Traffic Flow Modeling & Management
through
Conservation Laws

Rinaldo M. Colombo
Department of Mathematics
Brescia

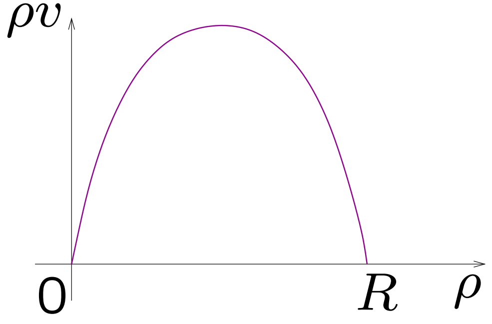
Traffic

Lighthill-Whitham (1955) & Richards (1956)

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$t \in [0, +\infty[$	time	$\rho = \rho(t, x)$	traffic density
$x \in \mathbf{R}$	space	$v = v(t, x)$	traffic speed

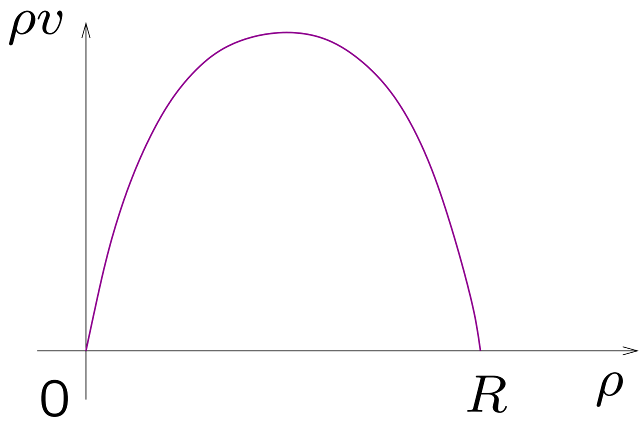
the number of vehicles is conserved and $v = v(\rho)$



R maximal density

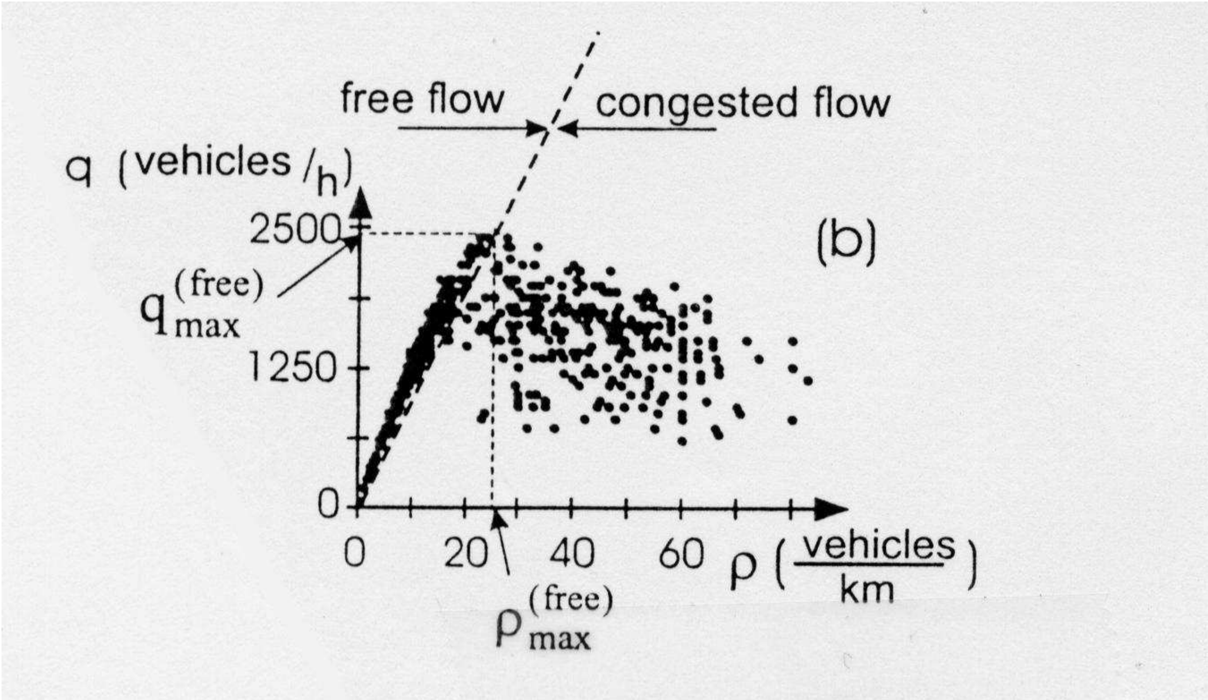
ρv traffic flow

Experimentally, the diagram



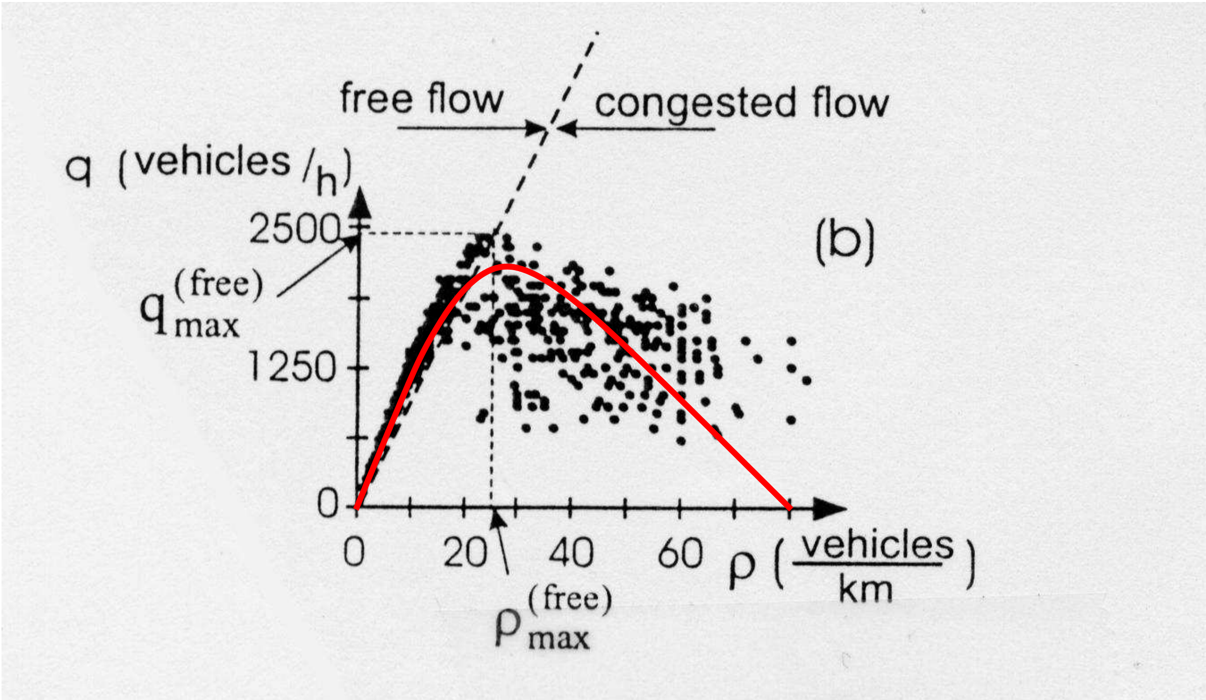
is not always observed!

A typical situation:



(Kerner (Daimler Benz AG), Phase Transitions in Traffic Flow, 2000)

A typical situation:



$$v = v(\rho)?$$

Free Flow

$\rho \in \Omega_f$

$$\partial_t \rho + \partial_x [\rho \cdot v] = 0$$

$$v(\rho) = \left(1 - \frac{\rho}{R}\right) \cdot V$$

(LWR)

Congested Flow

$(\rho, q) \in \Omega_c$

$$\begin{cases} \partial_t \rho + \partial_x [\rho \cdot v] = 0 \\ \partial_t q + \partial_x \left[(q - Q) \cdot v\right] = 0 \end{cases}$$

$$v(\rho, q) = \left(1 - \frac{\rho}{R}\right) \cdot \frac{q}{\rho}$$

(Colombo: Math.Comp.Mod. 2002)

$\rho \in [0, R]$ density; v speed; q weighted flow

R, Q, V road parameters

(Colombo: SIAM J.Appl.Math. 2002)

How to select *"good"* solutions to Riemann Problems?

Ω_f

$$\partial_t \rho + \partial_x [\rho \cdot v] = 0$$

Ω_c

$$\begin{cases} \partial_t \rho + \partial_x [\rho \cdot v] = 0 \\ \partial_t q + \partial_x \left[(q - Q) \cdot v\right] = 0 \end{cases}$$

Initially, the flow is free / congested $\forall\, x$

$\Downarrow$

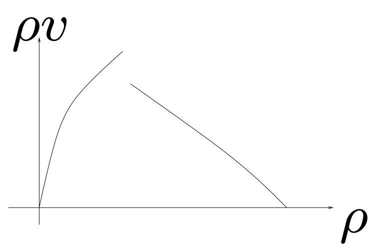
the solution is free / congested $\forall\, x$ e $\forall\, t$

(+ Phase transitions, Consistency, ...)

Qualitative properties:

Cauchy Problem:

- $\star$ existence and uniqueness for data in $\mathbf{BV}(\mathbf{R}, [0, R])$
  TV(initial data) not necessarily small
  independently from the number of initial phase boundaries
- $\star$ $\mathbf{L}^1$ Lipschitz dependence of solutions from the initial data
- $\star$ speed & density bounded and non negative
- $\star$ flow $= 0 \iff \rho \in \{0, R\}$
- $\star$ extends (Drake, Schofer, May; 1967)



(Colombo, Goatin, Priuli: preprint 2005)

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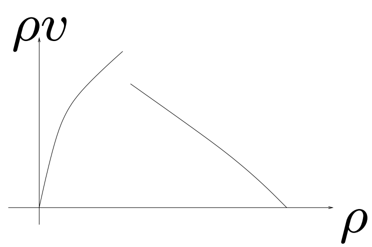
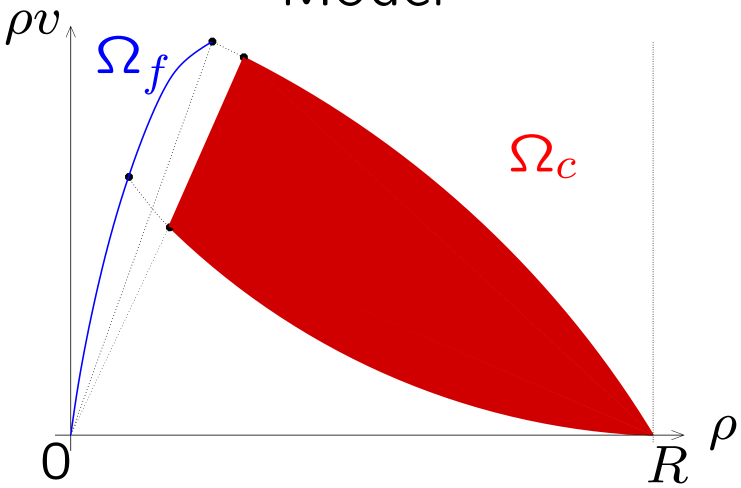
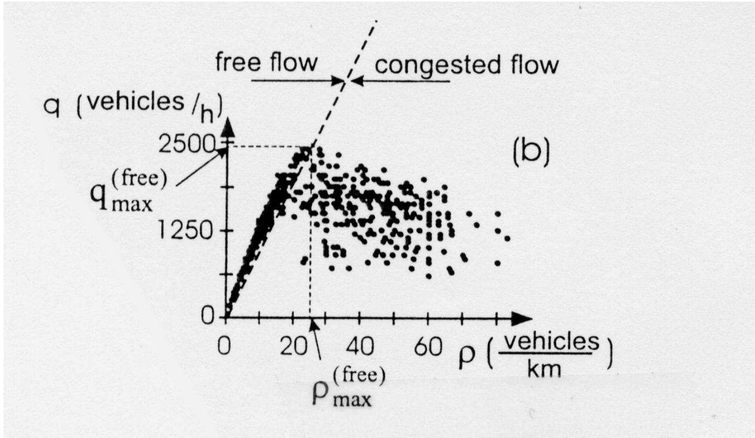


(Colombo, Goatin, Priuli: preprint 2005)

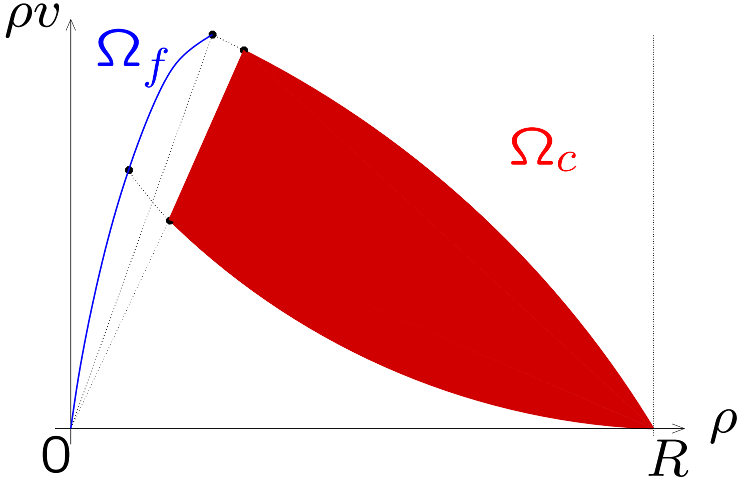
Model



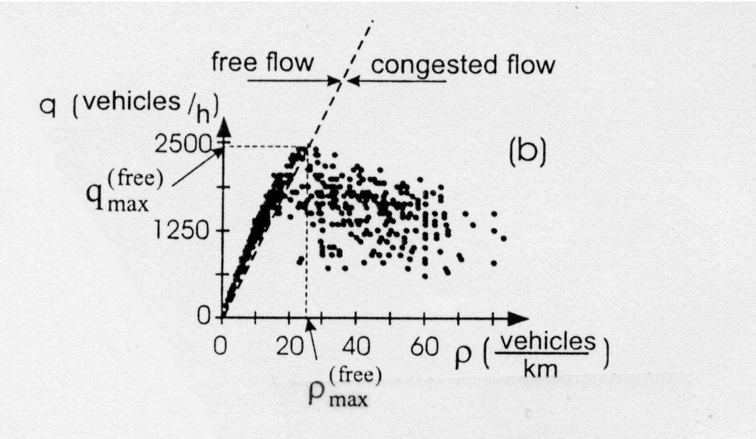
Experimental data



Model



Experimental data



⋆ Minimal requirements OK (flow = 0 $\iff \rho \in \{0, R\}$)

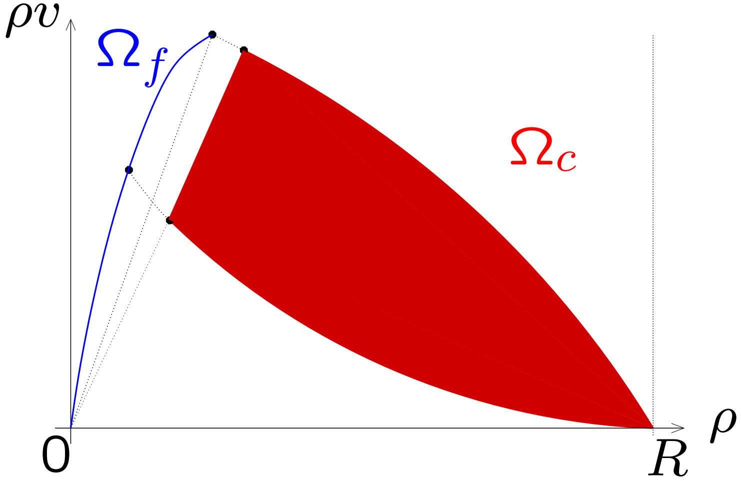
⋆ Different behaviors at a fixed density

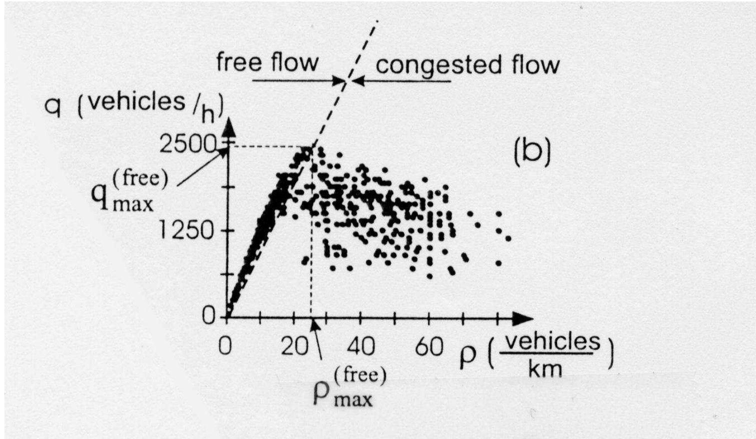
⋆ *Wide jams*

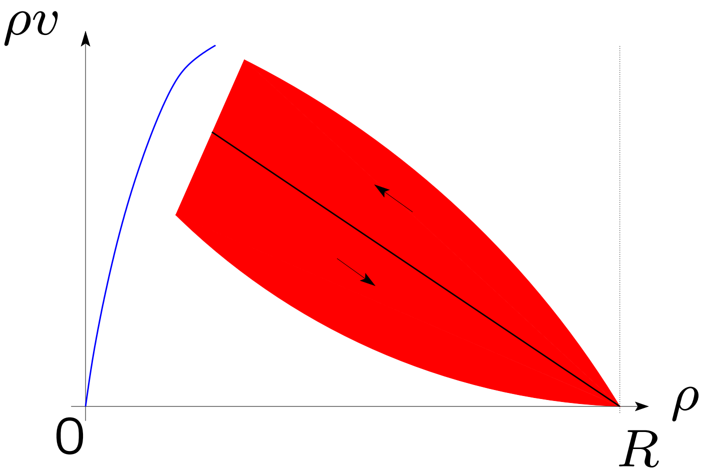
(Colombo: SIAM J.Appl.Math. 2002)

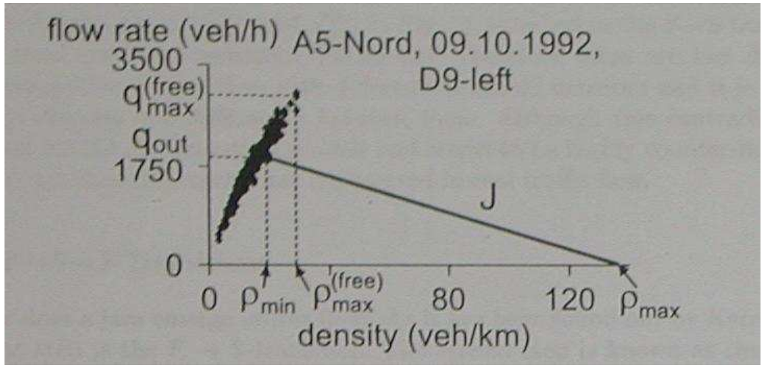
Model

Experimental data

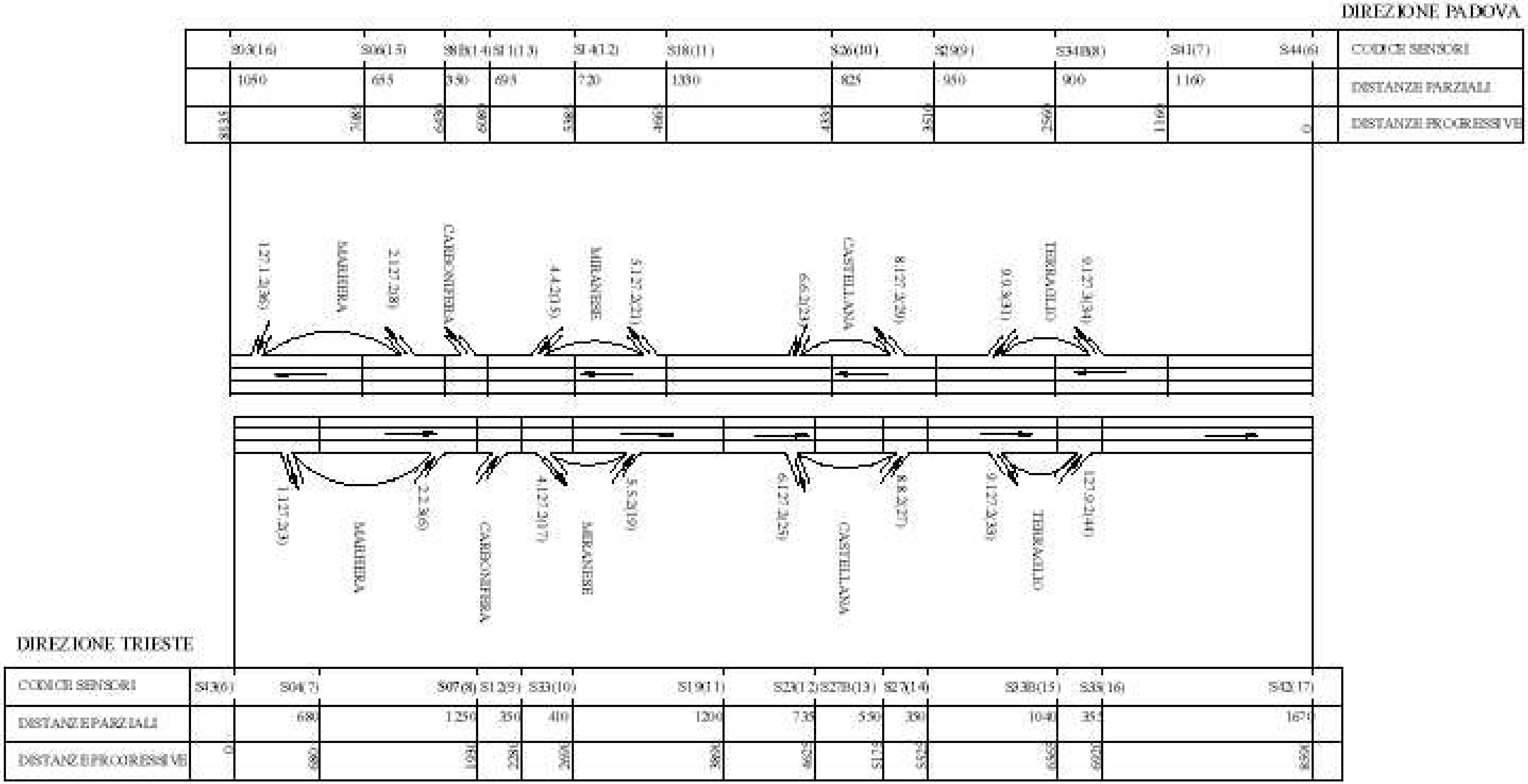




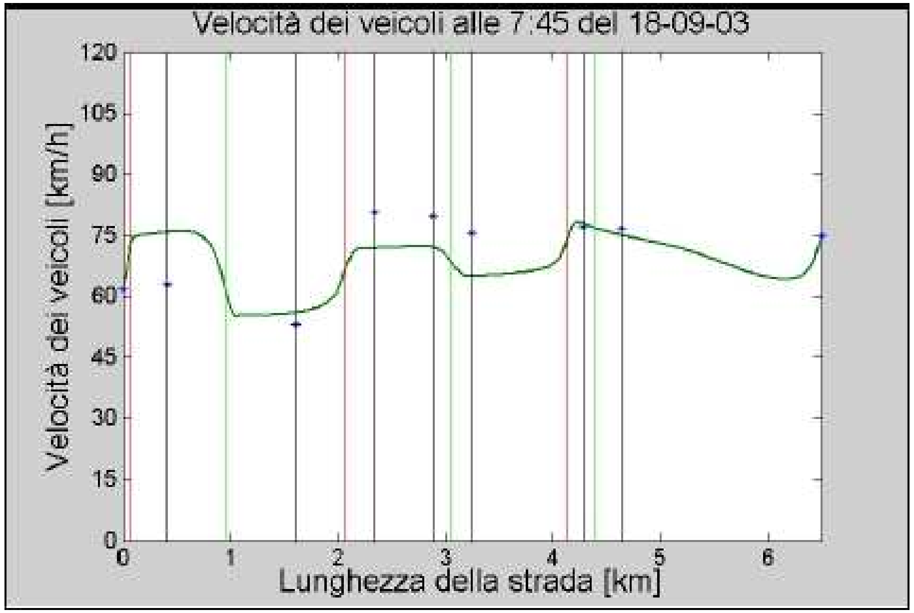




Venice Highway



Venice Highway



(with source terms! $\Rightarrow$ 5 parameters)

(Colombo, Maternini, Pedretti: Thesis, 2004)

Alternative construction:

Free Flow

$\rho \in \Omega_f$

$$\partial_t \rho + \partial_x [\rho \cdot v] = 0$$

$$v(\rho) = \left(1 - \frac{\rho}{R}\right) \cdot V_f$$

(LWR)

Congested Flow

$(\rho, q) \in \Omega_c$

$$\begin{cases} \partial_t \rho + \partial_x [\rho \cdot v] = 0 \\ \partial_t q + \partial_x [q \cdot v] = 0 \end{cases}$$

$$v(\rho, q) = \frac{q}{\rho} - V_c \ln \frac{\rho}{R}$$

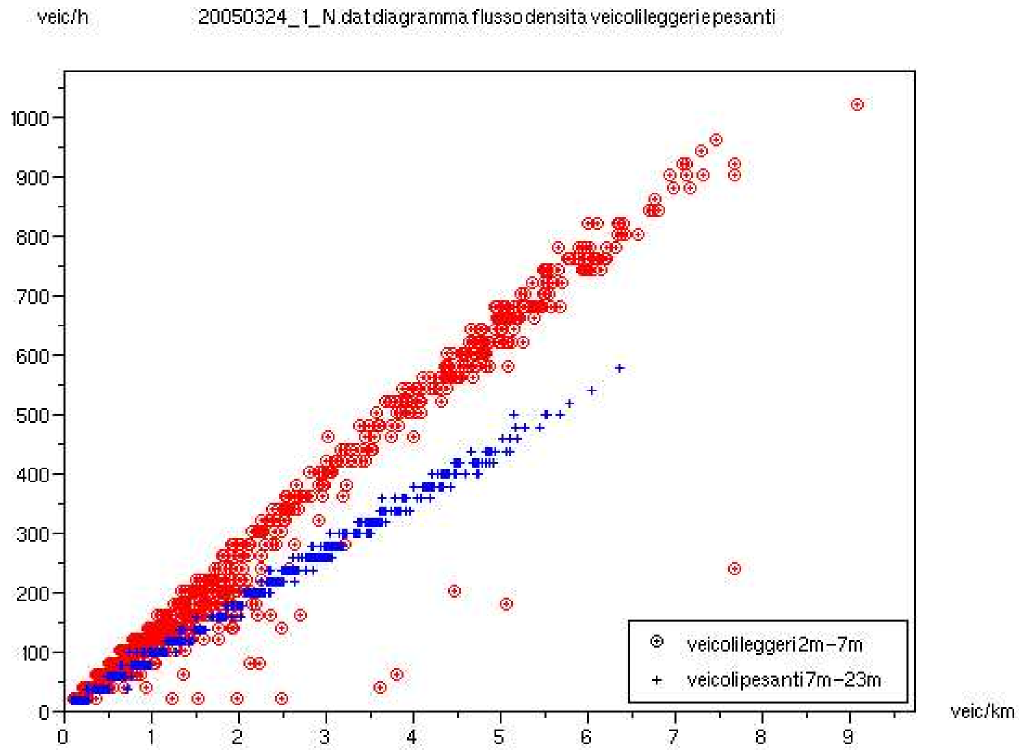
(Aw, Rascle: SIAM J.Appl.Math., 2000)

$\rho \in [0, R]$ density; v speed; $V \ln(\rho/R)$ pressure

R, V_f, V_c road parameters

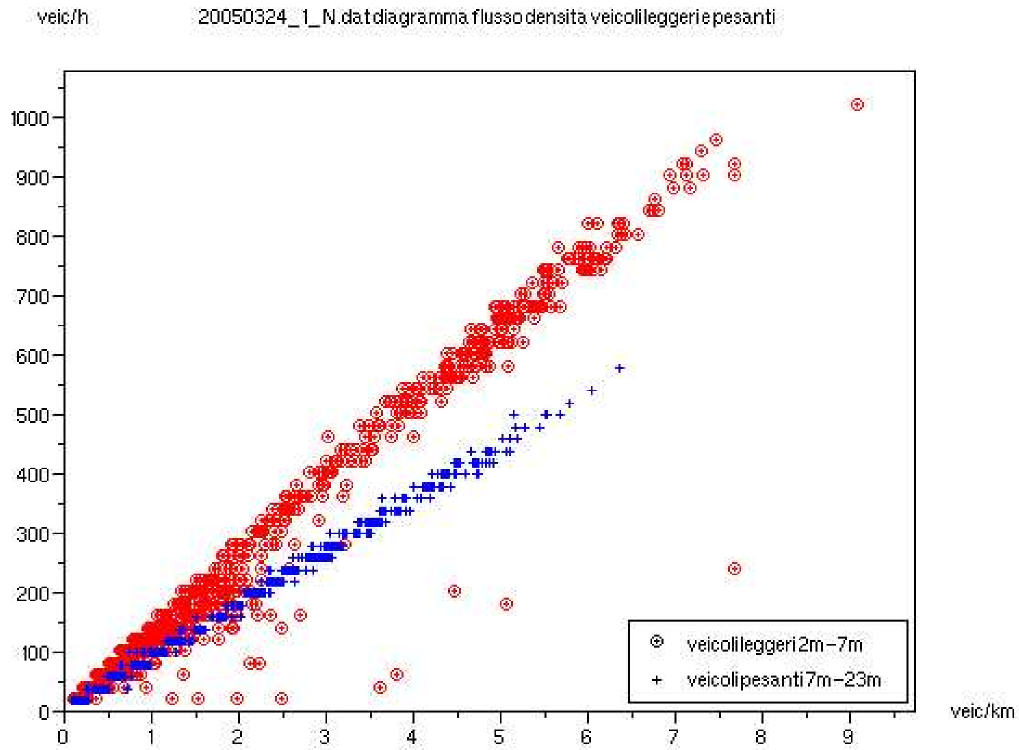
(Goatin, in preparation)

Another typical situation:



(Colombo, Maternini, Pedretti: work in progress.)

Another typical situation:



$$v = v(\rho)?$$

$$\text{Vehicles differ in} \quad \begin{cases} \text{speed} \\ \text{size} \\ \text{driving style} \\ \ldots \end{cases}$$

$$v = v(\rho)?$$

$$\Downarrow$$

$$\boxed{\boxed{\text{many populations}}}$$

1 population $\quad \partial_t \rho \quad + \quad \partial_x \left[\rho \cdot V \cdot \left(1 - \frac{\rho}{R}\right)\right] \quad = \quad 0$

1 population $\partial_t \rho + \partial_x \left[\rho \cdot V \cdot \left(1 - \frac{\rho}{R} \right) \right] = 0$

n populations $\partial_t \rho_i + \partial_x \left[\rho_i \cdot V_i \cdot \left(1 - \frac{\rho_1 + \cdots + \rho_n}{R} \right) \right] = 0$

1 population	$\partial_t \rho$	$+$	$\partial_x \left[\rho \cdot V \cdot \left(1 - \frac{\rho}{R}\right)\right]$	$=$	0
n populations	$\partial_t \rho_i$	$+$	$\partial_x \left[\rho_i \cdot V_i \cdot \left(1 - \frac{\rho_1 + \cdots + \rho_n}{R}\right)\right]$	$=$	0

where: $i = 1, \ldots, n$

ρ_i density of the i-th population

V_i maximal speed of the i-th population

in the simplex $\rho_i \geq 0$, $\sum_{i=1}^{n} \rho_i \leq R$.

$$\partial_t \rho_i + \partial_x \left[\rho_i \cdot V_i \cdot \left(1 - \frac{\rho_1 + \cdots + \rho_n}{R} \right) \right] = 0$$

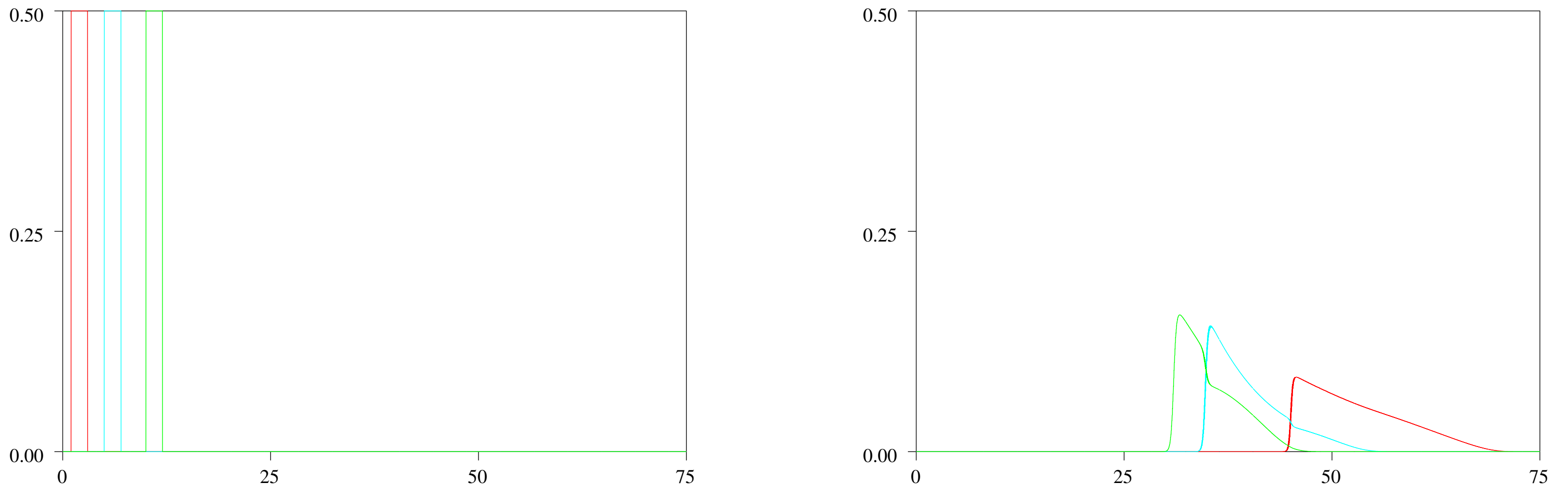
+ Simmetrizable ($\Rightarrow$ Hyperbolic);
+ Explicit Entropy – Entropy Flux;
+ Information carried by vehicles ($\max \lambda_i \leq \max v_i$);
+ If a population is missing ...

– Umbilic points – lines – surfaces;

– Multiple intersections between Lax curves;

+ Overtakings are possible!

Overtakings are possible!



$$(n = 3;\ V_1 = 1.3,\ V_2 = 0.9,\ V_3 = 0.6)$$

(Benzoni–Gavage & Colombo: Europ.J.Appl.Math. 2003)

“*Reverse*” Kinetic Limit

$$\partial_t \rho_i(t,x) + \partial_x \left[V_i \rho_i(t,x) \left(1 - \frac{\sum_j \rho_j(t,x)}{R} \right) \right] = 0$$

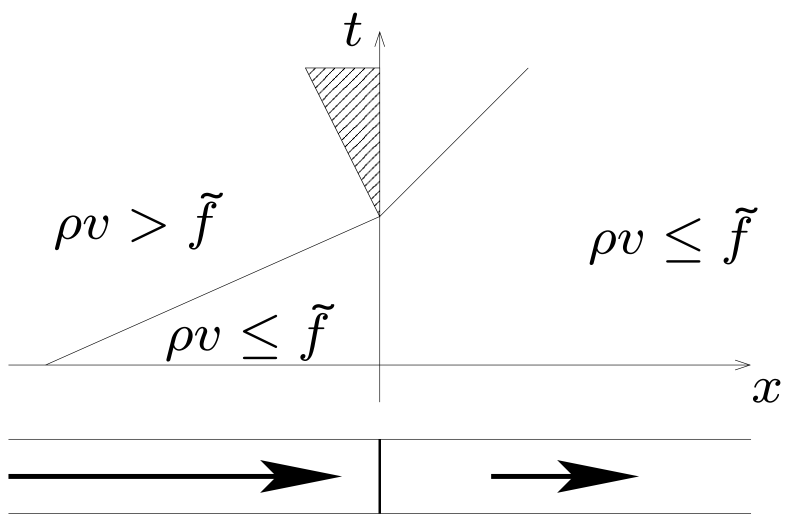
$$\downarrow \text{ for } \boxed{\boxed{n \to +\infty}}$$

$$\partial_t f(t,x;V) + \partial_x \left[V f(t,x;V) \left(1 - \frac{\int f(t,x;W)\, dW}{R} \right) \right] = 0$$

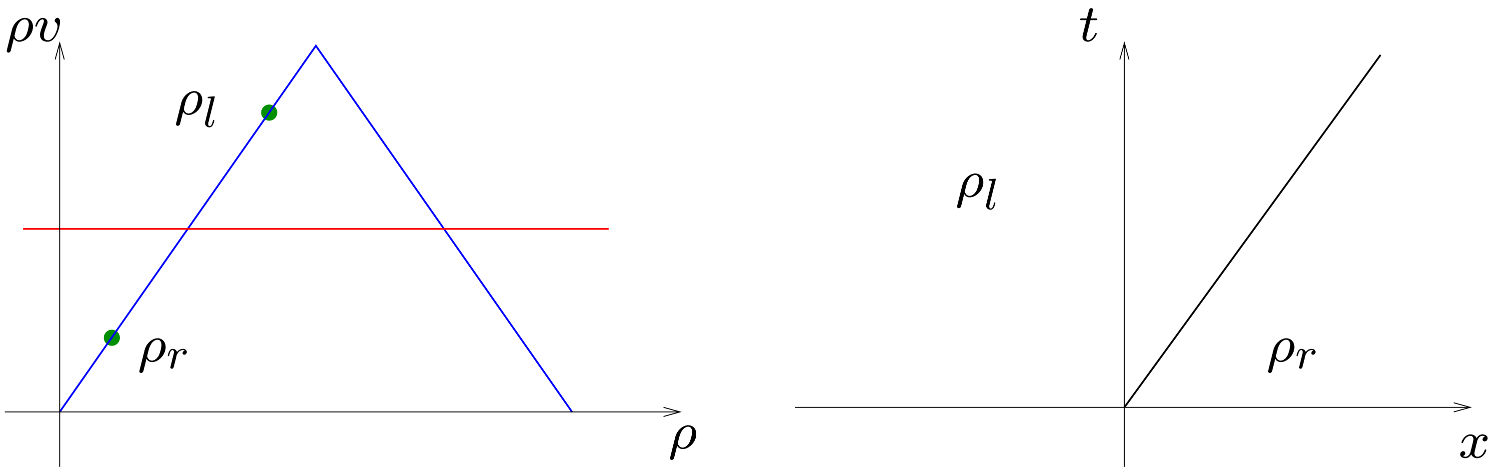
(Benzoni–Gavage, Colombo, Gwiazda: preprint 2004)

Toll Gate

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho v(\rho)\right) = 0 & (t,x) \in [0,+\infty[\times \mathbf{R} \\ \rho(0,x) = \rho_o(x) & x \in \mathbf{R} \\ (\rho\, v)\,(t,0) \leq \tilde f & t \in [0,+\infty[\end{cases}$$

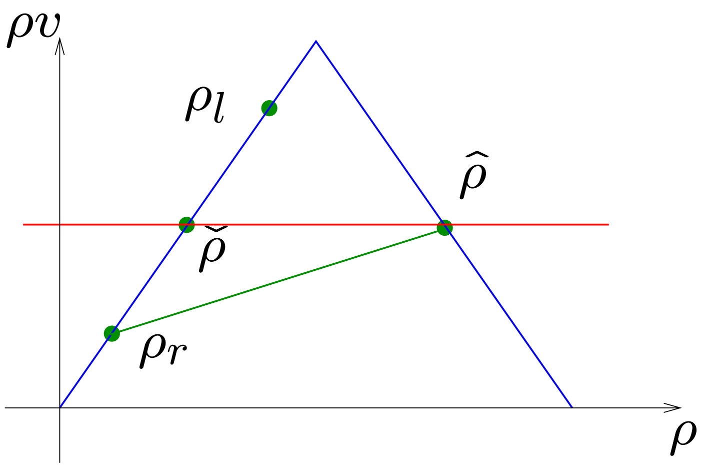


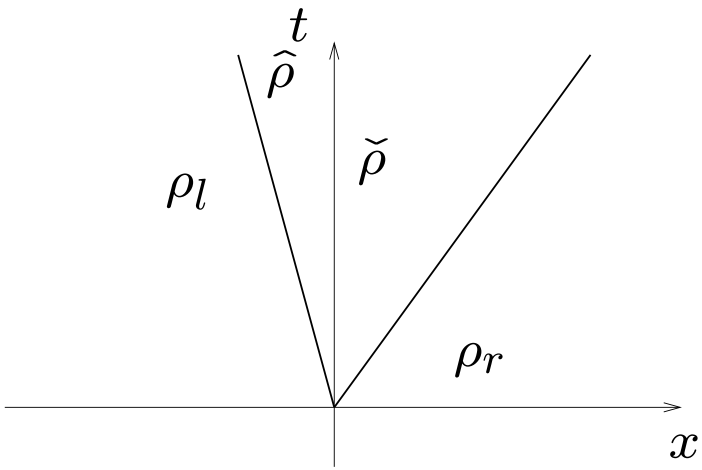
Toll Gate



Standard solution, $(\rho\, v)(t, 0) > \tilde{f}$

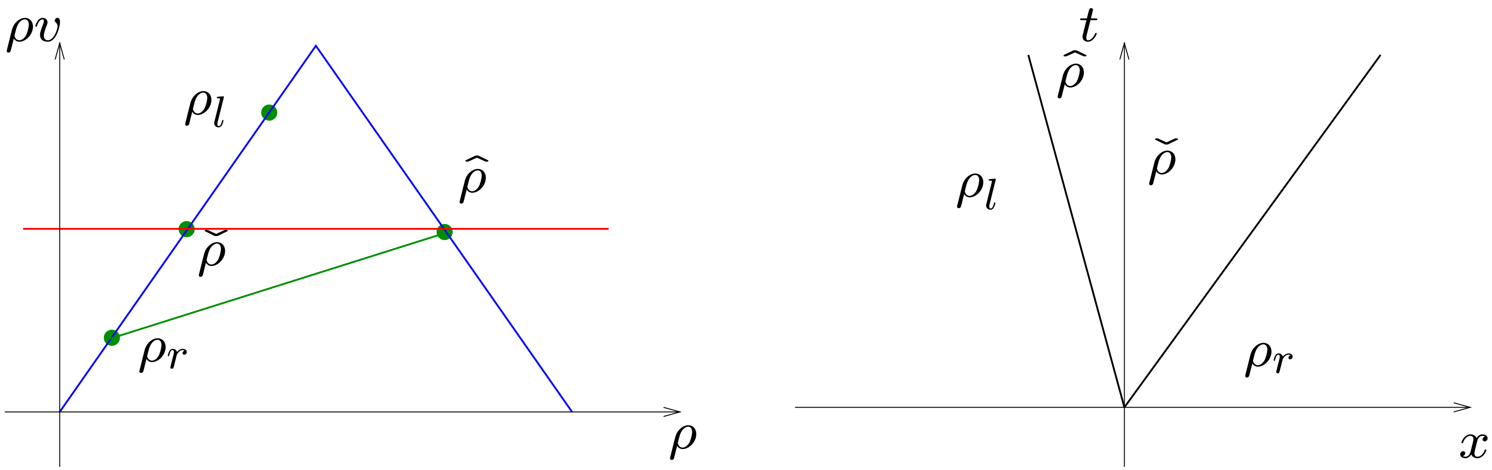
Toll Gate





$$(\rho\, v)(t, 0) < \tilde{f}$$

Toll Gate

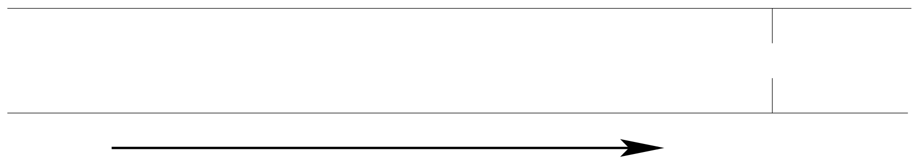


Existence for the Cauchy problem with bounded TV.

(Colombo, Goatin: work in progress)

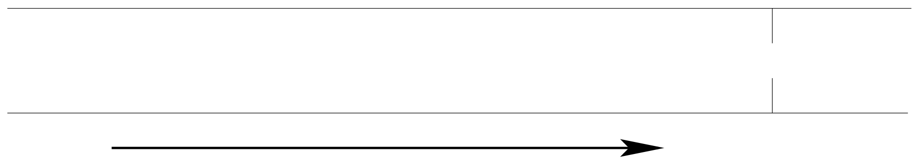
(Piccoli, Terracina: work in progress)

Pedestrian Flow

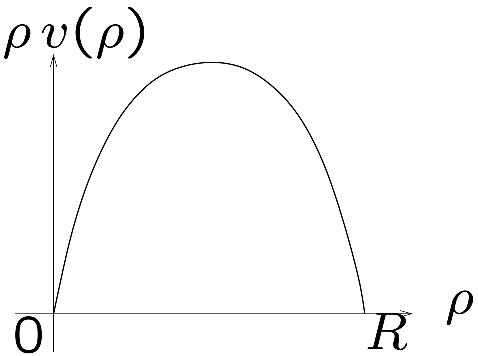


$$\partial_t \rho + \partial_x \left[\rho \, v(\rho) \right] = 0$$

Pedestrian Flow

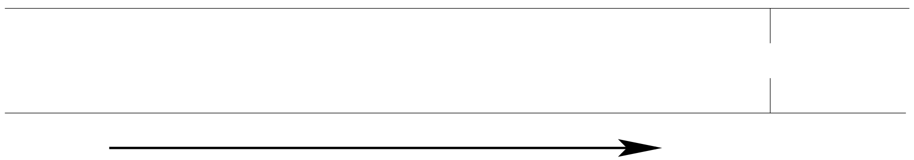


$$\partial_t \rho + \partial_x \left[\rho \, v(\rho)\right] = 0$$

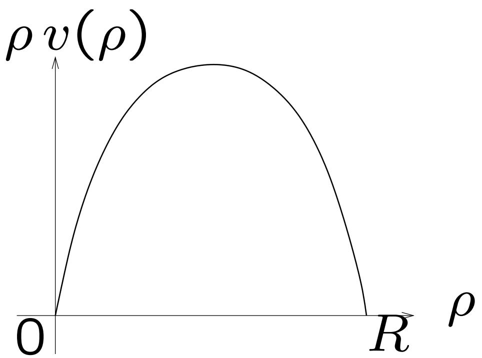


Maximum Principle

Pedestrian Flow



$$\partial_t \rho + \partial_x \left[\rho\, v(\rho)\right] = 0$$

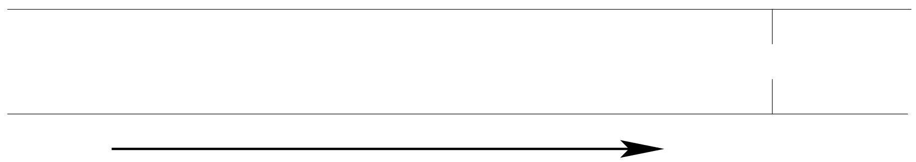


Maximum Principle

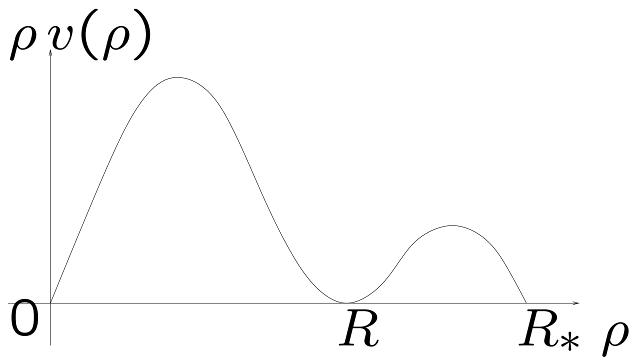
No panic effects!

(Helbing, Ferrara 2003)

Pedestrian Flow



$$\partial_t \rho + \partial_x \left[\rho\, v(\rho)\right] = 0$$

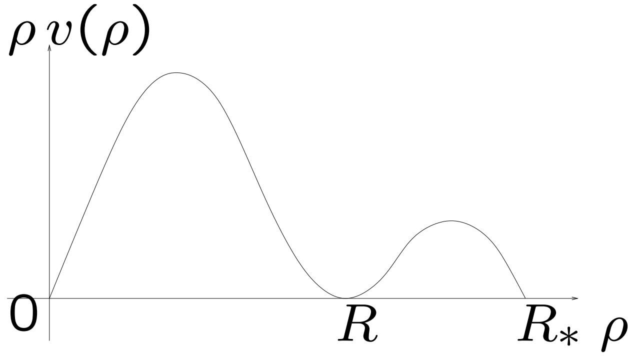


"overcompression"

NON Classical Shocks

No Maximum Principle

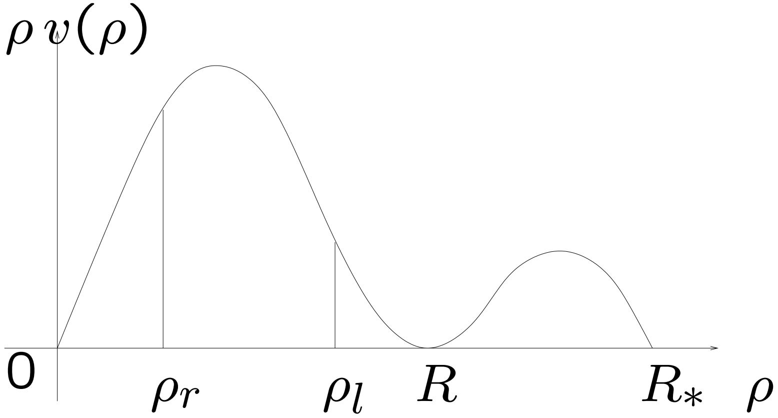
Riemann Problem for Pedestrian Flow

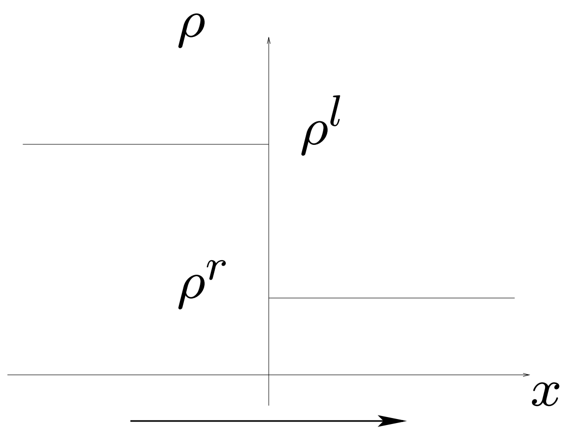


$[0, R]$ standard states

$]R, R_*]$ "panic" states

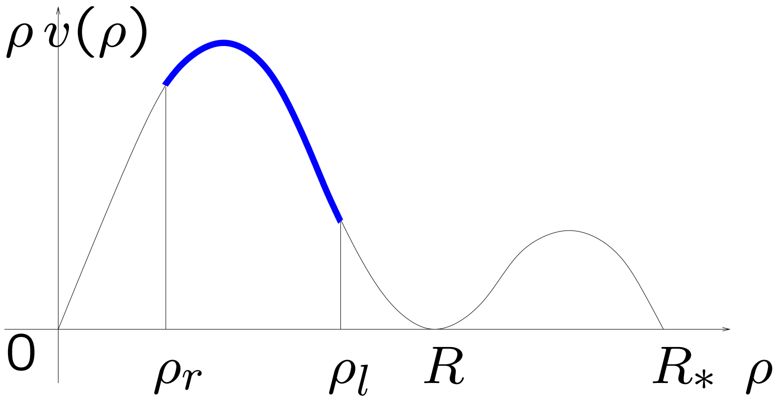
Riemann Problem for Pedestrian Flow

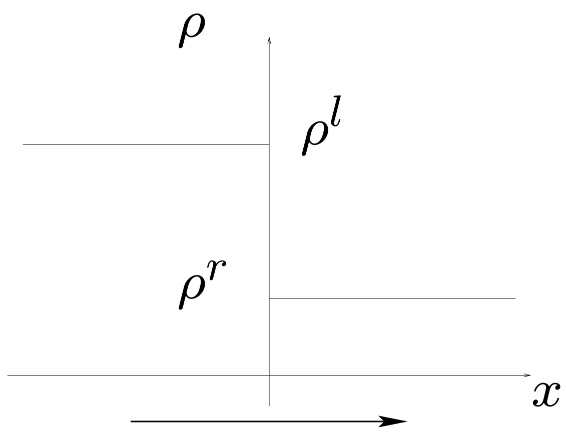




$$\rho_l > \rho_r \quad \Rightarrow \quad \text{standard solution}$$

Riemann Problem for Pedestrian Flow

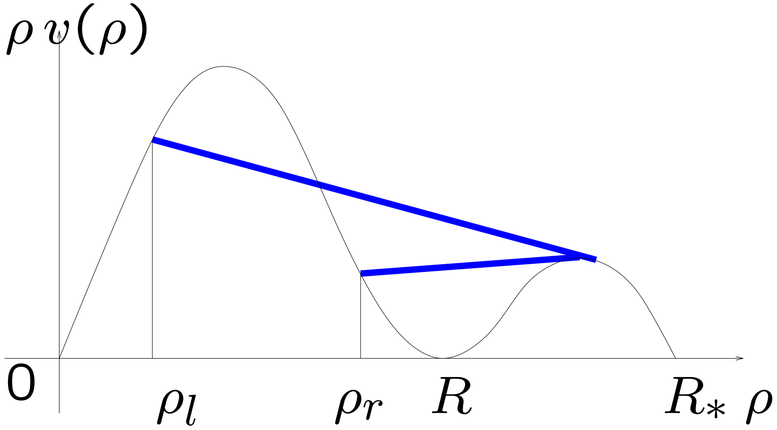


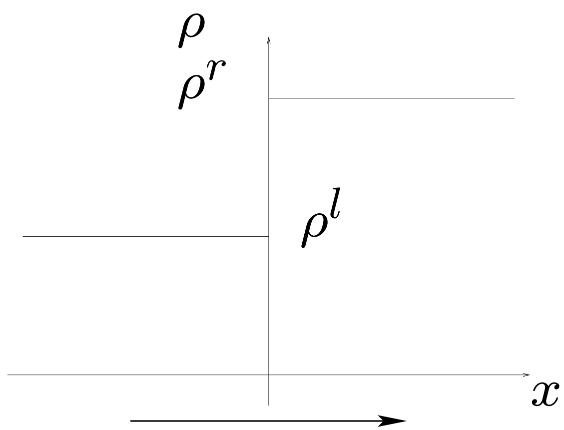


$\rho_l > \rho_r \quad \Rightarrow \quad$ standard solution

rarefaction

Riemann Problem for Pedestrian Flow



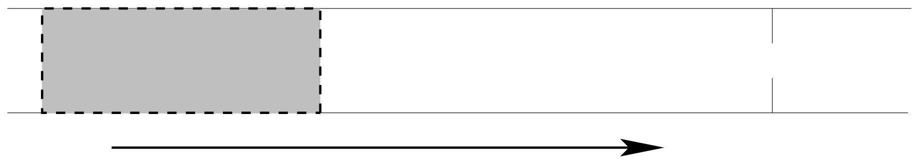


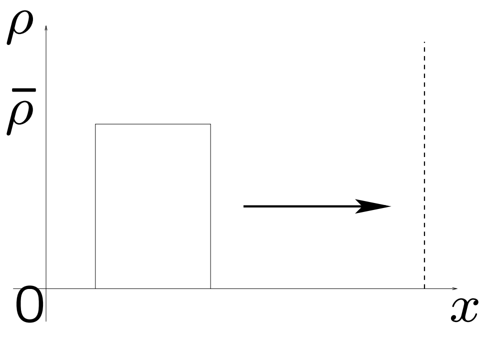
$$\left.\begin{array}{r} \rho_r - \rho_l > \text{ threshold} \\ \rho_l > \text{ threshold} \end{array}\right\} \Rightarrow$$

nonclassical solution

shock rarefaction shock

Pedestrian Flow

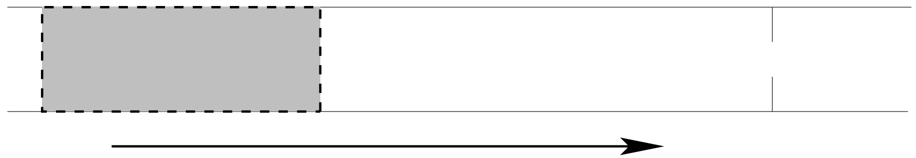


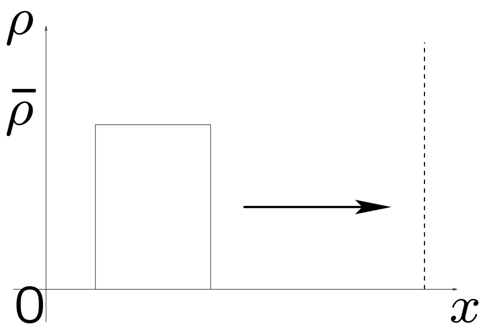


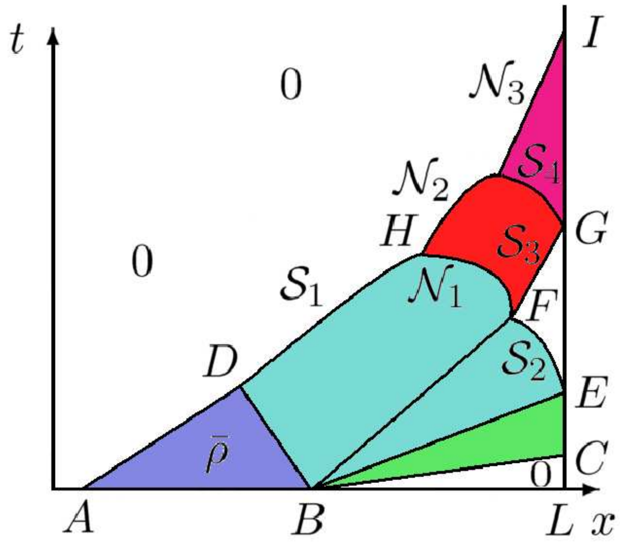
Riemann Problems

(Colombo, Rosini: MMAS, 2005. Chalons: preprint 2005)

Pedestrian Flow







(Colombo, Rosini: MMAS, 2005. Chalons: preprint 2005)

Modeling

Modeling

Control & Optimization

Modeling

Control & Optimization

Management

Control and Optimization for Traffic Flows

Control Parameters

1. Inflow
2. Maximal speed
3. (Road)

Control and Optimization for Traffic Flows

Control Parameters	Target
1. Inflow	1. Minimize travel time
2. Maximal speed	2. Maximize flow
3. (Road)	3. Optimize safety

(Ancona, Marson: SIAM J.Control Opt., 1998

Ancona, Marson: Nonlinear Analysis, 1999)

Control and Optimization for Traffic Flows

Control Parameters	Target
1. Inflow	1. Minimize travel time
2. Maximal speed	2. Maximize flow
3. (Road)	3. Optimize safety

$$\text{inflow} \mapsto \text{IBVP} \mapsto \text{solution} \mapsto \text{cost}$$

Control and Optimization for Traffic Flows

Control Parameters	Target
1. Inflow	1. Minimize travel time
2. Maximal speed	2. Maximize flow
3. (Road)	3. Optimize safety

inflow $\mapsto$ IBVP $\mapsto$ solution $\mapsto$ cost

$$\text{Cost} = \int_0^T \int_{\mathbf{R}} F\left(\rho(t,x)\right)\,dx\,dt$$

Necessary Conditions?

(Colombo, Groli: Calc.Var.P.D.E. 2004)

Control and Optimization for Traffic Flows

Control Parameters	Target
1. Inflow	1. Minimize travel time
2. Maximal speed	2. Maximize flow
3. (Road)	3. Minimize TV(speed) (Stop & go waves)

Control and Optimization for Traffic Flows

Control Parameters

1. Inflow
2. Maximal speed
3. (Road)

Target

1. Minimize travel time
2. Maximize flow
3. Minimize TV(speed) (Stop & go waves)

inflow $\mapsto$ IBVP $\mapsto$ solution $\mapsto$ cost

$$\text{Cost} = \int_0^T \int_{\mathbf{R}} \text{weight}(t,x)\ d\left|\partial_x v\left(\rho(t,x)\right)\right| dt$$

(Colombo, Groli: Appl.Math.Letters 2003)