The LWR is embedded in the ARZ model)

This property is easily demonstrated, by considering an equilibrium state as initial condition: $v = V_e(\rho)$. The second order model is reduced to the conservation equation at equilibrium (the LWR model):

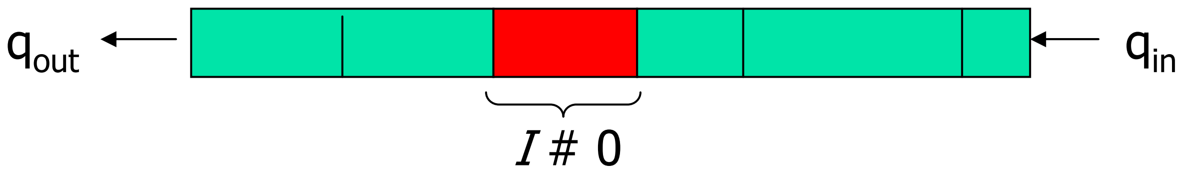
Simulation results (4) in Homogeneous case

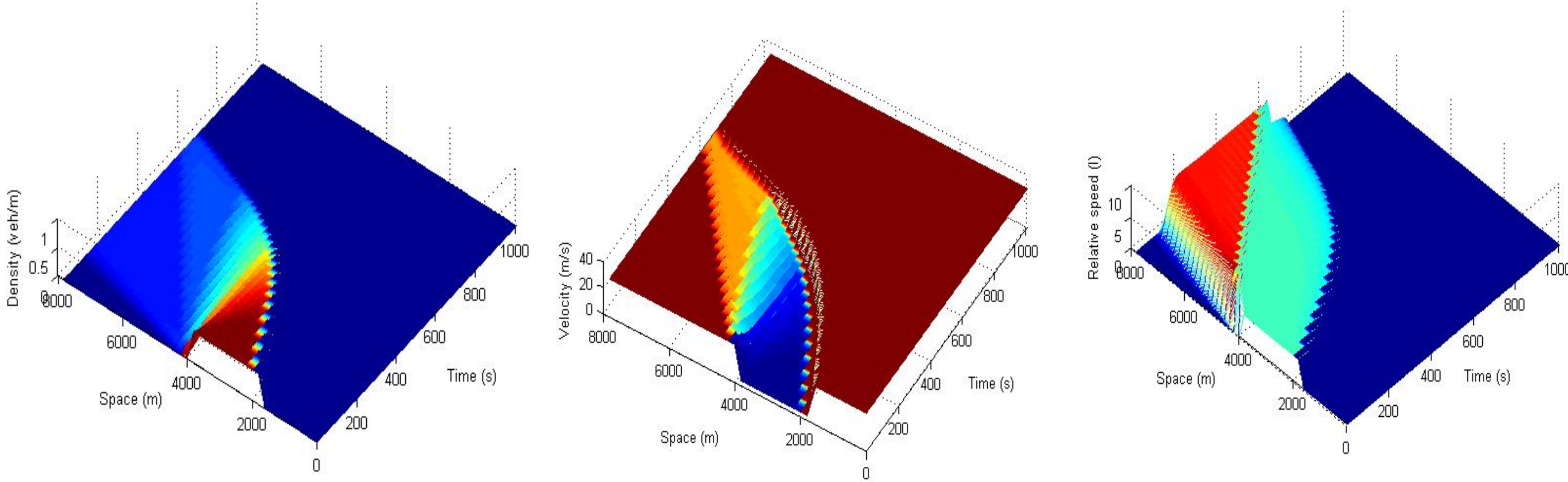






Simulation results (5) in Inhomogeneous case





Conclusion & next steps

- Consistent boundary conditions
- Godunov scheme
- General solution principle (based on LWR with modified FD)

Next steps:

- Development of the Intersection modeling
- Development of network models
- Hybridization (micro-macro)
- Integration of the model in MAESTRAU kernel