A traffic-like model for supply chains

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Summary

1. Introduction

A supply chain

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  - A supply chain is a string (or network) of processors (or stations) along which goods (or parts) are circulating

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  - A supply chain is a string (or network) of processors (or stations) along which goods (or parts) are circulating
- At each processor, the goods undergo a transformation.
- Each processor requires a certain throughput time T to process a given good
- Each processor has a limited capacity q (maximum number of goods it is able to deliver per unit of time)

A supply chain

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A supply chain

- In front of each processor, goods can be stored in buffer queues while waiting for being treated
- Goods are picked up in the queues according to a given policy.
  - The simplest policy: FIFO (first in first out)
  - More complex policies (e.g. tagged 'hot lots' to be processed with higher priority)

Discrete Event Simulators (DES)

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- The simplest model to describe a supply chain
- Provides a recursion formula for $\tau(m, n) =$ time at which the part P_n enters the buffer queue of Station S_m
- Discrete analog of a particle model in gas dynamics

Hierarchy of models

- Like in fluid dynamics, one can derive
  - fluid models
  - kinetic models

  for supply chains

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  for supply chains

- The goal of this talk
  - ’Rigorously’ derive a fluid model from a large particle limit of a DES model under FIFO policy
  - Refine this model into a kinetic model able to account for more complex policies (hot lots)

References

- DES simulation: [Banks, Carson II, Nelson]

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- Fluid models: [Anderson], [Billings, Hasenbein], [Newell]
  - Recently: [Klar et al]
- Review: [Daganzo]

2. A simple discrete event simulator

Data

- Station S_m
  - Capacity q_m (number of parts per unit of time)
  - Throughput time T_m (time needed to process a single part)

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- $\tau(m,n) =$ time at which the part P_n enters the buffer queue of Station S_m

- Buffer queues are of infinite size (can be relaxed)

Case distinction

- First case: buffer of S_m non-empty
  - S_m processes at full rate q_m

$$\Longrightarrow \tau(m+1,n) = \tau(m+1,n-1) + \frac{1}{q_m}$$

Case distinction

- First case: buffer of S_m non-empty
  - S_m processes at full rate q_m

$$\Longrightarrow \tau(m+1,n) = \tau(m+1,n-1) + \frac{1}{q_m}$$

- Second case: buffer of S_m empty
  - S_m processes part when it arrives

$$\Longrightarrow \tau(m+1,n) = \tau(m,n) + T_m$$

Recursion formula

- If buffer of S_m non-empty

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Recursion formula

- If buffer of S_m non-empty

$$\Longrightarrow \tau(m+1,n) \geq \tau(m,n) + T_m$$

- If buffer of S_m empty

$$\Longrightarrow \tau(m+1,n) \geq \tau(m+1,n-1) + \frac{1}{q_m}$$

- Collect the two cases into

$$\tau(m{+}1,n) \; = \; \max\{\, \tau(m{+}1,n{-}1) {+} \frac{1}{q_m} \,,\; \tau(m,n) {+} T_m \,\}$$

The continuum limit

- Investigate the 'thermodynamic limit' $M, N \to \infty$,
  - $M =$ Number of stations
  - $N =$ Number of parts

  and find a continuum model

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- Investigate the ’thermodynamic limit’
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  and find a continuum model

- Idea: find that the DES is a discrete version of a conservation law in Lagrangian variable
  - $n =$ part index $=$ ’mass’ variable
  - $m =$ station index $=$ ’space’ variable

3. Continuity equation

Position of part P_n

- 'Position' (or Station number m) at which part P_n is at time t given by

$$\mu(t,n) = \frac{1}{M} \sum_{m=1}^{M} H(t - \tau(m,n))$$

where $H(x) = 1$ for $x > 0$ and 0 otherwise (Heaviside fct)

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- ’Velocity’ of part P_n

$$v(t,n) = \frac{d}{dt}\mu(t,n) = \frac{1}{M}\sum_{m=1}^{M}\delta(t-\tau(m,n))$$

Velocity and specific volume

- 'Velocity' of part P_n

$$v(t,n) = \frac{d}{dt}\mu(t,n) = \frac{1}{M}\sum_{m=1}^{M}\delta(t-\tau(m,n))$$

- Spacing between the parts (= 'specific volume')

$$\theta(t,n) = -\frac{\mu(t,n+1)-\mu(t,n)}{1/N}\,, \quad \theta(t,N) = 0$$

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$$\theta(t,n) = -\frac{\mu(t,n+1)-\mu(t,n)}{1/N}\,, \qquad \theta(t,N) = 0$$

- By construction

$$\frac{d}{dt}\theta(t,n) + \frac{v(t,n+1)-v(t,n)}{1/N} = 0$$

- Define
  - $y =$ mass variable $\in [0,1]$ (part number)
  - $x =$ space variable $\in [0,1]$ (station number)

$$n = [Ny], \quad m = [Mx] \quad [\cdot] = \text{integer part}$$

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  - $x =$ space variable $\in [0,1]$ (station number)

$$n = [Ny]\,, \quad m = [Mx] \quad [\cdot] = \text{integer part}$$

- Assumption

$$\tau^{M,N}([Mx],[Ny]) \longrightarrow \tilde{\tau}(x,y)$$

as $M, N \to \infty$, as smoothly as we need

Position in the continuum limit

- We have

$$\mu^{M,N}(t, [Ny]) \longrightarrow X(t, y)$$

where $t \to X(t, y)$ is the inverse function of $x \to \tilde{\tau}(x, y)$ (which is increasing)

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- $X(t,y)$ is the position in Lagrangian variables

Velocity and Specific volume

- We have

$$v^{M,N}(t,[Ny]) = \frac{d}{dt}\mu^{M,N}(t,[Ny]) \to \frac{dX}{dt}(t,y) := v(t,y)$$

(velocity in Lagangian variables)

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(velocity in Lagangian variables)

- and

$$\begin{aligned} \theta^{M,N}(t,[Ny]) &= -\frac{\mu^{M,N}(t,[Ny]+1) - \mu^{M,N}(t,[Ny])}{1/N} \\ &\to -\frac{\partial X}{\partial y}(t,y) := \theta(t,y) \end{aligned}$$

(specific volume)

Continuity equation

- Since

$$\frac{\partial}{\partial y}\left(\frac{dX}{dt}\right) = \frac{d}{dt}\left(\frac{\partial X}{\partial y}\right)$$

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$$\frac{\partial}{\partial y}\left(\frac{dX}{dt}\right) = \frac{d}{dt}\left(\frac{\partial X}{\partial y}\right)$$

- The continuity equation is satisfied

$$\frac{\partial \theta}{\partial t} + \frac{\partial v}{\partial y} = 0$$

4. Constitutive relation

- was written

$$\tau([Mx]+1,[Ny]) \;=\; \max\{\,\tau([Mx]+1,[Ny]-1) \\ +\frac{1}{q([Mx])}\,,\; \tau([Mx],[Ny])+T([Mx])\,\}$$

Recursion formula (cont)

➠ After dividing by $1/N$ and some rearrangement

$$\max\{ \frac{\tau([Mx]+1,[Ny]-1)-\tau([Mx]+1,[Ny])}{1/N}+\frac{N}{q([Mx])}, \frac{N}{M}(\frac{\tau([Mx],[Ny])-\tau([Mx]+1,[Ny])}{1/M}+MT([Mx])) \}=0$$

Scaling hypotheses

- $N \to \infty$, $q^{M,N} \to \infty$ and

$$\frac{N}{q^{M,N}([Mx])} \to \frac{1}{\bar{q}(x)}$$

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$$MT^{M,N}([Mx]) \to \bar{T}q(x)$$

- $\dfrac{N}{M} \to 1$ Number of parts and stations are of the same order of magnitude

Recursion: continuum limit

- Under scaling assumptions, as $N, M \to \infty$:

$$\max\{-\frac{\partial\tilde{\tau}}{\partial y} + \frac{1}{\tilde{q}}, -\frac{\partial\tilde{\tau}}{\partial x} + \tilde{T}\} = 0$$

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- Using that $t \to X(t, y)$ is the inverse fct of $x \to \tilde{\tau}(x, y)$

$$\frac{\partial \tilde{\tau}}{\partial x}(X(t,y),y) = \frac{1}{v(t,y)}, \qquad \frac{\partial \tilde{\tau}}{\partial y}(X(t,y),y) = \frac{\theta(t,y)}{v(t,y)}$$

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- Gives

$$\max\{-\frac{\theta(t,y)}{v(t,y)} + \frac{1}{\tilde{q}(X(t,y))}, -\frac{1}{v(t,y)} + \tilde{T}(X(t,y))\} = 0$$

Resolution of the max

- Either max is attained for 1st argument and

$$v(t,y) = q(X(t,y))\theta(t,y) \leq \frac{1}{T(X(t,y))}$$

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$$v(t,y) = \frac{1}{T(X(t,y))} \leq q(X(t,y))\theta(t,y)$$

- Thus

$$v(t,y) = \min\{\frac{1}{T(X(t,y))}, \ q(X(t,y))\theta(t,y)\}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial v}{\partial y} = 0$$

$$v(t,y) = \min\{\frac{1}{T(X(t,y))}\,,\; q(X(t,y))\theta(t,y)\}$$

$$-\frac{\partial X}{\partial y}(t,y) = \theta(t,y)$$

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$$v(t,y) = \min\{\frac{1}{T(X(t,y))}, q(X(t,y))\theta(t,y)\}$$

$$-\frac{\partial X}{\partial y}(t,y) = \theta(t,y)$$

- Last eq. equivalent to:

$$X(t,y) = \int_y^1 \theta(t,z)dz$$

5. Passage to Eulerian variables

Goal

- Obtain a model in (t, x) rather than (t, y)

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- Obtain a model in (t, x) rather than (t, y)
- Classical procedure in gas dynamics
  - Coordinate change:

$$x = X(t, y) = \int_y^1 \theta(t, z) dz \quad \text{strictly} \searrow \text{ of } y$$

$$y = Y(t, x) \quad \text{inverse fct}$$

Eulerian unknowns

- Using that $y \longrightarrow Y(t,x)$ is the inverse fct of $x \longrightarrow X(t,y)$

$$\frac{\partial Y}{\partial x}(t,x) = -\frac{1}{\theta(t, Y(t,x))} := \rho(t,x)$$

Number density of parts at x at time t

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$$\frac{\partial Y}{\partial x}(t,x) = -\frac{1}{\theta(t,Y(t,x))} := \rho(t,x)$$

Number density of parts at x at time t

- and

$$\frac{\partial Y}{\partial t}(t,x) = -\frac{v(t,Y(t,x))}{\theta(t,Y(t,x))} := \rho(t,x)u(t,x)$$

$u(t,x) = v(t,Y(t,x))$ velocity in Eulerian coord.

- Since

$$\frac{\partial}{\partial t}\left(\frac{\partial Y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial Y}{\partial t}\right)$$

Continuity and constitutive eqs.

- Since

$$\frac{\partial}{\partial t}\left(\frac{\partial Y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial Y}{\partial t}\right)$$

- The continuity equation is satisfied

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$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

- From the constitutive relation in Lagrangian variable, we get

$$\rho u(t,x) = \min\{\frac{1}{T(x)}\rho(t,x)\ ,\ q(x)\}$$

Continuum supply chain model

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➠ Hyperbolic model with flux constraint

6. Kinetic model

- Fluid model

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\rho u = \min\{\rho V_0\,,\, q\}\,, \qquad V_0 = \frac{1}{T}$$

Particle interpretation of fluid model

- Fluid model

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\rho u = \min\{\rho V_0 \, , \, q\} \, , \qquad V_0 = \frac{1}{T}$$

- Particle interpretation

$$\dot{X} = \begin{cases} V_0 \\ 0 \end{cases} \qquad \dot{\rho} = \begin{cases} -\rho \partial_x V_0 & \text{if} \quad \rho V_0 \leq q \\ -\partial_x q & \text{if} \quad \rho V_0 > q \end{cases}$$

Particle motion

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- Either particle moves or is blocked according to whether the 'free' flux ρV_0 is below or exceeds the threshold q.
- Kinetic model: 'regularization' of this singular dynamics
- Introduce an attribute variable ξ to each particle
  - Particles move with actual velocity

$$V(t, x, \xi) \leq V_0$$

  - s.t. total flux $\leq q$

Distribution function

- $f(x, \xi, t)$ density of parts at time t, position x, with attribute ξ

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- Density ρ and flux ρu

$$\rho = \int f(x, \xi, t) \, d\xi \,, \quad \rho u = \int f(x, \xi, t) \, V(x, \xi, t) \, d\xi$$

Distribution function

- $f(x, \xi, t)$ density of parts at time t, position x, with attribute ξ
- Density ρ and flux ρu

$$\rho = \int f(x, \xi, t)\, d\xi \,, \quad \rho u = \int f(x, \xi, t)\, V(x, \xi, t)\, d\xi$$

- Maximal possible flux Q:

$$Q = \int f(x, \xi, t)\, V_0(x)\, d\xi = \rho V_0$$

Flux if there would be no capacity limitation

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  - Note: same methodology would apply for other policies

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  - Note: same methodology would apply for other policies
- Procedure: move parts by increasing attribute number with maximum allowed speed V_0 until processor capacity is reached
- Number of parts with attribute $\leq \alpha$ is

$$\int_{-\infty}^{\alpha} f(x,\xi,t)\,d\xi$$

Implementing policy I

- If parts with attribute $\leq \alpha$ all move with maximal speed V_0, the associated flux is

$$\beta(x,\alpha,t) = V_0(x) \int_{-\infty}^{\alpha} f(x,\xi,t)\, d\xi$$

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$$\alpha \in \mathbb{R} \to \beta(x,\alpha,t) \in [0,Q]$$

is increasing

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- The fonction

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is increasing

- Denote $\beta^{-1}(x, \cdot, t)$ its inverse

Implementing policy II

- Processors process all parts with attribute $\leq \alpha$
  - where α s.t. associated flux $=$ processor capacity

$$\beta(x, \alpha, t) = q \qquad (\Leftrightarrow \alpha = \beta^{-1}(x, q, t)\,)$$

Implementing policy II

- Processors process all parts with attribute $\leq \alpha$
  - where α s.t. associated flux = processor capacity

$$\beta(x, \alpha, t) = q \qquad (\Leftrightarrow \alpha = \beta^{-1}(x, q, t))$$

- except if maximal possible flux Q lower than processor capacity q
  - in which case $\alpha = \infty$

$$q > Q \quad \Longrightarrow \quad \alpha = \infty$$

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- except if maximal possible flux Q lower than processor capacity q
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$$q > Q \quad \Longrightarrow \quad \alpha = \infty$$

- Then

$$\beta(x, \alpha, t) = \min\{q, Q\}$$

Actual velocity $V(x, \xi, t)$

- Note: processor velocity V_0 can be attribute dependent

$$V_0 = V_0(x, \xi)$$

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- Actual velocity

$$V(x, \xi, t) = \begin{cases} V_0(x, \xi) & \text{if} \quad \xi \leq \alpha(x, t) \\ 0 & \text{if} \quad \xi > \alpha(x, t) \end{cases}$$

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- Actual velocity

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- or

$$V(x, \xi, t) = V_0(x, \xi) H(\alpha(x, t) - \xi) \qquad H = \text{ Heaviside}$$

- Since $\beta \nearrow$ fonction of α, we have

$$\begin{aligned} H(\alpha(x,t)-\xi) &= H(\beta(x,\alpha(x,t),t)-\beta(x,\xi,t)) \\ &= H(\min\{q,Q\}-\beta(x,\xi,t)) \\ &= H(q-\beta(x,\xi,t)) \quad (\text{since } \beta \leq Q) \end{aligned}$$

Actual velocity II

- Since $\beta \nearrow$ fonction of α, we have

$$\begin{aligned} H(\alpha(x,t)-\xi) &= H(\beta(x,\alpha(x,t),t)-\beta(x,\xi,t)) \\ &= H(\min\{q,Q\}-\beta(x,\xi,t)) \\ &= H(q-\beta(x,\xi,t)) \quad (\text{since } \beta \le Q) \end{aligned}$$

- and

$$\begin{aligned} \beta(x,\xi,t) &= \int_{-\infty}^{\xi} V_0(x,\xi')\, f(x,\xi',t)\, d\xi' \\ &= \int_{\mathbb{R}} V_0(x,\xi')\, f(x,\xi',t)\, H(\xi-\xi')\, d\xi' \end{aligned}$$

Actual velocity III

➠ Finally

$$V(x,\xi,t) = V_0(x,\xi)H(q-\beta(x,\xi,t))$$

$$\beta(x,\xi,t) = \int_{\mathbb{R}} V_0(x,\xi')\, f(x,\xi',t)\, H(\xi-\xi')\, d\xi'$$

Part dynamics

- By analogy with fluid model

$$\dot{X} = V(X, \Xi, t) = V_0(X, \Xi) H(q - \beta(X, \Xi, t))$$

$$\dot{f} = -f(\partial_x V)|_{(X,\Xi,t)}$$

$$\dot{\Xi} = 0$$

Part dynamics

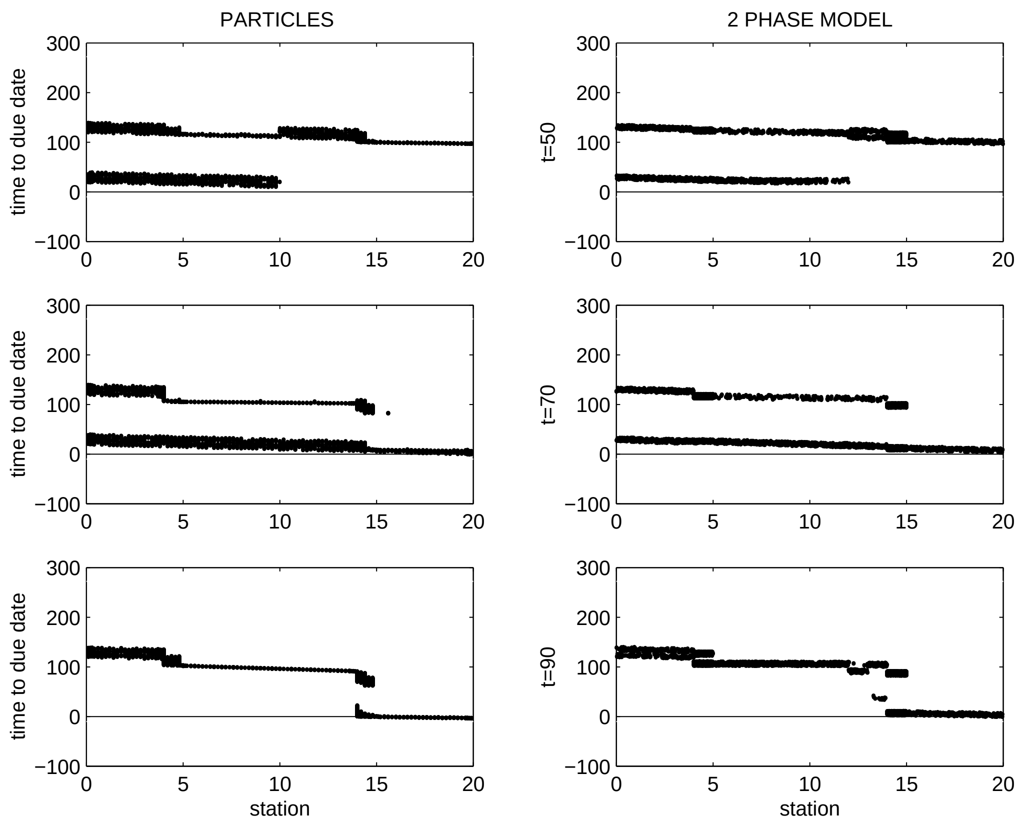
- By analogy with fluid model

$$\dot{X} = V(X, \Xi, t) = V_0(X, \Xi) H(q - \beta(X, \Xi, t))$$
$$\dot{f} = -f(\partial_x V)|_{(X,\Xi,t)}$$
$$\dot{\Xi} = 0$$

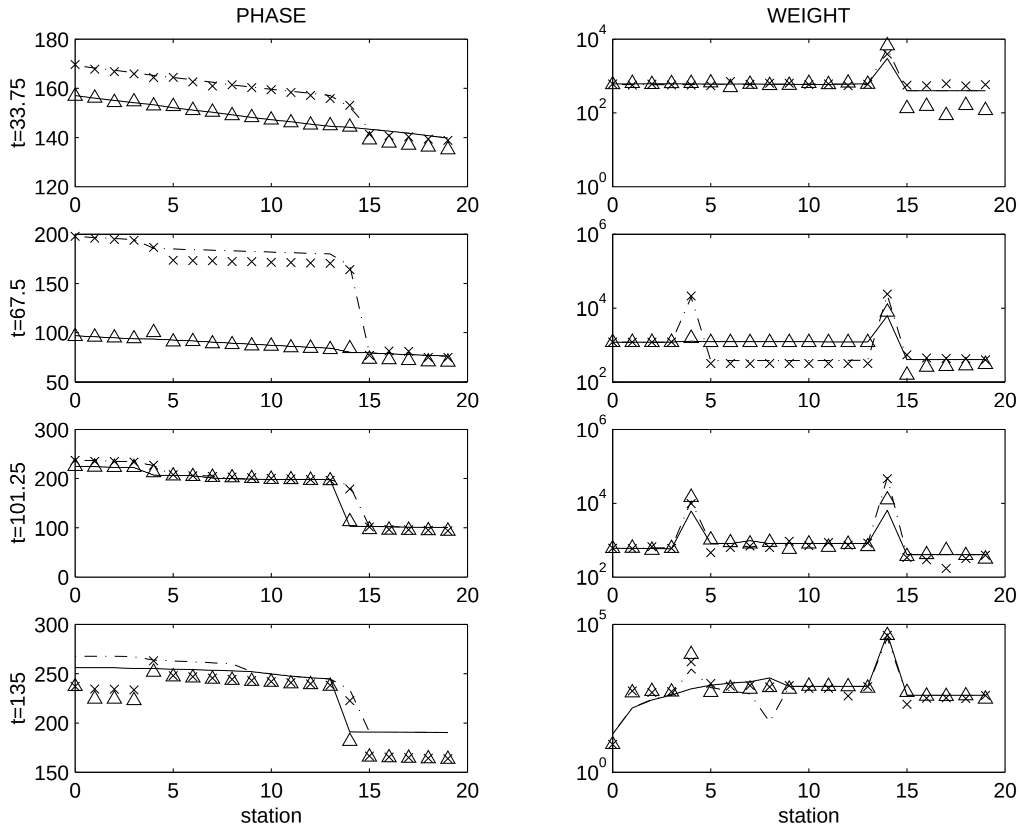
- Characteristics of the first order kinetic eq.

$$\partial_t f + \partial_x (V f) = 0$$

When attribute value evolves



Phases as a function of 'space' (or DOC)
at various times (top to down)
Left: kinetic model   Right: two-phase model

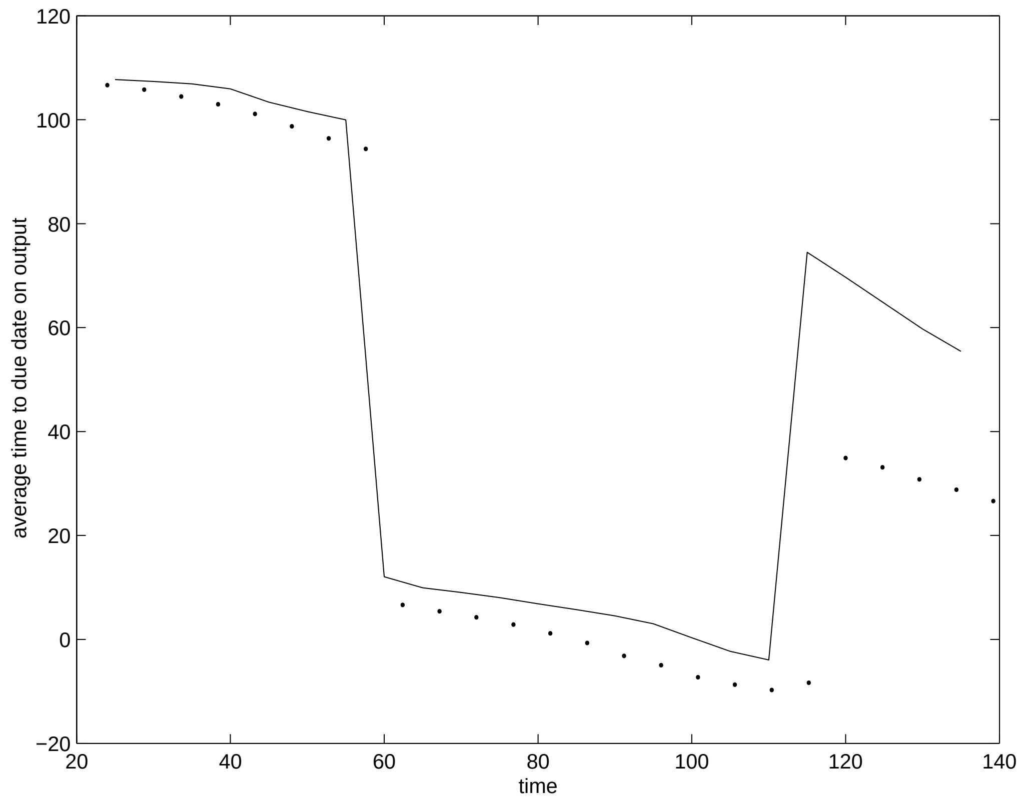


Phases (left) and densities (right) as a function of 'space'

at various times (top to down)

Kinetic ($\times$, Δ) and two-phase (—, – · –) models

Case 2: Multiphase and kinetic models



Expected time to due-date m_1/m_0 in the last cell as a function of time

Kinetic ($\cdots$) and two-phase (—) models

9. Conclusion

Summary

- (Rigorous) derivation of continuum model from Discrete Event Simulator
  - Nonlinear hyperbolic model with saturated flux
  - Reproduces DES model satisfactorily even in situations where capacity has large variations

Summary

- (Rigorous) derivation of continuum model from Discrete Event Simulator
  - Nonlinear hyperbolic model with saturated flux
  - Reproduces DES model satisfactorily even in situations where capacity has large variations
- Kinetic model with an internal variable (policy attribute)
  - Simple closure recovers the fluid model
  - Multiphase closure allows to implement policies
  - Correct agreement between two-phase and kinetic models

Work in progress

- Fluid model
  - Networks

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- Fluid model
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- Kinetic model
  - Randomness (random failures)

Work in progress

- Fluid model
  - Networks
- Kinetic model
  - Randomness (random failures)
- More complex models
  - Orders, payments, etc.