A traffic flow model for the dynamics of traffic jams: a Pressureless Gas Dynamics system under a maximal constraint

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Joint work with

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Summary

1. Traffic models: overview on fluid models
2. Modified Aw-Rascle model
3. Limit $\varepsilon \longrightarrow 0$: Constrained Pressureless Gas Dynamics
4. CPGD: additional laws
5. Existence theorem for CPGD
6. Numerical simulations
7. Conclusion

1. Traffic models: overview on fluid models

Fluid models (1)

- Conservation of car density

$$\partial_t n + \partial_x q = 0$$

- What expression for the flux q ?

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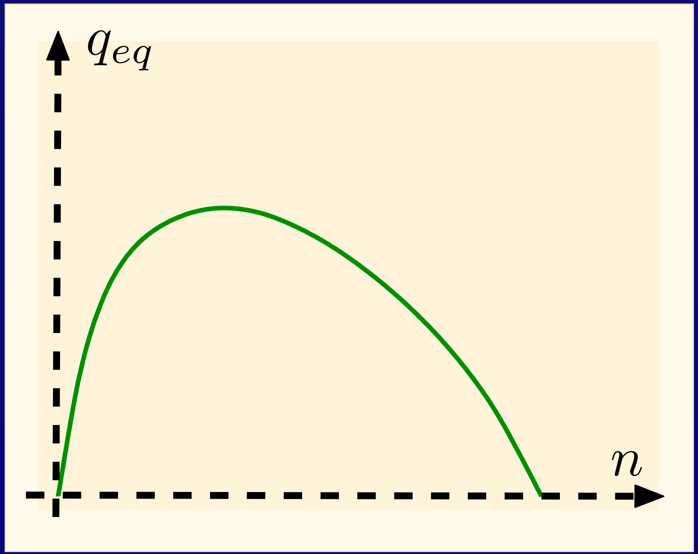
- First order models:

$$q = q_{eq}(n)$$

[Lighthill, Witham (1955)], …



- Second order models: $q = nu$ and gas dynamics-like eq. for u:

$$\partial_t nu + \partial_x(nu^2 + p) = -\frac{nu - q_{eq}(n)}{\tau}$$

  - [Payne (1971)], ...

Fluid models (2)

- Second order models: $q = nu$ and gas dynamics-like eq. for u:

$$\partial_t nu + \partial_x (nu^2 + p) = -\frac{nu - q_{eq}(n)}{\tau}$$

  - [Payne (1971)], . . .

- [Daganzo (1995)]: Inacceptable properties (e.g. Vehicles going backwards)
  - Fluid $\Rightarrow$ sound propagation is isotropic in a comoving frame
  - Traffi c: information propagates backwards

The Aw-Rascle model (1)

- Modified 2nd order model (see also [Zhang (2002)])
- Preferred velocity w is a Lagrangian quantity:

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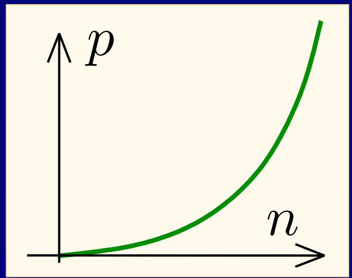
$$\dot{w} := (\partial_t + u\partial_x)w = 0$$

- The actual velocity u offsets the preferred velocity w by a quantity $p(n)$ which increases with n

$$w = u + p(n)\,, \quad p \nearrow \text{ as } n \nearrow$$

- Typically $p(n) = n^\gamma$, $\gamma > 0$



AR model (2)

$$\partial_t n + \partial_x(nu) = 0$$
$$(\partial_t + u\partial_x)(u + p(n)) = 0$$

- Second eq. equivalent to

$$(\partial_t + (u - np'(n))\partial_x)u = 0$$

- Two characteristic velocities:
  - $\lambda_1 = u - np'(n)$ (assoc.w. u, GNL)
  - $\lambda_2 = u$ (assoc.w. $w = u + p(n)$, LD)

Properties of AR model

- Invariant regions: (u, w) - rectangles
  - If

$$a < u_0 < b \quad \text{and} \quad c < w_0 < d$$

then for all times

$$a < u(t) < b \quad \text{and} \quad c < w(t) < d$$

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- AR model in Lagrangian coordinates = continuous version of Follow-the-Leader model [Aw, Klar, Materne, Rascle (2002)]

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- Problem: there is no invariant region for n
  - $n > 0$ BUT:
  - n can exceed the upper limit n^* (if any) even if initially $n < n^*$)
- Goal:
  - Modify AR model s.t. it enforces an upper limit constraint on the density $n \leq n^*$
  - Investigate the dynamics of the clustered regions where $n = n^*$

2. Modified Aw-Rascle model

Modified AR model (M-AR)

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$$n < n^*$$

at all times

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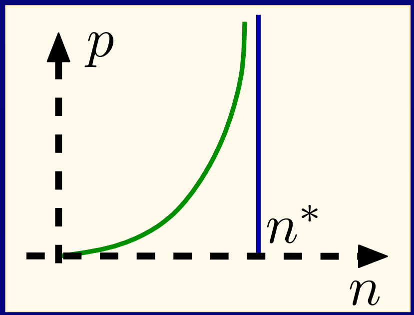
- Modify $p(n)$ s.t.

$$p(n) \longrightarrow \infty \quad \text{as} \quad n \longrightarrow n^*$$

- For instance

$$p(n) = \frac{1}{\left(\frac{1}{n} - \frac{1}{n^*}\right)^\gamma}$$



Density constraint

- M-AR has the same properties as the standard AR model
  - Hyperbolicity
  - Invariant regions
- Satisfies the density constraint

$$n < n^*$$

at all times

A singular situation

- In practice: two traffic regimes:
  - Uncongested traffic ($n < n^*$): driver goes its preferred velocity
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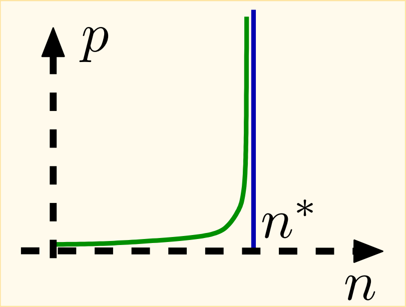
  - $p(n)$ large (and possibly ∞) only when $n \overset{\sim}{<} n^*$
- Modeled by the rescaling:

$$p(n) = \varepsilon \tilde{p}(n)$$



Rescaled Modified AR model (RM-AR)

- Perturbed AR system

$$\partial_t n^\varepsilon + \partial_x(n^\varepsilon u^\varepsilon) = 0$$
$$(\partial_t + u^\varepsilon \partial_x)(u^\varepsilon + \varepsilon p(n^\varepsilon)) = 0$$

- with modifi ed velocity offset:

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- Question: what happens in the limit

$$\varepsilon \longrightarrow 0$$

3. Limit $\varepsilon \to 0$: Constrained Pressureless Gas Dynamics

$\varepsilon \longrightarrow 0$: Case I (uncongested)

- Suppose $n^\varepsilon \longrightarrow n < n^*$ (uncongested case)
  - Then $\varepsilon p(n^\varepsilon) \longrightarrow 0$ in (RM-AR) model:

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- Limit system = Pressureless Gas Dynamics

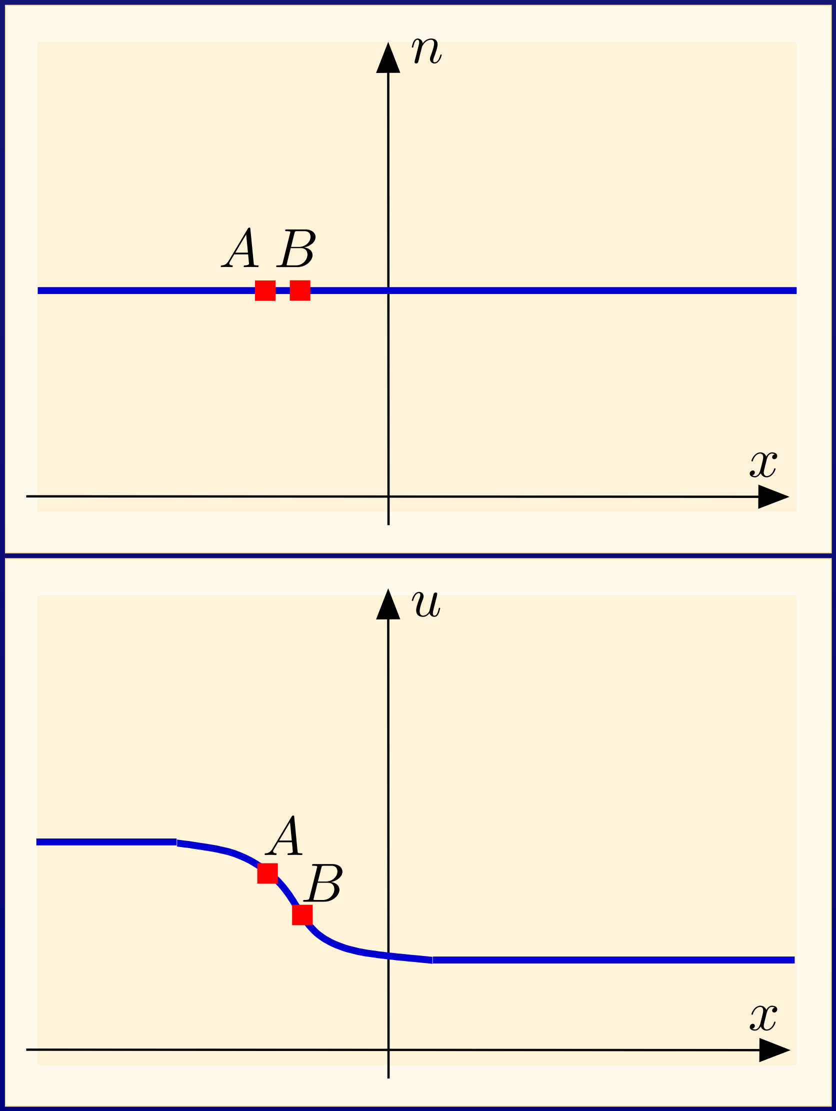
$$\partial_t n + \partial_x(nu) = 0$$
$$(\partial_t + u\partial_x)u = 0$$

-
  - Mass conservation
  - Burger's eq. for the velocity

Pressureless Gas Dyn. properties

- Not strictly hyperbolic
  - 2 identical eigenvalues u
  - But not diagonalizable: Jacobian $= \begin{pmatrix} u & n \\ 0 & u \end{pmatrix}$
- Weak instability:
  - linearized solution increase like $O(t)$
- Generates mass concentrations

Pressureless Gas Dyn. concentrations



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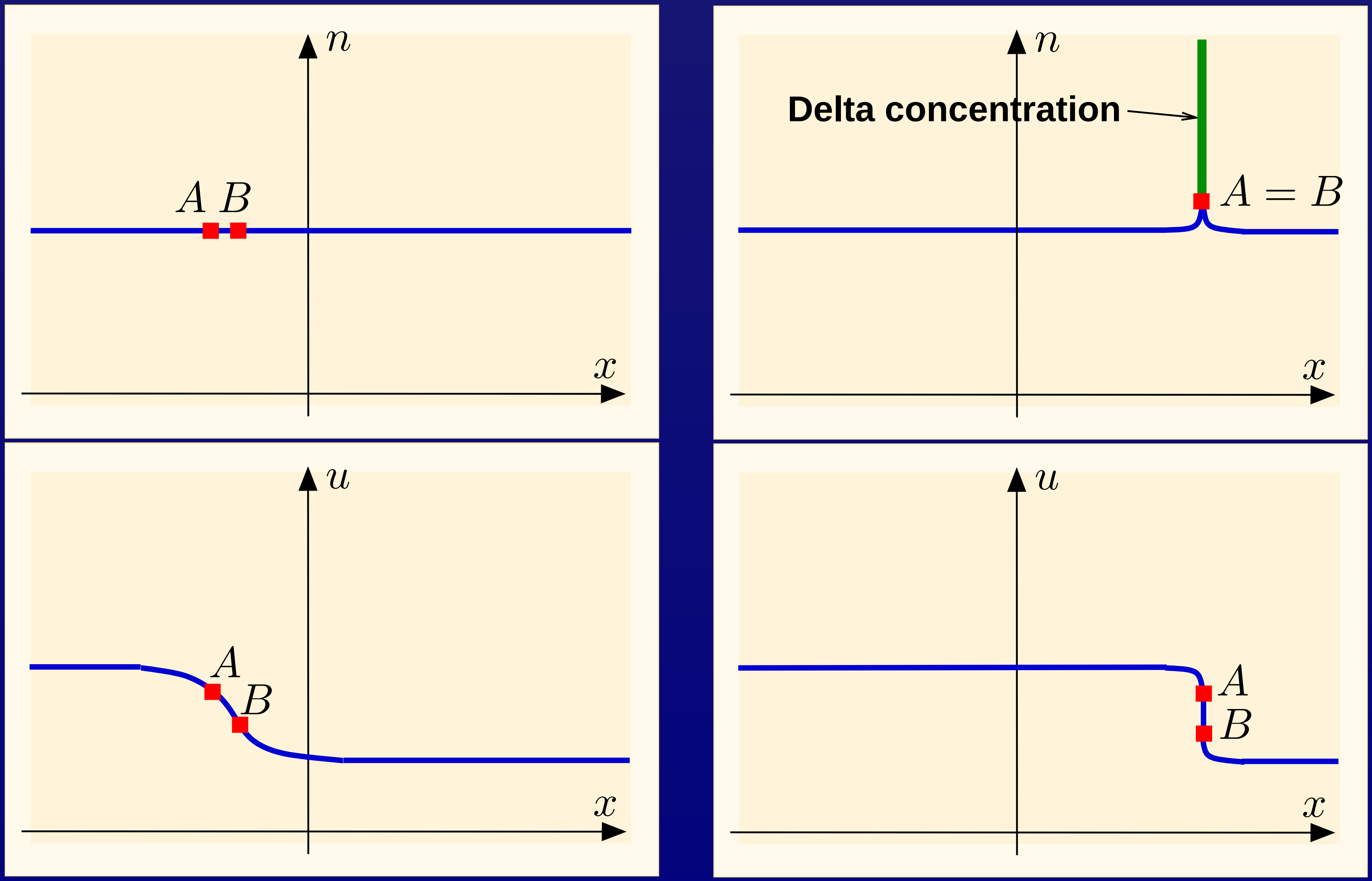

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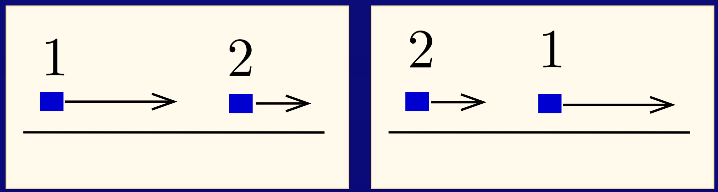

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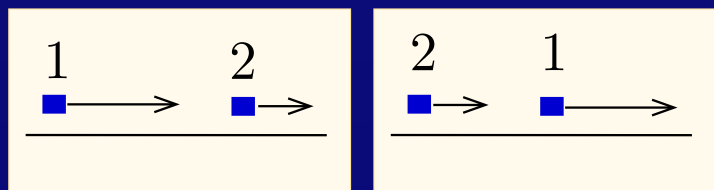
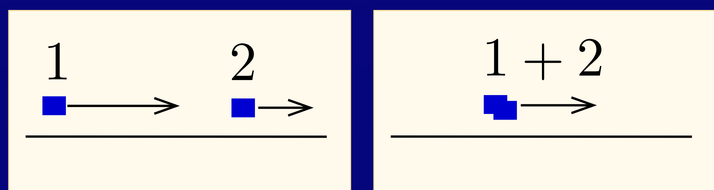


  - Sticky particles (Zeldowitch, E, ...)



- see e.g. [Bouchut (94)], [Grenier (95)], [Rykov, Sinai (96)], [Brenier, Grenier (98)], . . .

Here: no concentrations

- Density constraint: no concentration formation
  - No need to define a particle dynamics
- Instead: formation of ’clusters’ (traffic jams)
  - Cluster dynamics follows from the asymptotic limit

$\varepsilon \longrightarrow 0$: Case II (congested)

- Suppose $n^\varepsilon \to n^*$ (then $p(n^\varepsilon) \to \infty$)
  - Suppose $\varepsilon p(n^\varepsilon) \to \bar{p} < \infty$
- Then $\varepsilon \to 0$ in (RM-AR) model:

$$\partial_t n^\varepsilon + \partial_x (n^\varepsilon u^\varepsilon) = 0$$
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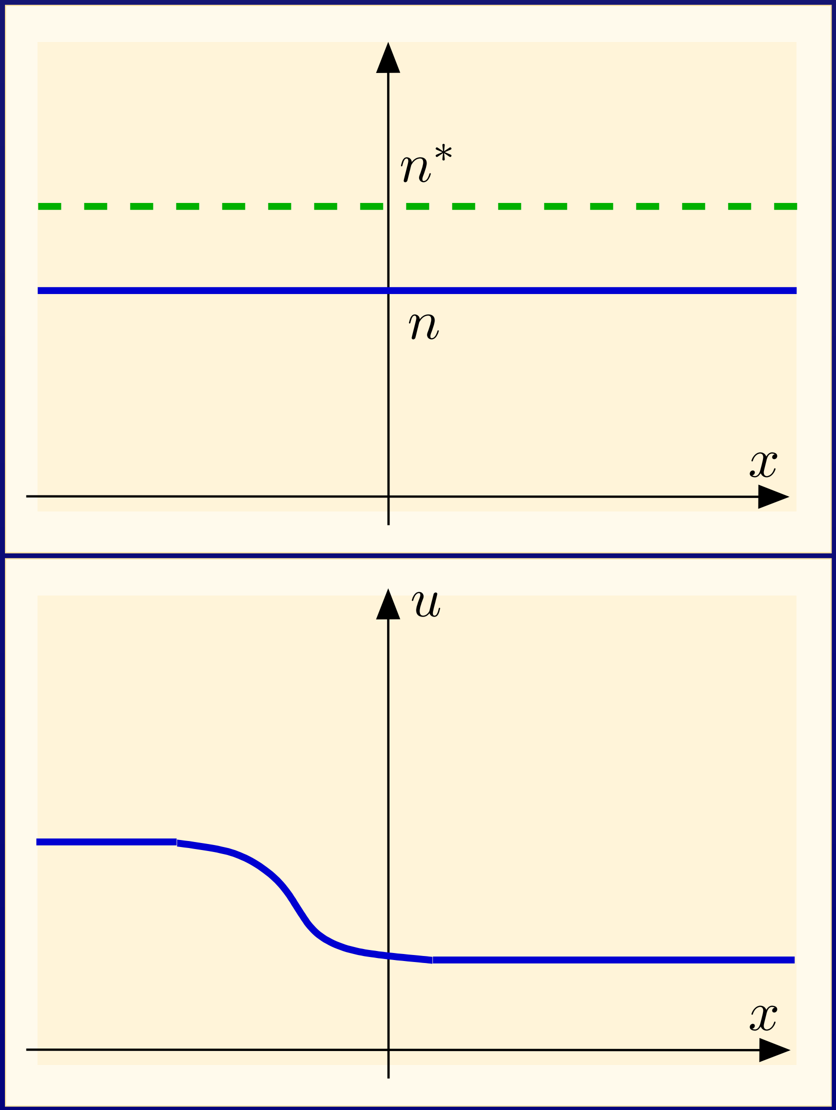
- Gives

$$\partial_t n + \partial_x (nu) = 0$$
$$(\partial_t + u\partial_x)(u + \bar{p}) = 0$$
$$n = n^*$$

- $\bar{p}$ unknown: Lagrange multiplier

Cluster formation



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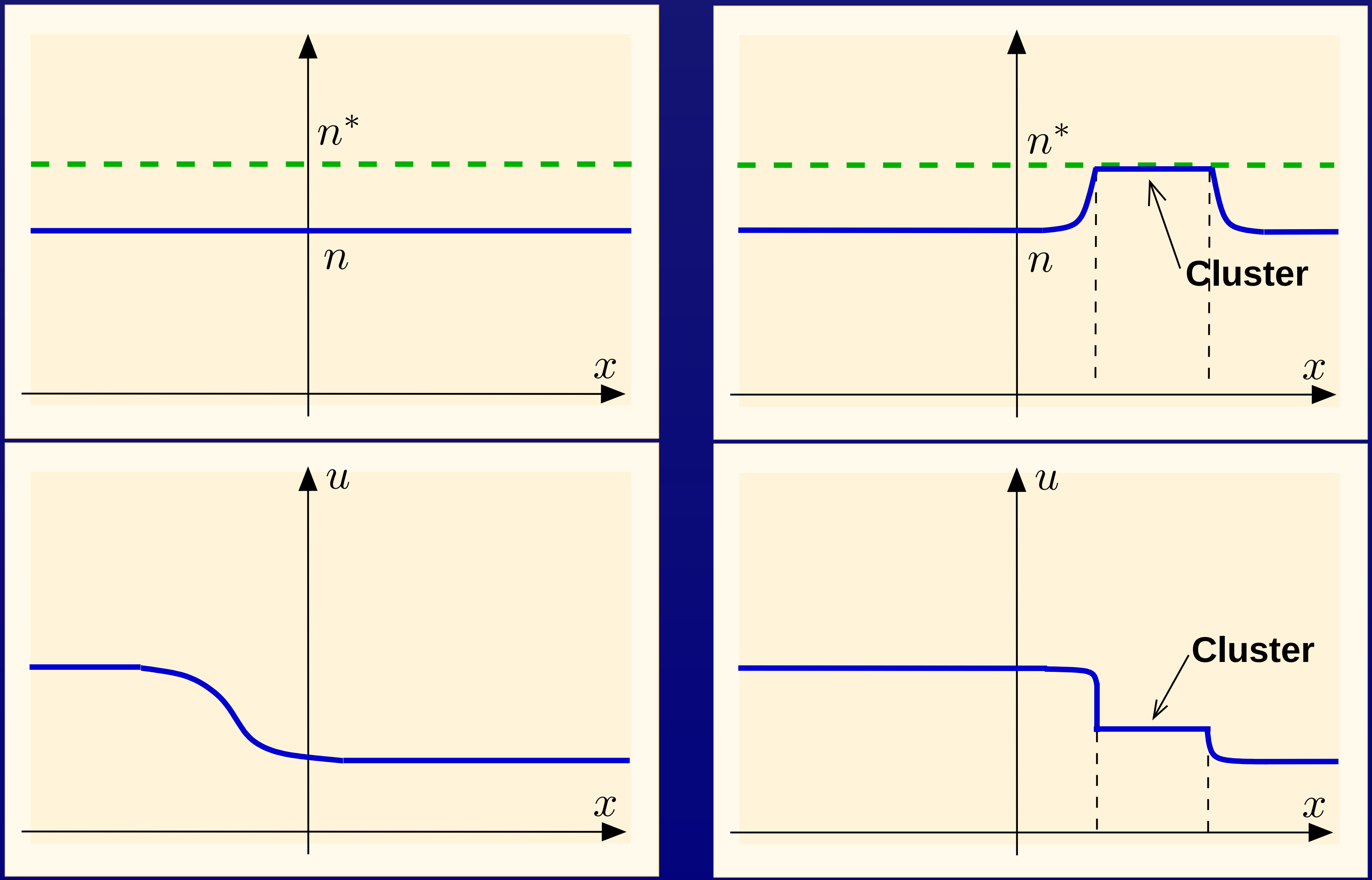

Unified formulation

- Constrained Pressureless Gas Dynamics (CPGD)

$$
\begin{aligned}
&\partial_t n + \partial_x (nu) = 0 \\
&(\partial_t + u\partial_x)(u + \bar{p}) = 0 \\
&\bar{p}(n^* - n) = 0 \\
&\bar{p} \geq 0\,, \quad 0 \leq n \leq n^*
\end{aligned}
$$

- see e.g. [Brenier, ...], [B. and Bouchut] for gaseous corks in pipes

4. Constrained Pressureless Gas Dynamics: additional laws

CPGD model

- CPGD formulation ill-posed
  lack of information for defi ning a unique solution

CPGD model

- CPGD formulation ill-posed
  lack of information for defi ning a unique solution
- To be defi ned
  - Cluster dynamics
  - Value of $\bar{p}$ inside clusters
  - What if clusters meet ?

Characteristic velocities

- If $n^\varepsilon \to n^*$ with $\varepsilon p(n^\varepsilon) \to \bar{p} < \infty$, then:

$$\varepsilon n^\varepsilon p'(n^\varepsilon) \to \infty$$

- Characteristic velocities:
  - $\lambda_1^\varepsilon = u^\varepsilon - n^\varepsilon p'(n^\varepsilon) \to -\infty$
  - $\lambda_2^\varepsilon = u^\varepsilon \to u$

- Riemann invariant associated with λ_1 is u:

$$\partial_t u^\varepsilon + \lambda_1^\varepsilon \partial_x u^\varepsilon = 0$$

- In the limit $\varepsilon \to 0$, u is constant in a cluster

$$\partial_x u = 0$$

- In a cluster, all cars have the same velocity (the cluster velocity)
- Since $\lambda_1^\varepsilon \longrightarrow -\infty$, all variations of the cluster velocity originate from the leading vehicle of the cluster
- Any variation of the velocity of the leading vehicle instantaneously propagates to the whole cluster

How to complete CPGD formulation

- Limit (RM-AR) $\longrightarrow$ (CPGD) is formal
  - Gives no information about cluster dynamics beyond what has been noticed above

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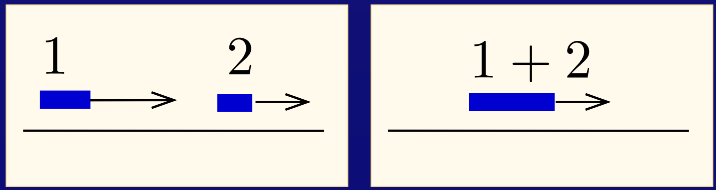
- But Riemann problem solutions of (RM-AR) are explicit
  - Limit $\varepsilon \longrightarrow 0$ in these solutions give information about cluster dynamics

Cluster dynamics (from Riemann pbm)

- When two clusters meet, they merge
  - The resulting cluster takes the velocity of the front cluster (the slowest one)



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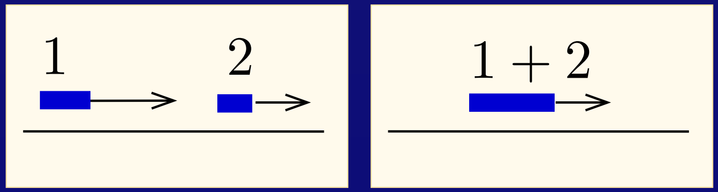

- At the right-most end of a cluster, there is
  - either vacuum (and $\bar{p} = 0$ at the right-most point of the cluster)
  - or the velocity is continuous accross the right most point of the cluster

Cluster dynamics (2)

- These rules allow to determine
  - $\bar{p}$ everywhere
  - The velocity of the extreme points of the cluster

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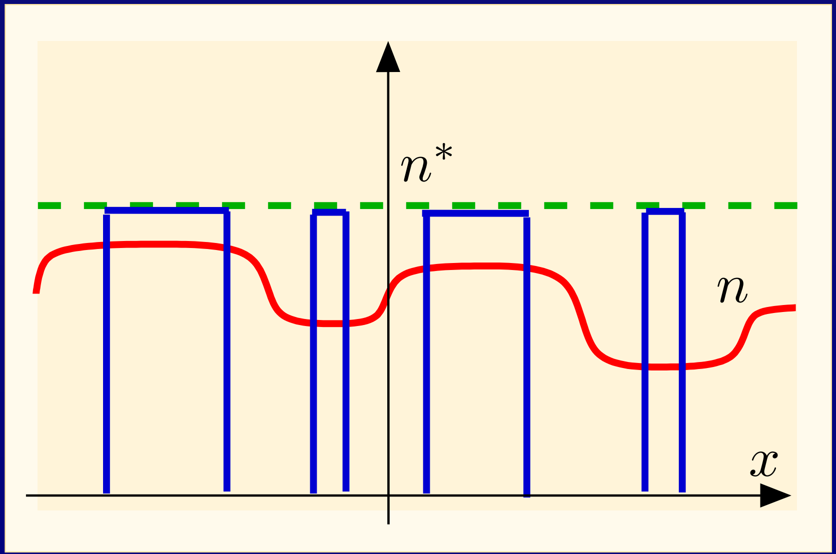
- These rules allow to determine
  - $\bar{p}$ everywhere
  - The velocity of the extreme points of the cluster
- Cluster dynamics $\neq$ from that of [Brenier, ...] and [B. and Bouchut]
  - no momentum conservation at cluster collapse

5. Existence theorem for CPGD

Cluster approximate solution

- Idea (follows from [B. and Bouchut (2002, 2003)]),
  - Approximate (in $\mathcal{D}'$) the solution by clusters

$$\begin{pmatrix} n(x,t) \\ (nu)(x,t) \end{pmatrix} \approx \sum_{1}^{N} \begin{pmatrix} n^* \\ n^* u_i(t) \end{pmatrix} \chi_{a_i(t) \leq x \leq b_i(t)}$$



Cluster dynamics

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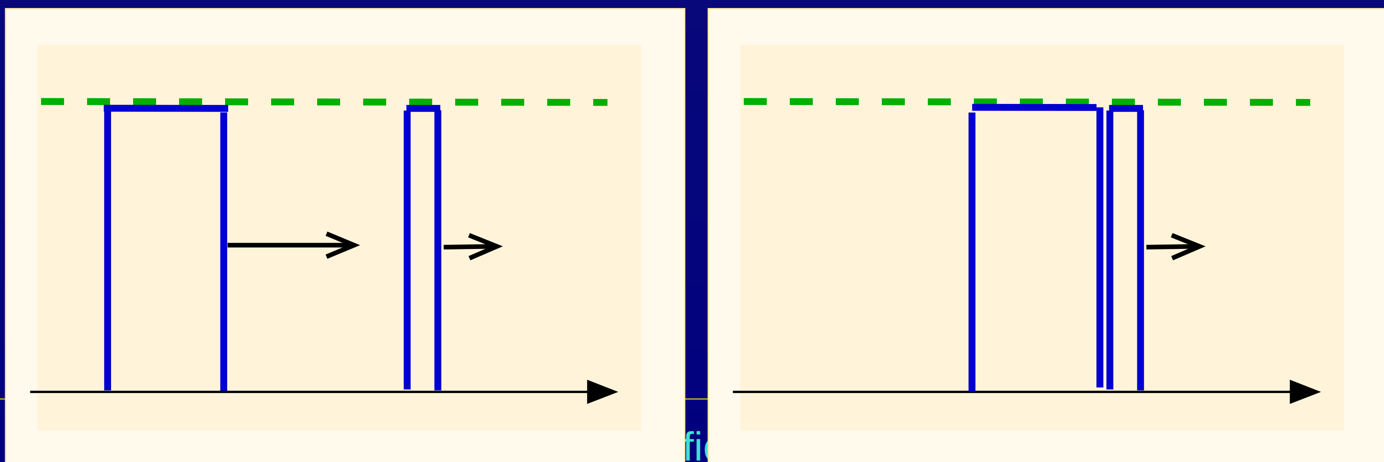

Properties of cluster dynamics

- Is a weak solution of (CPGD)
- Satisfi es an 'Oleinik' condition

$$\partial_x u \leq \frac{1}{t}$$

- Satisfi es L^∞ and BV bounds:

$$\inf u_0 \leq u(.,t) \leq \sup u_0$$
$$TV_K(\bar{p}) \leq 2TV(u_0)\,, \quad \forall K \text{ compact}$$
$$0 \leq \bar{p} \leq 2 \sup u_0$$

Properties of cluster dynamics (2)

- Satisfies an 'entropy' equality
  - For any $S \in C^1$, $\exists \bar{p}^S$ s.t.

$$\partial_t(nS(u) + n\bar{p}^S) + \partial_x(nuS(u) + nu\bar{p}^S) = 0$$

- $\bar{p}^S$ has a BV bound:

$$TV_K(\bar{p}^S) \leq 2|S'|_\infty TV u_0$$

Proof of existence

- Step 1: approximate initial condition (n_0, u_0) by a converging sequence of clusters (n_0^k, u_0^k) [B. (02)]
  - Defines a sequence of cluster sol. (n^k, u^k) satisfying the above a priori bounds

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- Key point: an extension of the compensated compactness lemma using Oleinik estimate [B. (02)]

Existence result

- Suppose
  - $n_0 \in L^1 \quad , \qquad 0 \leq n_0 \leq n^*$
  - $u_0 \in L^\infty \cap BV$

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  - a solution of CPGD
  - satisfying Oleinik estimate
  - satisfying entropy equality for any $S \in C^1$
- No uniqueness so far
- No convergence proof (RM-AR) $\longrightarrow$ (CPGD)

6. Numerical simulations

- Rescaled Modifi ed Follow-the-Leader

$$\dot{x}_i = v_i \,, \qquad \dot{v}_i = \frac{\varepsilon}{\gamma} \frac{v_{i+1} - v_i}{(x_{i+1} - x_i - d)^{\gamma+1}}$$

Method: modified Follow-the-Leader

- Rescaled Modifi ed Follow-the-Leader

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- Limit $\varepsilon \to 0$

$$\dot{x}_i = v_i\,, \qquad \dot{v}_i = \begin{cases} \dot{v}_i = 0 & \text{if} \quad x_{i+1} - x_i > d \\ v_i = v_{i+1} & \text{if} \quad x_{i+1} = x_i + d \end{cases}$$

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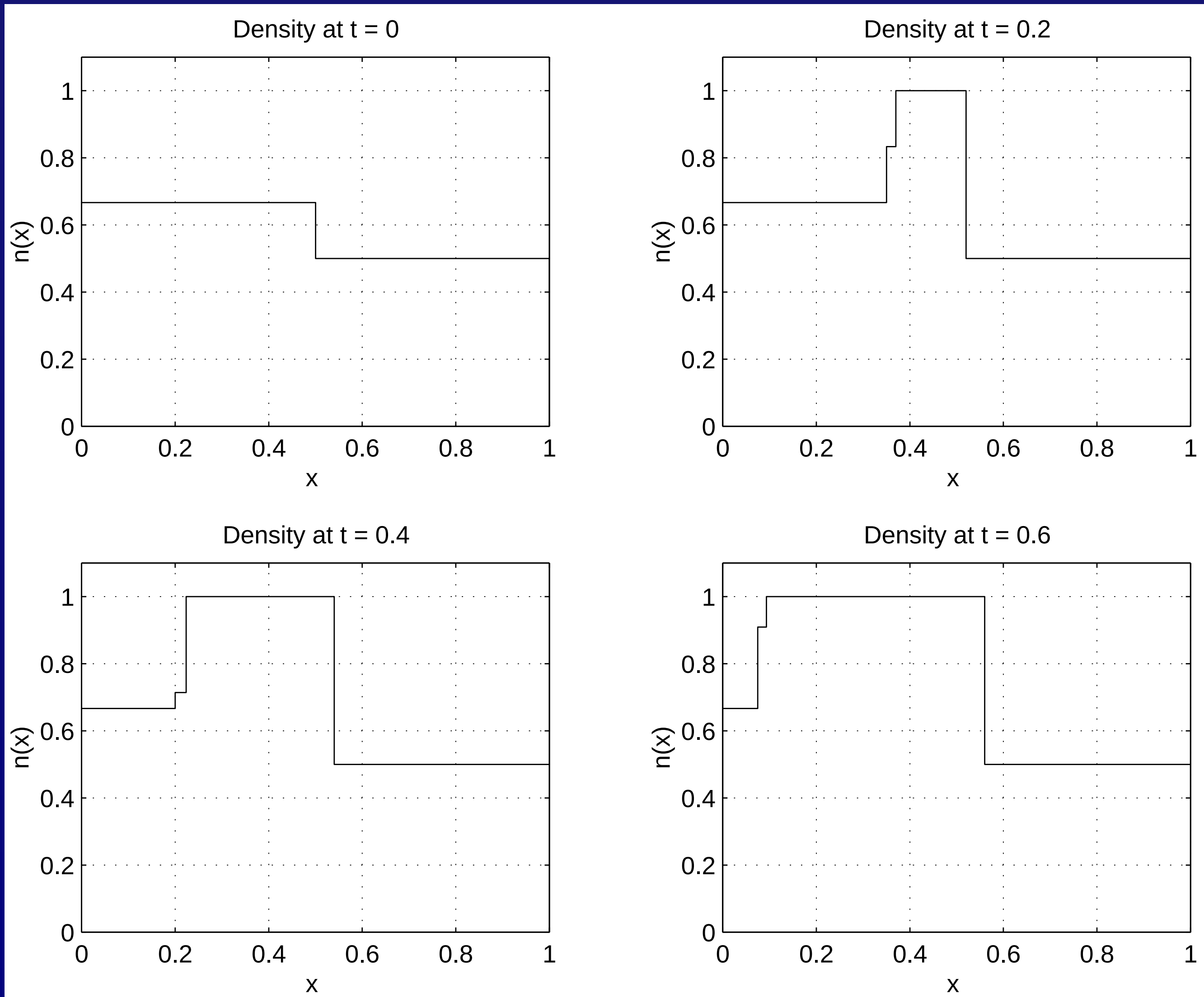
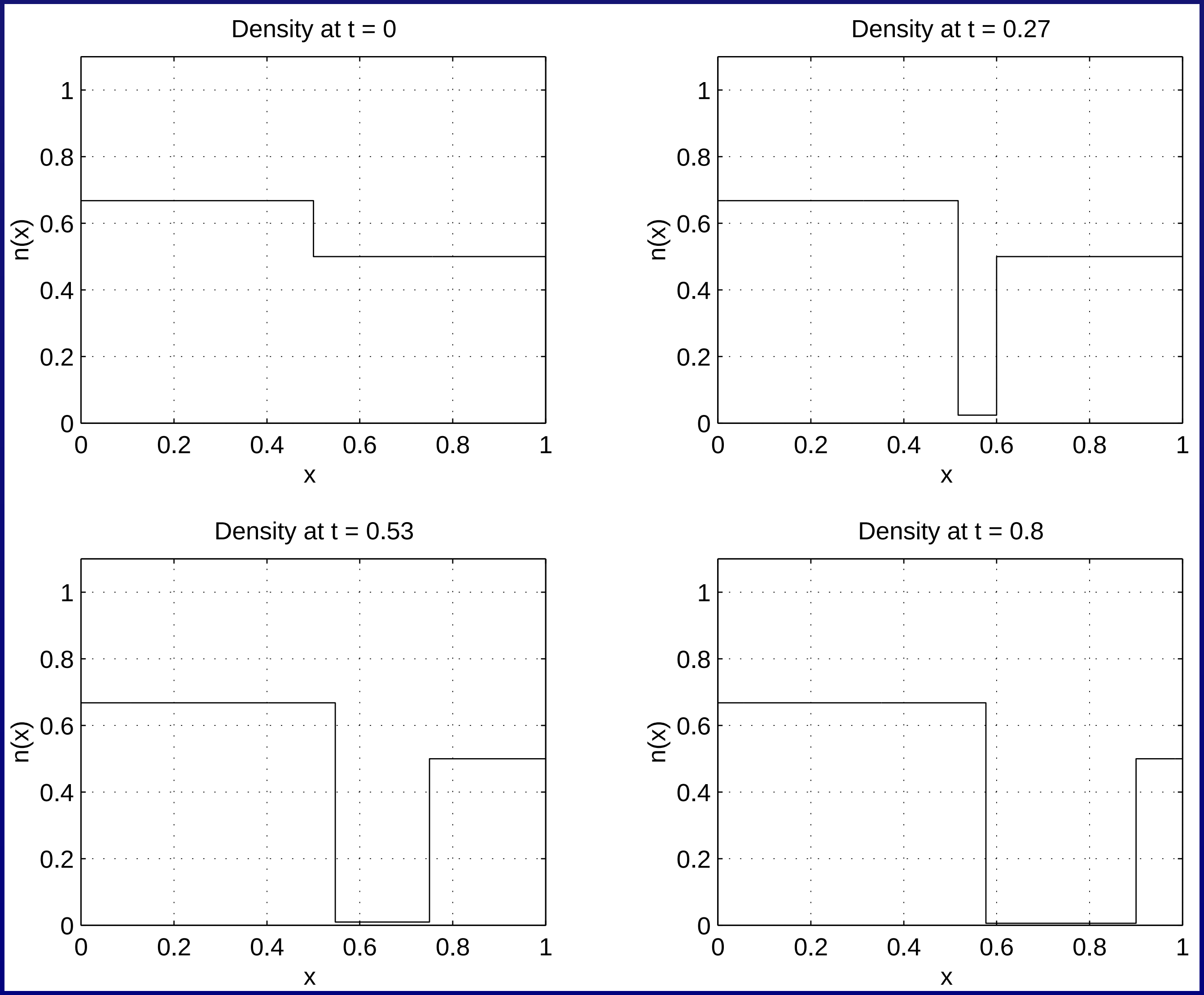
- Cluster = sequence of vehicles separated by d

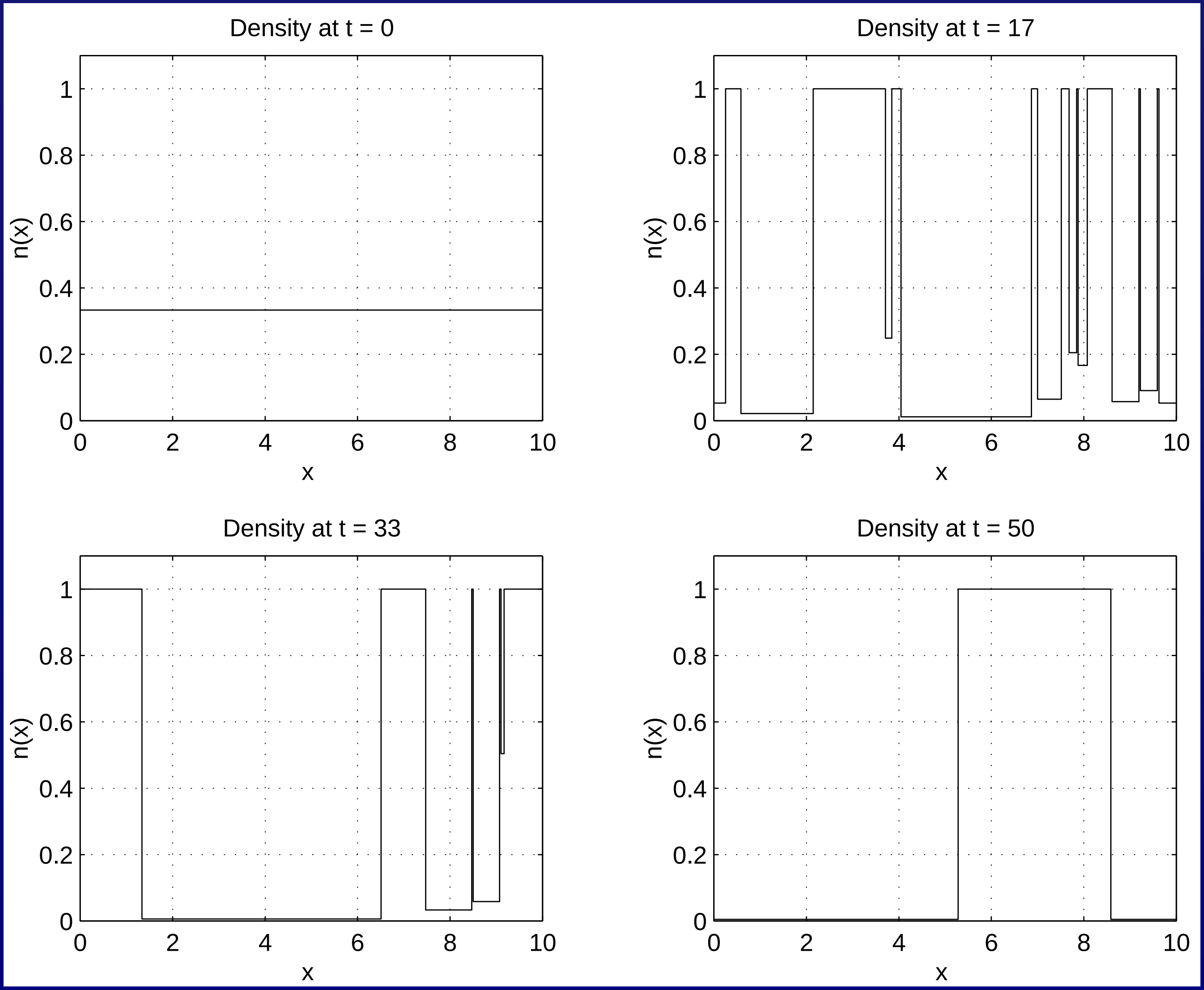
Riemann problem: Cluster formation



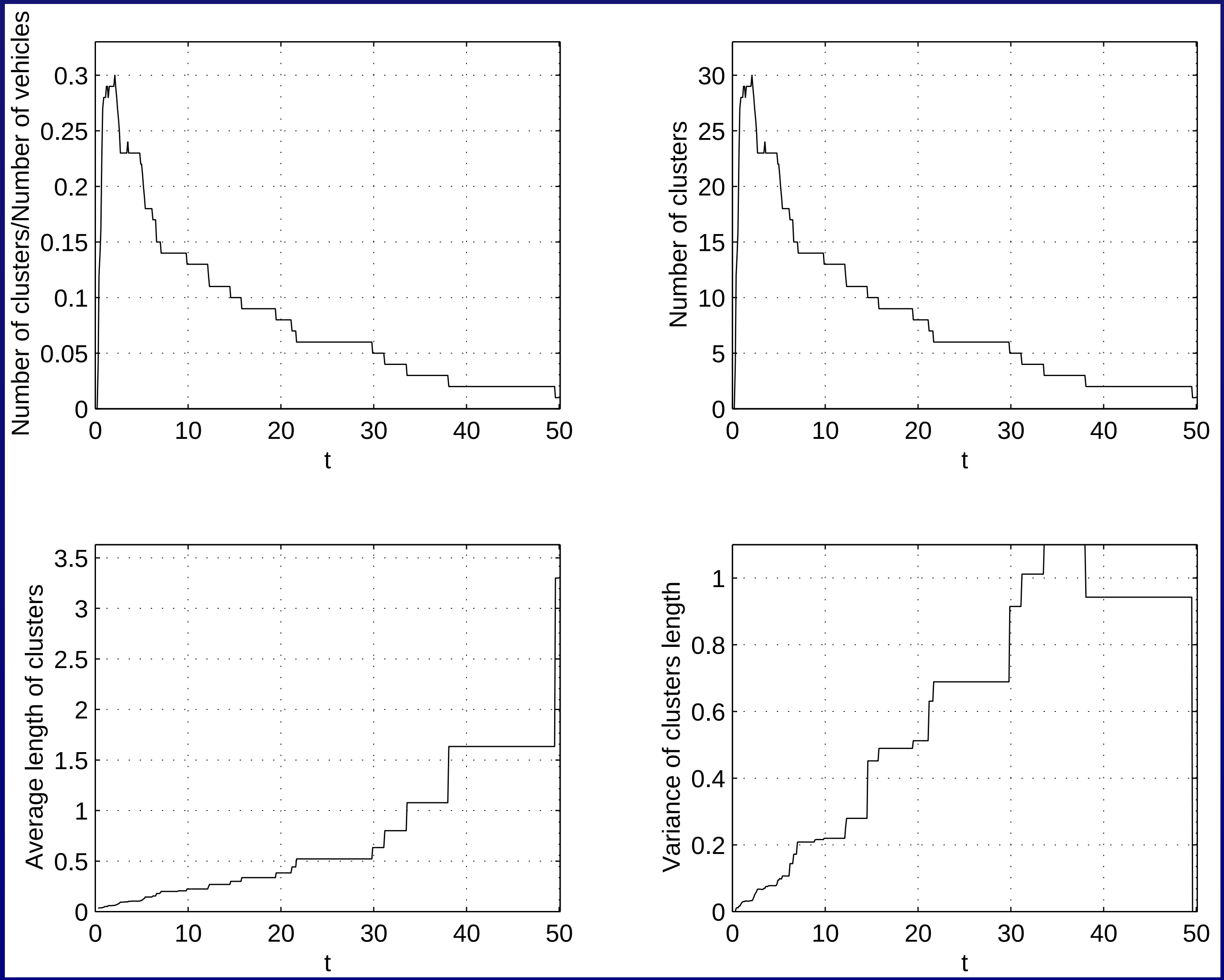
Riemann problem: vacuum formation



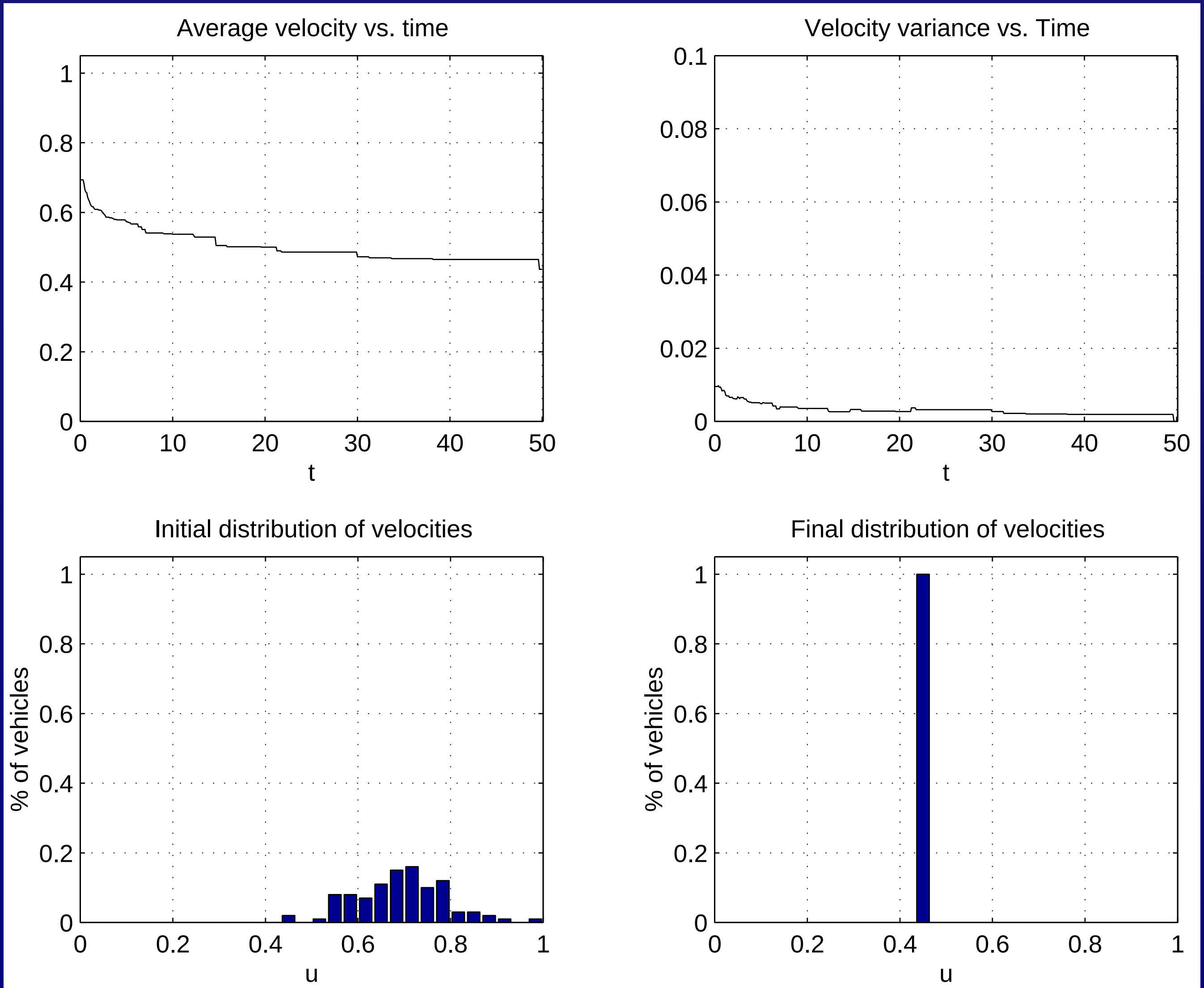
Random initial velocity: cluster formation



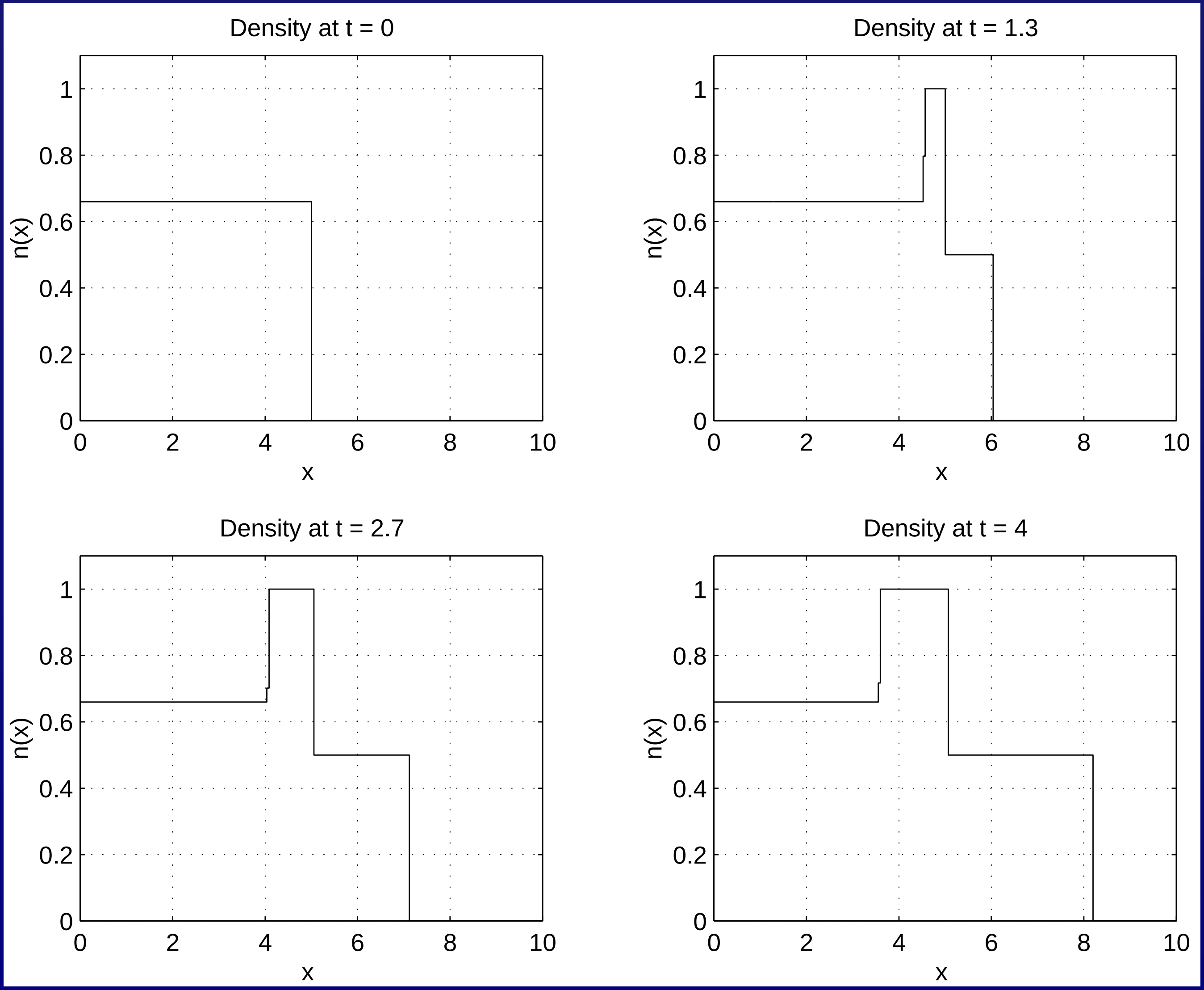
Random initial velocity: cluster statistics 43



Random initial velocity: velocity distribution



Bottelneck: jam formation



7. Conclusion

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- Modifi ed Aw-Rascle model
  - Density constraint
  - Rescaled for small difference between preferred velocity and actual velocity in uncongested situations

Summary

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  - Density constraint
  - Rescaled for small difference between preferred velocity and actual velocity in uncongested situations
- Limit model
  - Constrained Pressureless Gas Dynamics
  - Describes well cluster formation and dynamics
  - Existence theorem
  - Numerical simulations

Perspectives

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  - Convergence proof (RM-AR) $\longrightarrow$ (CPGD)
  - Lagrangian formulation and scheme

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- CPGD:
  - Convergence proof (RM-AR) $\longrightarrow$ (CPGD)
  - Lagrangian formulation and scheme
- More elaborate model
  - Density constraint depends on velocity (work in progress with P. degond, V. Leblanc, M. Rascle and J. Royer)
  - Multi-lane
  - Multi-class
  - etc.