A categorical duality for algebras of partial functions

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Abstract
We prove a categorical duality between a class of abstract algebras of partial functions and a class of (small) topological categories. The algebras are the isomorphs of collections of partial functions closed under the operations of composition, antidomain, range, and preferential union (or ‘override’). The topological categories are those whose space of objects is a Stone space, source map is a local homeomorphism, target map is open, and all of whose arrows are epimorphisms.

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1 Introduction

Variants and extensions of Stone duality are pervasive in logic and computer science: for example in modal [11], intuitionistic [6], substructural [5], and many-valued [10] logic, and in semantics [1], formal language theory [9], and logics for static analysis [3]. In its most basic form—between Boolean algebras and Stone spaces—it provides a duality for the isomorphs of algebras of unary relations equipped with the union and complement operations. One extremely prominent ‘real world’ example of such an algebra is the set of regular languages (for a fixed finite alphabet). Indeed, recently it has been shown how an extended Stone duality is the explanation behind many of the great successes of algebraic language theory [9, 8].

Another genre of algebras of relations that has been studied is algebras of partial functions [4, 13, 12, 15]. Here the algebras are formed of collections of partial functions closed under certain natural operations such as composition or ‘preferential union’. Again, preexisting examples can be found in automata/formal language theory. Transducers are finite state machines that take words as input and produce words as output. In general they realise word-to-word relations, but certain collections of word-to-word partial functions defined via transducers are considered important—we think particularly of the rational functions, and the regular functions [7].

In this paper, motivated by the utility of duality applied to regular languages, we give a description of a duality (Theorem 3.6) applicable to both the rational functions and the regular functions. Specifically, on one side is the category of isomorphs of algebras of partial functions equipped with four operations: composition, antidomain, range, and preferential union (see Section 2). On the other side is the category of (small) categories each equipped with a topology and satisfying certain extra conditions (see Section 3). This duality is a generalisation of a duality due to Mark Lawson between a certain subclass of inverse semigroups and certain topological groupoids [14]. Recall that the inverse semigroups are the isomorphs of injective partial functions. In fact the duality presented here is not only a generalisation with respect to algebras encompassed, but also the morphisms of algebras. In
our duality, the morphisms of algebras are exactly the homomorphisms, whereas in [14] they are a restricted type of homomorphism only.

In Section 2 we define formally our algebras of functions, and the category they constitute. In Section 3 we do the same for the small topological categories on the other side of our duality. In Section 4, we describe one half of the duality: the functor from algebras to topological categories. In Section 5, we describe the remaining half of the duality: the functor from topological categories to algebras. In Section 6, we prove that these two functors do indeed form a contravariant equivalence of categories. In Section 7, we say a little about the duality as it applies to the rational and regular functions.

2 Algebras of functions

Given an algebra $A$, when we write $a \in A$ or say that $a$ is an element of $A$, we mean that $a$ is an element of the domain of $A$. Similarly for the notation $S \subseteq A$ or saying that $S$ is a subset of $A$. We follow the convention that algebras are always nonempty. If $S$ is a subset of the domain of a map $\theta$ then $\theta[S]$ denotes the set $\{ \theta(s) \mid s \in S \}$, and similarly for operations with multiple arguments. As is common in algebraic logic, compositions denoted with the symbol $;$ are written with the first composee on the left, that is, contrary to the usual mathematical convention.

We begin by making precise what is meant by partial functions and algebras of partial functions.

Definition 2.1. Let $X$ be a set. A partial function on $X$ is a subset $f$ of $X \times X$ satisfying

$$(x, y) \in f \text{ and } (x, z) \in f \implies y = z. \quad (1)$$

Definition 2.2. Let $\sigma \subseteq \{ ;, A, R, \sqcup \}$ be a functional signature. An algebra of partial functions of the signature $\sigma$ is a universal algebra $A = (A, \sigma)$ where the elements of the universe $A$ are all partial functions on some (common) set $X$, the base, and the interpretations of the symbols are given as follows:

- The binary operation $;$ is composition of partial functions.
- The unary operation $A$ is the operation of taking the diagonal of the antidomain of a partial function:

\[
A(f) := \{(x, x) \in X^2 \mid \exists y \in X : (x, y) \in f\}.
\]

- The unary operation $R$ is the operation of taking the diagonal of the range of a partial function:

\[
R(f) := \{(y, y) \in X^2 \mid \exists x \in X : (x, y) \in f\},
\]

- the binary operation $\sqcup$ is preferential union:

\[
(f \sqcup g)(x) = \begin{cases} 
  f(x) & \text{if } f(x) \text{ defined} \\
  g(x) & \text{if } f(x) \text{ undefined, but } g(x) \text{ defined} \\
  \text{undefined} & \text{otherwise}
\end{cases}
\]

Note that (despite the symmetry of the symbol $\sqcup$) preferential union is not a commutative operation.
Definition 2.3. An algebra $\mathfrak{A}$ of the signature $\sigma$ is **representable** by partial functions if it is isomorphic to an algebra of partial functions of the signature $\sigma$. An isomorphism from $\mathfrak{A}$ to an algebra of partial functions is a **representation** of $\mathfrak{A}$.

We begin by looking at representable $\{; A, R\}$-algebras, but we will soon see that the algebras we are interested in are equivalent to the representable $\{; A, R, \sqcup\}$-algebras.

Remark 2.4. Note that the constants $0$ (empty function) and $1'$ (identity function), the operation $D$ (domain), and the relation $\leq$ (subset), are all definable in the signature $\{; A, R\}$. That is, the term $0 := A(a) : a$, the term $1' := A(0)$, and the term $D(a) := A(A(a))$, are all necessarily represented in the intended way by any representation, and the relation

$$a \leq b \iff D(a) : b = a$$

(2)

corresponds via any representation precisely to the subset relation on the image of the representation.

Statements involving order will always be with respect to $\leq$.

Remark 2.5. The representable $\{; A, R\}$-algebras form a proper quasivariety, axiomatised by a finite number of quasiequations [12, Theorem 4.1].

Definition 2.6. Two elements $a$ and $b$ of an algebra of the signature $\{; A, R\}$ are **compatible** if $D(a) : b = D(b) : a$.

Clearly in any representable $\{; A, R\}$-algebra the compatibility relation expresses precisely that for any representation the representing functions agree on their common domain. In such an algebra compatibility is necessary for the existence of a least upper bound of a pair $a$ and $b$. If $a$ and $b$ have an upper bound, $c$ say, they have a least upper bound ($\sqcup$) given by $A(A(a) : A(b)) : c$. From this term, we see that in concrete algebras any binary joins must be given by binary unions.


Proof. Let $h : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of $\{; A, R\}$-algebras, and suppose $a, b \in \mathfrak{A}$ have an upper bound. Since the algebras are representable, we may assume they are algebras of partial functions. Clearly $h$ is order preserving, so $h(a \sqcup b)$ is an upper bound for $\{h(a), h(b)\}$, that is, $h(a), h(b) \subseteq h(a \sqcup b)$. On the other hand

$$h(a \sqcup b) = h(D(a) \sqcup D(b)) ; (a \sqcup b) = (D(h(a)) \cup D(h(b)) ; h(a \sqcup b).$$

(We know joins correspond to unions on the subalgebra of elements of the form $D(-)$, since join is expressible there, as $A(A(-) : A(-))$. Hence $h(a \sqcup b) \subseteq h(a) \cup h(b)$, and so $h(a \sqcup b)$ is the smallest possible upper bound for $h(a)$ and $h(b)$. $h(a) \cup h(b)$.

Let $\mathfrak{A}$ be the subclass of the representable $\{; A, R\}$-algebras consisting of those validating the first-order condition that every compatible pair has an upper bound.

Corollary 2.8. The category consisting of $\mathfrak{A}$ with $\{; A, R\}$-homomorphisms is isomorphic to the category of representable $\{; A, R, \sqcup\}$-algebras with $\{; A, R, \sqcup\}$-homomorphisms.

Proof. In any representable $\{; A, R\}$-algebra in which compatible pairs have upper bounds, the operation $\sqcup$ is definable as $a \sqcup b := a \sqcup A(a) : b$. And there is an inverse interpretation of any representable $\{; A, R, \sqcup\}$-algebra as a representable $\{; A, R\}$-algebra with compatible joins, since for compatible $a$ and $b$ we have $a \sqcup b = a \sqcup b$. It remains to see that the morphisms are the same. Since $\{; A, R\}$-homomorphisms preserve binary joins, they must preserve $\sqcup$, since $\sqcup$ is then definable in terms of preserved operations.
A categorical duality for algebras of partial functions

Let $\Sigma$ be a finite alphabet. The rational functions are the partial functions from $\Sigma^*$ to $\Sigma^*$ realisable by a one-way transducer. The regular functions are the partial functions from $\Sigma^*$ to $\Sigma^*$ realisable by a two-way transducer. The rational and the regular functions are both closed under composition, antiodomain, and range, and also under the partial operation of compatible union. These classes of partial functions are not closed under other familiar operations that we may be tempted to include in the signature, such as intersection and relative complement. This is the reason for our interest in the class $\mathbf{A}$.

We mention one other important class of partial functions important to the theory of transducers. The sequential functions are those partial functions realised by one-way input-deterministic transducers. However, the sequential functions do not fit within our framework for the reason that they are not closed under compatible unions. (For example, the sequential functions $a^n \mapsto a^n$ and $a^nb \mapsto b^n$ have disjoint domains, hence are compatible, but their union is not sequential.)

In view of Corollary 2.8, we can choose to work with the representable $\{; A, R, \sqcup\}$-algebras, in lieu of $\mathbf{A}$, and henceforth that is what we will do. This pays off immediately: the class has a syntactically simple finite axiomatisation (and therefore is algebraically well behaved).

$\blacktriangleright$ Theorem 2.9 (Hirsch, Jackson, and Mikulás [12, Corollary 4.2 + Lemma 3.6]). The representable $\{; A, R, \sqcup\}$-algebras form a proper quasivariety, axiomatised by the following finite list of equations and quasiequations.

\[
a ; (b ; c) = (a ; b) ; c \quad (3)
\]
\[
A(a) ; a = A(b) ; b \quad (4)
\]
\[
1' ; a = a \quad (5)
\]
\[
a ; A(b) = A(a ; b) ; a \quad (6)
\]
\[
D(a) ; b = D(a) ; c \land A(a) ; b = A(a) ; c \quad \Rightarrow \quad b = c \quad (7)
\]
\[
D(R(a)) = R(a) \quad (8)
\]
\[
a ; R(a) = a \quad (9)
\]
\[
a ; b = a ; c \quad \Rightarrow \quad R(a) ; b = R(a) ; c \quad (10)
\]
\[
D(a) ; (a \sqcup b) = a \quad (11)
\]
\[
A(a) ; (a \sqcup b) = A(a) ; b \quad (12)
\]

The category of representable $\{; A, R, \sqcup\}$-algebras and their homomorphisms is the first of the two categories between which we will exhibit a duality.

We now introduce a small running example by starting with an eight-element $\{; A, R, \sqcup\}$-algebra. Though finite algebras cannot inform very much about the topological aspect of our duality—their duals all have discrete topologies—the example will be sufficient to grasp the essence of the duality.

$\blacktriangleright$ Example 2.10. Let $\mathfrak{A}$ be the following collection of partial functions on the set $\{1, 2, 3\}$. The empty function, $\emptyset$, the identity $id_{\{1,2\}}$ on $\{1, 2\}$, the identity $id_{\{3\}}$ on $\{3\}$, the identity $id_{\{1,2,3\}}$ on $\{1, 2, 3\}$, the 'swap' $s := \{1 \mapsto 2, \ 2 \mapsto 1\}$, the function $\{1 \mapsto 2, \ 2 \mapsto 1, \ 3 \mapsto 3\}$, the constant function $c := \{1 \mapsto 3, \ 2 \mapsto 3\}$, and the constant function $\{1 \mapsto 3, \ 2 \mapsto 3, \ 3 \mapsto 3\}$.

Then one can check that $\mathfrak{A}$ is closed under the operations of composition, antiodomain, range, and preferential union, and is therefore a $\{; A, R, \sqcup\}$-algebra of partial functions.

Later we will take a particular interest in homomorphisms that are what we call locally proper, though they are not essential to our duality. To define locally proper homomorphisms, we first need the notion of a prime filter in a representable $\{; A, R, \sqcup\}$-algebra.
Definition 2.11. A homomorphism of representable \( \{ ;, A, R, \sqcup \} \)-algebras is \textit{locally proper} if the inverse image of every prime filter (see Definition 4.1) is nonempty.

### 3 Stone étale categories

In this section we describe the other (large) category participating in our duality. It is a category of small categories with extra structure.\(^1\) To reduce the potential for confusion, we will call the ‘object level’ morphisms of the small categories arrows (which underlines their abstract nature), and reserve \textit{morphism} for the ‘meta level’ morphisms of the large category. Composition in the small categories is denoted \( \cdot \) and like the \( ; \) notation, the first composee appears on the left-hand side.

Definition 3.1. A \textit{topological category}\(^2\) is a (small) category whose sets of objects \( O \) and arrows \( M \) are both topological spaces, and such that

1. the source map \( d : M \to O \) is continuous,
2. the target map \( r : M \to O \) is continuous,
3. the composition map \( \cdot : M \times_O M \to M \) is continuous,\(^3\)
4. the identity assigning map \( x \mapsto 1_x \) sending each object to its identity arrow is continuous.

Put concisely, for us a topological category is a category internal to the category \( \text{Top} \) of topological spaces. Note that a topological category is a particular type of \textit{topological partial algebra}—a partial algebra on a topological space whose (possibly) partial operations are continuous when considered as functions on their domains of definition (equipped with the subspace topology).

Definition 3.2. A \textit{local homeomorphism} \( \pi : X \to Y \) of topological spaces is a continuous map such that for every \( x \in X \) there exists an open neighbourhood \( U \) of \( x \) such that

1. \( \pi(U) \) is open,
2. \( \pi|_U : U \to \pi(U) \) is a homeomorphism.

Definition 3.3. An \textit{étale category} is a topological category such that

1. the source map \( d : M \to O \) is a local homeomorphism,
2. the target map \( r : M \to O \) is an open map.

An étale category is \textit{Stone} if its space of objects is a Stone space (also known as a Boolean space), that is, a compact and totally separated space.

The condition that \( d \) is a local homeomorphism, and the condition, coming from the definition of a category, that \( d \) is surjective, together say that in an étale category, \( d \) gives \( M \) the structure of an \textit{étale space} (of sets) over \( O \) (also known as a sheaf space).

One might expect to take functors given by continuous maps as the morphisms of topological categories. However we require a more general definition in order to capture all the duals of algebra homomorphisms.

Definition 3.4. Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. A \textit{partial functor} \( F : \mathcal{C} \to \mathcal{D} \) consists of a partial function from the objects of \( \mathcal{C} \) to the objects of \( \mathcal{D} \) and another from the arrows of \( \mathcal{C} \) to the arrows of \( \mathcal{D} \) (both denoted \( F \)) such that

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\(^1\) Unlike algebras, small categories are allowed to be empty.

\(^2\) Not to be confused with the various other (unrelated) usages of this term.

\(^3\) The topology on the pullback \( M \times_O M \) is the initial topology with respect to the two projections. That is, the topology generated by sets of the form \( \{ (x, y) \in M \times_O M \mid x \in U \} \) and \( \{ (x, y) \in M \times_O M \mid y \in U \} \) for open subsets \( U \) of \( M \). In other words, it is the subspace topology on \( M \times_O M \subseteq M \times M \).
1. If $f : x \to y$ is an arrow of $C$ and $F(f)$ is defined, then both $F(x)$ and $F(y)$ are defined, and $F(f) : F(x) \to F(y)$.

2. If $F(x)$ is defined, for an object $x$, then $F(1_x)$ is defined and $1_{F(x)} = F(1_x)$.

3. If $f \cdot g$, $F(f)$, and $F(g)$ are defined, for arrows $f$ and $g$, then $F(f \cdot g)$ is defined and $F(f \cdot g) = F(f) \cdot F(g)$.

For $F$ to be a partial functor between topological categories we also require that

(i) $F$ is continuous,

(ii) the domain of definition of $F$ is an open set of arrows.\(^4\)

We call a partial functor total on objects if it is defined on every object, and a total functor if it is a functor in the usual (total) sense.

We now pick out certain special partial functors to account for the structure of the algebras for whose homomorphisms they are to provide duals.

Definition 3.5. A functor $f : C \to D$ between categories is star injective if for every object $x$ of $D$, it restricts to an injection on the ‘star’ $\text{Hom}(x, -)$. The same functor is star surjective if its restrictions to stars are surjections. It is star bijective if both star injective and star surjective. It is co-star surjective if $f^{\text{op}}$ is star surjective. It is sesqui-star bijective if it is star bijective and co-star surjective.

We are now finally ready to state our duality theorem.

Theorem 3.6. There is a categorical duality between the following two categories.

- The category $A$ with
  - objects the $\{; A, R, \sqcup\}$-algebras representable by partial functions,
  - morphisms the homomorphisms of $\{; A, R, \sqcup\}$-algebras.

- The category $C$ with
  - objects the Stone étale categories all of whose arrows are epimorphisms,
  - morphisms the sesqui-star-bijective partial functors of topological categories that are total on objects.

We will also show that the duality restricts to the following sub-duality.

Theorem 3.7. There is a categorical duality between the following two categories.

- The category $A'$ with
  - objects the $\{; A, R, \sqcup\}$-algebras representable by partial functions,
  - morphisms the locally proper homomorphisms of $\{; A, R, \sqcup\}$-algebras.

- The category $C'$ with
  - objects the Stone étale categories all of whose arrows are epimorphisms,
  - morphisms the sesqui-star-bijective total functors of topological categories.

4 From algebras to topological categories

In this section, we will define a contravariant functor in the direction from algebras to topological categories that forms one half of our duality. Following [14], we present the functor in terms of certain filters. That is, given an algebra, the entities used to construct a topological category—the entities that will constitute the arrows—will be filters satisfying a

\(^4\) By this second condition, it is unambiguous what it means for a partial functor to be continuous, because that inverse images are open in $C$ is equivalent to them being open in the domain of definition equipped with the subspace topology.
primality condition. We mention, however, that an alternative presentation is possible using the sort of algebraic distillation of germs of functions found in [2], for example.

We start with some easily verifiable remarks. In a representable \( \{; \cdot, A, R, \sqcup\} \)-algebra \( \mathfrak{A} \), the elements of the form \( A(\cdot) \) form a subalgebra. We call an element of this subalgebra a domain element (since it is equivalently of the form \( D(\cdot) \)). This subalgebra, \( D[\mathfrak{A}] \), is a Boolean algebra, with least element 0, greatest element \( 1' \), meet given by \( ; \cdot \), complement given by \( A \), and join given either by De Morgan or by \( \sqcup \).

\[ \text{Definition 4.1.} \text{ Let } \mathfrak{A} \text{ be a representable } \{; \cdot, A, R, \sqcup\} \text{-algebra. A filter of } \mathfrak{A} \text{ is a nonempty, upwards-closed, downward-directed subset of } \mathfrak{A}. \text{ A filter } F \text{ is prime if it is proper and whenever } a \sqcup b \in F, \text{ either } a \in F \text{ or } b \in F. \]

Our definition of filters being the standard one, many basic facts are already known to us. For example there is a smallest filter including any given subset; that is, the notion of the filter generated by a subset is well defined. Many of the properties of prime filters that we need have been proven in [12, Section 4] (where a prime filter is called an ultrasubset). The proofs there apply to any representable \( \{; \cdot, A, R\} \)-algebra, so in particular the representable \( \{; \cdot, A, R, \sqcup\} \)-algebras.

\[ \text{Lemma 4.2. In representable } \{; \cdot, A, R, \sqcup\} \text{-algebras, the prime and maximal filters coincide.}\]

\[ \text{Proof.} \text{ Take first a prime filter } P. \text{ Let } a \text{ be an arbitrary element not in } P. \text{ Since } P \text{ is nonempty, we can find } p \in P. \text{ The filter generated by } P \cup \{a\} \text{ must contain a lower bound for } \{a, p\}, \text{ call this } b. \text{ Since } b \leq p \text{ it follows, by reasoning about partial functions, that } p = b \lor (A(b) \uplus p). \text{ Hence either } b \in P \text{ or } A(b) \uplus p \in P. \text{ But } b \text{ is not in } P, \text{ else } a \text{ would be in } P. \text{ Hence } A(b) \uplus p \in P, \text{ and so the filter generated by } P \cup \{a\} \text{ contains both } b \text{ and } A(b) \uplus p, \text{ and hence some lower bound for this pair—necessarily 0. Hence any extension of } P \text{ is improper, so } P \text{ is maximal.}

\text{For the converse, we first establish:}

\[ \text{for any upwards-closed set } S, \text{ the set } D[S] \text{ is upwards closed in } D[\mathfrak{A}]. \] (13)

If \( a \in S \) and \( D(b) \geq D(a) \), then \( a \sqcup b \geq a \), so \( a \sqcup b \in S \). It is a property of partial functions that \( D(b) \geq D(a) \implies D(a \sqcup b) = D(b) \), hence \( D[S] \) contains \( D(b) \). We conclude that \( D[S] \) is upwards closed.

Now take a maximal filter \( M \), and suppose \( a \sqcup b \in M \). By [12, Lemma 4.5(ii)] and that \( D[M] \) is upwards closed in \( D[\mathfrak{A}] \), the set \( D[M] \) is an ultrafilter of \( D[\mathfrak{A}] \). Then either \( D(a) \in D[M] \) or \( A(a) \in D[M] \). Suppose the former, that is, there is a \( c \in M \) with \( D(c) = D(a) \). As \( M \) is downward directed, there is a \( d \in M \) with \( d \leq a \sqcup b, c \). It follows, by reasoning about partial functions, that \( d \leq a \), and hence \( a \in M \). By a similar argument, if \( A(a) \in D[M] \) then \( b \in M \). Hence \( M \) is prime.

By [12, Lemma 4.5(ii)] we now know that the following conditions are equivalent.
1. \( P \) is a prime filter.
2. \( P \) is a maximal filter.
3. \( P = (\mu ; a)^\uparrow \) for some \( a \in \mathfrak{A} \) and ultrafilter \( \mu \) of \( D[\mathfrak{A}] \) such that \( 0 \notin \mu ; a \).
4. for some ultrafilter \( \mu \) of \( D[\mathfrak{A}] \), for all \( a \in P \), we have \( P = (\mu ; a)^\uparrow \).

\[ \text{This is equivalent to the condition that whenever } a \vdash b \in F, \text{ either } a \in F \text{ or } b \in F. \]

\[ \text{Of course ‘maximal filter’ will always mean maximal proper filter.} \]
an element is a prime filter if and only if it is maximal, and if and only if it is of the form \((\mu ; a)\) for some ultrafilter \(\mu \in \mathbf{D}[\mathfrak{A}]\) and element \(a \in \mathfrak{A}\) such that \(0 \notin \mu ; a\).

In the following lemmas, let \(\mathfrak{A}\) be a representable \(\{; , \mathbf{A}, \mathbf{R}, \sqcup\}\)-algebra. The notation \(S^\uparrow\) denotes the upwards closure of the set \(S\) in \(\mathfrak{A}\), or in \(\mathbf{D}[\mathfrak{A}]\) if specified.

**Lemma 4.3.** \(\Rightarrow \quad \text{Let } P \text{ be a prime filter of } \mathfrak{A}. \text{ Then } \mathbf{D}[P] \text{ and } \mathbf{R}[P]^\uparrow \text{ are both ultrafilters of } \mathbf{D}[\mathfrak{A}], \text{ where the upwards closure is taken in } \mathbf{D}[\mathfrak{A}].\)

**Proof.** As \(P\) is nonempty, \(\mathbf{D}[P]\) is too. Now suppose \(D(a), D(b) \in \mathbf{D}[P]\), with \(a, b \in P\). Then as \(P\) is downward directed, it contains a \(c \leq a, b\). This inequality implies (for any partial functions) that \(D(c) \leq D(a), D(b)\). Hence \(\mathbf{D}[P]\) is downward directed. If \(\mathbf{D}[P]\) contained \(0\), then \(P\) would have to too, since \(D(a) = 0 \implies a = 0\) for partial functions. So \(\mathbf{D}[P]\) must be a proper filter. It remains to show that for any \(a \in \mathfrak{A}\), either \(D(a)\) or \(\mathbf{A}(a)\) belongs to \(\mathbf{D}[P]\).

Take any element \(b \in P\). Then \(b = \mathbf{D}(a) ; b \uplus \mathbf{A}(a) ; b\), so either \(D(a) ; b \in P\) or \(\mathbf{A}(a) ; b \in P\). Then we can obtain an element of \(\mathbf{D}[P]\) that is less than or equal to either \(D(a)\) or \(\mathbf{A}(a)\), respectively.

As \(P\) is nonempty, \(\mathbf{R}[P]^\uparrow\) is too. It is upwards closed by definition. Suppose \(\mathbf{R}(a), \mathbf{R}(b) \in \mathbf{R}[P]\), with \(a, b \in P\). Then as \(P\) is downward directed, it contains a \(c \leq a, b\). The inequality implies that \(\mathbf{R}(c) \leq \mathbf{R}(a), \mathbf{R}(b)\). Hence \(\mathbf{R}[P]\) is downward directed, so \(\mathbf{R}[P]^\uparrow\) is too. Since \(\mathbf{R}[P]\) cannot contain \(0\), neither can \(\mathbf{R}[P]^\uparrow\), so it is proper. Given any \(a \in \mathfrak{A}\), take any \(b \in P\).

Then \(b = b ; D(a) \sqcup b ; \mathbf{A}(a)\), so either \(b ; D(a) \in P\) or \(b ; \mathbf{A}(a) \in P\). But \(\mathbf{R}(b) ; D(a) \leq D(a)\) and \(\mathbf{R}(b) ; \mathbf{A}(a) \leq \mathbf{A}(a)\). So either \(D(a) \in \mathbf{R}[P]^\uparrow\) or \(\mathbf{A}(a) \in \mathbf{R}[P]^\uparrow\), respectively.

**Lemma 4.4.** \(\Rightarrow \quad \text{Let } \mu \text{ be an ultrafilter of } \mathbf{D}[\mathfrak{A}]. \text{ Then } \mu^\uparrow, \text{ where the upwards closure is taken in } \mathfrak{A}, \text{ is a prime filter of } \mathfrak{A}.\)

**Proof.** We have \(\mu^\uparrow = (\mu ; 1)^\uparrow\), and by [12, Lemma 4.5(ii)] this is a maximal filter. By Lemma 4.2, it is a prime filter.

**Lemma 4.5.** \(\Rightarrow \quad \text{Let } \mu \text{ be an ultrafilter of } \mathbf{D}[\mathfrak{A}]. \text{ Then } \mathbf{D}[\mu^\uparrow] = \mu, \text{ where the upwards closure is taken in } \mathfrak{A}.\)

**Proof.** We know \(\mu \subseteq \mathbf{D}[\mu^\uparrow]\) because \(\mathbf{D}\) fixes all domain elements. Conversely, take an element of \(\mathbf{D}[\mu^\uparrow]\); an element \(D(b)\) such that \(b \geq \alpha\) for some \(\alpha \in \mu\). Then \(D(b) \geq D(\alpha)\). But \(D(\alpha) = \alpha\) and \(\mu\) is upwards closed; hence \(D(b) \in \mu\). This proves the reverse inclusion.

**Lemma 4.6.** \(\Rightarrow \quad \text{Let } P, Q \text{ be prime filters of } \mathfrak{A}. \text{ Then } (P ; Q)^\uparrow \text{ is a prime filter if and only if } \mathbf{D}[Q] = \mathbf{R}[P]^\uparrow. \text{ Otherwise } (P ; Q)^\uparrow \text{ contains } 0 \text{ and hence is all of } \mathfrak{A}.\)

**Proof.** Let \(P = (\mu ; a)^\uparrow\) and \(Q = (\nu ; b)^\uparrow\) for \(\mu, \nu\) ultrafilters of domain elements. By [12, Lemma 4.6(i)], the set \((P ; Q)^\uparrow\) is of the form \((\mu ; a ; \nu ; b)^\uparrow\). By [12, Lemma 4.6(ii)], the set \((\mu ; a ; \nu ; b)^\uparrow\) is a prime filter if and only if it does not contain \(0\). So it remains to show that \(0 \in (P ; Q)^\uparrow\) if and only if \(\mathbf{D}[Q] \neq \mathbf{R}[P]^\uparrow\).

First suppose \(0 \in (P ; Q)^\uparrow\), so \(0 \in P ; Q\), and so \(0 = a ; b\) for some \(a \in P\) and \(b \in Q\). It follows, by reasoning about partial functions, that \(\mathbf{R}(a) ; D(b) = 0\). Since \(\mathbf{R}(a) \in \mathbf{R}[P]^\uparrow\) and \(D(b) \in \mathbf{D}[Q]\), and ; is meet on domain elements, these two ultrafilters cannot be equal. Conversely, suppose \(\mathbf{D}[Q] \neq \mathbf{R}[P]^\uparrow\). Since they are ultrafilters, we can then find Boolean complements \(\alpha\) and \(\mathbf{A}(\alpha)\) with \(\alpha \in \mathbf{R}[P]^\uparrow\) and \(\mathbf{A}(\alpha) \in \mathbf{D}[Q]\). That is, there are \(\alpha \in P\) with \(\mathbf{R}(a) \leq \alpha\) and \(b \in Q\) with \(D(b) = \mathbf{A}(\alpha)\). Then \(\mathbf{R}(a) ; D(b) = 0\), and it follows by reasoning about partial functions that \(a ; b = 0\); hence \(0 \in (P ; Q)^\uparrow\).

**Lemma 4.7.** \(\Rightarrow \quad \text{Let } P, Q \text{ be prime filters of } \mathfrak{A}. \text{ If } (P ; Q)^\uparrow \text{ is proper, then } \mathbf{R}[P ; Q]^\uparrow = \mathbf{R}[Q]^\uparrow.\)
Proof. First we show \( R[P; Q] \subseteq R[Q] \), giving \( R[P; Q]^\uparrow \subseteq R[Q]^\uparrow \). Take an \( R(a; b) \in R[P; Q] \), with \( a \in P \) and \( b \in Q \). Now \( b = (R(a); b) \cup (A(R(a)); b) \), so, since \( Q \) is a prime filter, either it contains \( R(a); b \) or \( A(R(a)); b \). If the latter, then \( P; Q \) contains \( a; A(R(a)); b \), equal to 0, contradicting the hypothesis that \( (P; Q)^\uparrow \) is proper. Hence \( Q \) contains \( R(a); b \), so \( R[Q] \) contains \( R(R(a); b) \). But \( R(R(a); b) = R(a; b) \) is a property of partial functions (it is axiom (r.VII) in [12]). Hence \( R(a; b) \in R[Q] \) and we have our first inclusion.

Conversely, suppose \( R(b) \in R[Q] \), with \( b \in Q \). Take any \( a \in P \). As before, it must be the case that \( R(a); b \in Q \). Then \( a; (R(a); b) \in P; Q \), that is, \( a; b \in P ; Q \). Hence \( R(a;b) \in R[P; Q] \).

By a property of partial functions, \( R(b) \geq R(a; b) \), hence \( R(b) \in R[P; Q]^\uparrow \). Since \( R(b) \) was an arbitrary element of \( R[Q] \) we have \( R[Q] \subseteq R[P; Q]^\uparrow \) and hence \( R[Q]^\uparrow \subseteq R[P; Q]^\uparrow \).  

- **Lemma 4.8.** If \( P \) and \( Q \) are nondisjoint prime filters with \( D[P] = D[Q] \), then \( P = Q \).

Proof. This is [12, Lemma 4.5(iv)].

- **Lemma 4.9.** Let \( P, Q, R \) be prime filters of \( \mathfrak{A} \). Suppose \( (P; Q)^\uparrow \) and \( (P; R)^\uparrow \) are proper, and equal. Then \( Q = R \).

Proof. If \( (P; Q)^\uparrow \) and \( (P; R)^\uparrow \) are proper, then by Lemma 4.6, we have \( D[Q] = R[P]^\uparrow = D[R] \).

So by Lemma 4.8 it is sufficient to show that \( Q \) and \( R \) are nondisjoint. Let \( a \in P \) and \( b \in Q \). So \( a; b \in (P; Q)^\uparrow = (P; R)^\uparrow \), that is, \( a; b \geq a'; c \) for some \( a' \in P \) and \( c \in R \). By definition this means \( D(a'; c); a; b = a'; c \). But \( D(a'; c) \) must belong to the ultrafilter \( D[P] \)—it cannot be that its Boolean complement \( A(a'; c) \) is in \( D[P] \), since \( (P; R)^\uparrow \) is proper. Hence \( D(a'; c); a \in P \). Pick some \( a'' \in P \) with \( a'' \leq D(a'; c); a \) and \( a'' \leq a' \). It is a property of partial functions that if \( d; b = d''; c \) and \( d'' \leq d, d'' \) then \( d''; b = d''; c \). Hence, from \( D(a'; c); a; b = a'; c \) we obtain \( a''; b = a''; c \). By axiom (10), this gives \( R(a''); b = R(a''); c \).

Now \( R(a'') \) is an element of \( D[Q] = R[P]^\uparrow = D[R] \), so \( R(a''); b \in Q \) and \( R(a''); c \in R \). That is, we have found our element common to \( Q \) and \( R \).

4.1 The functor \( \text{PF} \) on objects

We now define the functor \( \text{PF}: \mathcal{A} \to \mathcal{C} \) used for one half of the duality. For \( \mathfrak{A} \) a representable \( \{; , \mathcal{A}, \mathcal{R}, \sqcup \} \)-algebra, let \( \text{PF}(\mathfrak{A}) \) be the following topological category.

- The objects of \( \text{PF}(\mathfrak{A}) \) are the ultrafilters of \( D[\mathfrak{A}] \).
- The arrows of \( \text{PF}(\mathfrak{A}) \) are the prime filters of \( \mathfrak{A} \).
- The source and target of an arrow \( P \) are \( D[P] \) and \( R[P]^\uparrow \) respectively. By Lemma 4.3 these are objects.
- The identity arrow for an object \( \mu \) is \( \mu^\uparrow \), where the upwards closure is taken in \( \mathfrak{A} \). By Lemma 4.4, this is an arrow. By Lemma 4.5, its source is \( \mu \). By Lemma 4.6, the target of \( \mu^\uparrow \) must also be \( \mu \) since \( \mu^\uparrow \) is proper.
- For composable arrows \( P \) and \( Q \), the composition is given by \( P : Q \Rightarrow (P ; Q)^\uparrow \). By Lemma 4.6, this is an arrow, evidently with the same source as \( P \). By Lemma 4.7 it has the same target as \( Q \).

The confirmation that the structure so defined validates the axioms for categories is the content of Lemma 4.10, which follows shortly. By Lemma 4.9 all arrows are epimorphisms. Let \( \text{uf}(D[\mathfrak{A}]) \) denote the ultrafilters of \( D[\mathfrak{A}] \), and let \( \text{pf}(\mathfrak{A}) \) denote the prime filters of \( \mathfrak{A} \).

- The topology on the objects is the topology generated by \( \{ \alpha | \alpha \in D[\mathfrak{A}] \} \), where \( \alpha := \{ \mu \in \text{uf}(D[\mathfrak{A}]) | \alpha \in \mu \} \).
- The topology on the arrows is the topology generated by \( \{ a^\alpha | a \in \mathfrak{A} \} \), where \( a^\alpha := \{ P \in \text{pf}(\mathfrak{A}) | a \in P \} \).
The confirmations that the source, target, composition, and identity-assigning maps are continuous with respect to these topologies is the content of the following Lemma 4.11. The confirmation that the source map is a local homeomorphism is the following Lemma 4.12, and the confirmation that the target map is an open map is Lemma 4.13. It is immediate that the objects form a Stone space, since we have used for this space exactly the standard construction of the Stone dual of the Boolean algebra $D[\mathfrak{A}]$.

**Lemma 4.10.** Let $\mathfrak{A}$ be a representable $\{; A, R, \sqcup\}$-algebra. Then $\text{PF}(\mathfrak{A})$ satisfies the associativity and identity axioms for categories.

**Proof.** First the identity laws: let $P$ be an arrow (a prime filter) and let $\mu = D[P]$ be its source, so $P$ is of the form $(\mu; a)^\uparrow$ for some $a$. Then the identity arrow at $\mu$ is $\mu^\uparrow = (\mu; 1')^\uparrow$, and...

**Lemma 4.11.** Let $\mathfrak{A}$ be a representable $\{; A, R, \sqcup\}$-algebra. The source, target, composition, and identity-assigning maps on the category $\text{PF}(\mathfrak{A})$ are continuous with respect to the topologies generated by $\{\hat{\alpha} \mid \alpha \in D[\mathfrak{A}]\}$ and $\{a^\theta \mid a \in \mathfrak{A}\}$.

**Proof.** First $d$: we take $\hat{\alpha}$ and show that $d^{-1}(\hat{\alpha})$ is open. So let $P$ be a prime filter with $\alpha \in D[P]$. Take any $a \in P$. Then $b := a; \alpha$ is also in $P$, so $P \subseteq b^\uparrow$. Now for any $Q \subseteq b^\uparrow$, we have $D(b) \in D[Q] = d(Q)$, and $D(b) \subseteq \alpha$, so $\alpha \in d(Q)$. That is, $Q \subseteq d^{-1}(\hat{\alpha})$. So $P \subseteq b^\uparrow \subseteq d^{-1}(\hat{\alpha})$. Since $P$ was an arbitrary element of $d^{-1}(\hat{\alpha})$ and $b^\uparrow$ is by definition open, we are done.

Next $r$: we take $\hat{\alpha}$ and show that $r^{-1}(\hat{\alpha})$ is open. So let $P$ be a prime filter with $\alpha \in R[P]^\uparrow$. Take any $a \in P$. Then $b := a; \alpha$ is also in $P$—because $a = (a; \alpha) \sqcup (a; A(\alpha))$, but $a; A(\alpha)$ cannot be in $P$, else $R[P]^\uparrow$ (which contains $\alpha$) would contain 0. Hence $P \subseteq b^\uparrow$. Now for any $Q \subseteq b^\uparrow$, we have $R(b) \in R[Q]^\uparrow = r(Q)$, and $R(b) \subseteq \alpha$, so $\alpha \in r(Q)$. That is, $Q \subseteq r^{-1}(\hat{\alpha})$. So $P \subseteq b^\uparrow \subseteq r^{-1}(\hat{\alpha})$. Since $P$ was an arbitrary element of $r^{-1}(\hat{\alpha})$ and $b^\uparrow$ is by definition open, we are done.

For composition: we take $a^\theta$ and show that the inverse image under $\cdot$ is open. So let $P$ and $Q$ be two prime filters such that $P \cdot Q \subseteq a^\theta$, that is, $a \in (P \cdot Q)^\uparrow$. Then there are $b \in P$ and $c \in Q$ such that $b; c \subseteq a$. Now $D(b; c) : b$ must also be in $P$, for if $A(b; c) : b$ were in $P$ then $P \cdot Q$ would contain $A(b; c) : b; c = 0$. So we have open sets $(D(b; c) : b)^\uparrow$ containing $P$, and $a^\theta$ containing $Q$, and for any two composable arrows $R \subseteq (D(b; c) : b)^\uparrow$ and $S \subseteq e^\theta$, their composition $R \cdot S = (R \cdot S)^\uparrow$ contains $D(b; c) : b; c = b; c \subseteq a$ and therefore lies in $a^\theta$. Since $P$ and $Q$ were arbitrary subject to $P \cdot Q \subseteq a^\theta$, this proves that $a^\theta$ is open.

For the identity-assigning map: we take $a^\theta$ and confirm that the set $\{\mu \in \text{uf}(D[\mathfrak{A}]) \mid a \in \mu^\uparrow\}$ is open. Take $\nu$ in this set; then $a \geq \alpha$ for some $\alpha \in \nu$. It follows that the open set $\hat{\alpha}$ of objects contains $\nu$ and is included in $\{\mu \in \text{uf}(D[\mathfrak{A}]) \mid a \in \mu^\uparrow\}$, so we are done.

**Lemma 4.12.** The map $d: P \mapsto D[P]$ is a local homeomorphism from the arrows to the objects of $\text{PF}(\mathfrak{A})$.

**Proof.** We know from Lemma 4.11 that $d$ is continuous. Next we establish that $d$ is an open map by showing that $d[a^\theta] = D(a)$ for any $a \in \mathfrak{A}$. Clearly if $a$ belongs to a prime filter $P$ then $D(a)$ belongs to $d(P) = D[P]$, hence $d[a^\theta] \subseteq D(a)$. Conversely, any $\mu \in D(a)$ is the image under $d$ of the element $(\mu; a)^\uparrow$ of $a^\theta$ (for $D(a) \in \mu$ ensures $0 \notin \mu; a$). Hence $d$ is an open map.

Now any open and continuous map $f$ is a local homeomorphism if every point in its domain has an open neighbourhood $U$ such that the restriction of $f$ to $U$ is injective. For $d$
we take, for any $P$ in its domain, any $a \in P$ we wish and use the open neighbourhood $a^\theta$ of $P$. The map $d$ is injective on $a^\theta$ by Lemma 4.8.  

$\blacktriangleright$ **Lemma 4.13.** The map $r: P \mapsto R[P]^\uparrow$ is an open map from the arrows to the objects of $PF(\mathfrak{A})$.

Proof. We show that $r[a^\theta] = \widehat{R(a)}$ for any $a \in \mathfrak{A}$. Clearly if $\mu \in r[a^\theta]$, that is, $\mu = R[P]^\uparrow$ for some $P$ containing $a$, then $\mu$ contains $R(a)$, so $\mu \in \widehat{R(a)}$. Conversely, suppose $\mu \in \widehat{R(a)}$, that is, $R(a) \in \mu$. Consider the subset $D(\mu)$ of the Boolean algebra $D[\mathfrak{A}]$. This set $D(\mu)$ is nonempty (because $\mu$ is nonempty), and downward directed—because given $D(\alpha; \alpha)$ and $D(\beta)$, for $\alpha, \beta \in \mu$, we know $\alpha; \beta \in \mu$, and it is a property of partial functions that $D(\alpha; \alpha; \beta)$ is a lower bound for $\{D(\alpha; \alpha), D(\alpha; \beta)\}$ (in fact it is the meet). Further, $D(\alpha; \mu)$ does not contain 0, since that would imply $0 \in \alpha; \mu$, which implies that $A(R(\alpha)) \in \mu$, but this is prohibited, since $\mu$ is an ultrafilter containing $R(a)$. We have shown that $D(\alpha; \mu)$ is a proper filter of $D[\mathfrak{A}]$. Extend it to an ultrafilter $\nu$. Now $(\nu; a)^\uparrow$ is a prime filter because $0 \notin \nu; a$, by the following reasoning. The filter $D(\alpha; \mu)^\uparrow$ contains $D(\alpha; 1^\uparrow) = D(\alpha)$, and hence $\nu$ does too, meaning $\nu$ does not contain $A(\alpha)$, which is a necessary condition for $\nu; a$ to contain 0. Our prime filter $(\nu; a)^\uparrow$ contains $a$, so $r((\nu; a)^\uparrow) \in r[a^\theta]$. Finally, we claim that $r((\nu; a)^\uparrow) = \mu$. For any $\alpha \in \mu$, we know that $D(\alpha; \alpha) \in \nu$ and therefore $R(D(\alpha; \alpha); \alpha) \in r((\nu; a)^\uparrow)$. But it is a property of partial functions that $R(D(\alpha; \alpha); \alpha) \leq \alpha$ (given that $\alpha$ is a domain element). Since $r((\nu; a)^\uparrow)$ is upwards closed and $\alpha$ was an arbitrary element of $\mu$, we obtain $r((\nu; a)^\uparrow) \subseteq \mu$. Since $r((\nu; a)^\uparrow)$ and $\mu$ are ultrafilters, they are equal, as required.  

$\blacktriangleright$ **Example 4.14.** Let $\mathfrak{A}$ be the example from Example 2.10. The Boolean subalgebra $D[\mathfrak{A}]$ consists of the four elements $\emptyset, id_{\{1,2\}}, id_{\{3\}}$, and $id_{\{1,2,3\}}$. The ultrafilters of $D[\mathfrak{A}]$ (the objects of the dual) are $\{id_{\{1,2\}}, id_{\{1,2,3\}}\}$ and $\{id_{\{3\}}, id_{\{1,2,3\}}\}$, which we call ‘1, 2’ and ‘3’ respectively. The prime filters of $\mathfrak{A}$ (the arrows of the dual) are the upsets of the minimal nonzero elements of $\mathfrak{A}$, and there are four of these: $id_{\{1,2\}}, id_{\{3\}}, s,$ and $c$. Those that correspond to identity arrows in the dual are $id_{\{1,2\}}$ and $id_{\{3\}}$. We can calculate that both $id_{\{1,2\}}$ and $s$ have source ‘1, 2’ and target ‘1, 2’, that $s \cdot s = id_{\{1,2\}}$, and so on. A suggestive diagram of the dual $PF(\mathfrak{A})$ of $\mathfrak{A}$ follows.

![Diagram](image)

4.2 The functor $PF$ on morphisms

Let $f: \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of representable $\{\cdot, A, R, \sqcup\}$-algebras. It follows immediately that $f$ restricts to a Boolean algebra homomorphism from $D[\mathfrak{A}]$ to $D[\mathfrak{B}]$. The action of $PF$ on morphisms is given by inverse image. In more detail, the continuous partial functor $PF(f): PF(\mathfrak{B}) \to PF(\mathfrak{A})$ is given

- on an object $\mu \in uf(D[\mathfrak{A}])$ by $\mu \mapsto f^{-1}(\mu),$
- on an arrow $P \in pf(\mathfrak{A})$, by $P \mapsto f^{-1}(P)$ if this is nonempty; the value of $PF(f)$ is undefined at $P$ otherwise.

That the object component of $PF(f)$ is well defined and continuous follows from classical Stone duality. By definition, $PF(f)$ is total on objects. By Lemma 4.15, if $f^{-1}(P)$ is nonempty it is indeed a prime filter of $\mathfrak{A}$, so $PF(f)$ is well defined overall. The proof that $PF(f)$ is a partial functor from the category $PF(\mathfrak{B})$ to the category $PF(\mathfrak{A})$ is Lemma 4.16. The proof that the arrow component of $PF(f)$ is continuous and defined on an open set of arrows is Lemma 4.17. The proof that $PF(f)$ is sesqui-star bijective is Lemma 4.18. It is
clear that PF is itself functorial because composition of inverse images is the inverse image of the composition, and taking inverse images of an identity function induces the identity.

**Lemma 4.15.** For any homomorphism \( f : \mathfrak{A} \to \mathfrak{B} \) and any prime filter \( P \) of \( \mathfrak{B} \), if \( f^{-1}(P) \) is nonempty it is a prime filter of \( \mathfrak{A} \).

**Lemma 4.16.** For any homomorphism \( f : \mathfrak{A} \to \mathfrak{B} \), the partial function \( PF(f) \) is a partial functor between the categories \( \mathfrak{A} \) and \( \mathfrak{B} \).

**Proof.** Let \( P : D[P] \to R[P]^{\uparrow} \) be an arrow in \( PF(\mathfrak{B}) \) (a prime filter of \( \mathfrak{B} \)) and suppose \( PF(f)(P) \) is defined. We already know \( PF(f) \) is defined on \( D[P] \) and \( R[P]^{\uparrow} \), since it is defined on all objects, but we want to show also that these are mapped to the source and target of \( PF(f)(P) \) respectively. That is, we want to show \( f^{-1}(D[P]) = D[f^{-1}(P)] \) and \( f^{-1}(R[P]^{\uparrow}) = R[f^{-1}(P)]^{\uparrow} \).

Next we want to show that for every ultrafilter \( \mu \) of \( D[\mathfrak{A}] \) the value of \( PF(f) \) on the corresponding identity arrow \( (\mu; \{1\})^{\uparrow} \) is defined, and equals the identity arrow on \( PF(\mu) \). The value is clearly defined, since \( (\mu; \{1\})^{\uparrow} \) includes \( \mu \), so \( f^{-1}((\mu; \{1\})^{\uparrow}) \) is certainly nonempty. The identity arrow on \( PF(\mu) \) is \( (f^{-1}(\mu); \{1\})^{\uparrow} \), which is a witness to the nonemptiness of \( f^{-1}((\mu; \{1\})^{\uparrow}) \), so by Lemma 4.8 these prime filters are equal, as required.

Finally, if \( P \cdot Q, PF(P), \) and \( PF(Q) \) are defined, then taking \( a \in f^{-1}(P) \) and \( b \in f^{-1}(Q) \) provides \( a \ni b \) as a witness to the nonemptiness of \( f^{-1}(P \cdot Q) = f^{-1}(P; Q) \), so \( PF(P \cdot Q) \) is defined. Clearly \( f^{-1}(P \cdot Q) \subseteq f^{-1}(P; Q) \), as \( f \) is a homomorphism. But then it follows that \( f^{-1}(P \cdot Q) = f^{-1}(P; Q) \), since both are prime, therefore by Lemma 4.2 maximal, filters.

**Lemma 4.17.** For any homomorphism \( f : \mathfrak{A} \to \mathfrak{B} \), the arrow component of the partial functor \( PF(f) : PF(\mathfrak{B}) \to PF(\mathfrak{A}) \) is continuous. It is defined on an open set of arrows.

**Proof.** Since sets of the form \( a^{\theta} \) form a subbasis for the topology on \( \mathfrak{A} \), for the first part it suffices to show that given \( a_1, \ldots, a_n \in \mathfrak{A} \) it is true that \( PF(f)^{-1}(\bigcap_i a_i^{\theta}) \) is open in \( \mathfrak{B} \). Suppose \( P \in PF(f)^{-1}(\bigcap_i a_i^{\theta}) \). Then \( PF(f)(P) \subseteq \bigcap_i a_i^{\theta} \), that is, \( f^{-1}(P) \subseteq \bigcap_i a_i^{\theta} \). So \( a_1, \ldots, a_n \in f^{-1}(P) \). Since \( f^{-1}(P) \) is a filter, there is an \( a \in f^{-1}(P) \) with \( a \ni a_1, \ldots, a_n \). Then \( f(a) \in P \), that is, \( P \ni f(a)^{\theta} \), and for any prime filter \( Q \) of \( \mathfrak{B} \),

\[
Q \in f(a)^{\theta} \quad \Rightarrow \quad f(a) \in Q \\
\quad \Rightarrow \quad f(a_1), \ldots, f(a_n) \in Q \\
\quad \Rightarrow \quad a_1, \ldots, a_n \in f^{-1}(Q) \\
\quad \Rightarrow \quad f^{-1}(Q) \in a_1^{\theta}, \ldots, a_n^{\theta} \\
\quad \Rightarrow \quad PF(f)(Q) \in \bigcap_i a_i^{\theta} \\
\Rightarrow \quad Q \in PF(f)^{-1}(\bigcap_i a_i^{\theta})
\]

so the open set \( f(a)^{\theta} \) is contained entirely within \( PF(f)^{-1}(\bigcap_i a_i^{\theta}) \).

To see that \( PF(f) \) is defined on an open set, suppose it is defined on \( P \in PF(\mathfrak{B}) \). That is \( f^{-1}(P) \neq \emptyset \). Take any \( a \in f^{-1}(P) \). Then \( f(a) \in P \), that is, \( P \ni f(a)^{\theta} \), and the open \( f(a)^{\theta} \) lies entirely within the domain of definition of \( PF(f) \), since \( Q \ni f(a)^{\theta} \) implies \( f^{-1}(Q) \) contains \( a \), so is certainly not empty. Since \( P \) was arbitrary, we are done.

**Lemma 4.18.** For any homomorphism \( f : \mathfrak{A} \to \mathfrak{B} \), the partial functor \( PF(f) : PF(\mathfrak{B}) \to PF(\mathfrak{A}) \) is sesqui-star bijective.
Proof. For star injectivity, suppose \( P, Q \in \text{PF}(B) \) are prime filters with the same source—\( D[P] = D[Q] \)—and such that \( \text{PF}(f)(P) = \text{PF}(f)(Q) \). By definition, this means \( f^{-1}(P) = f^{-1}(Q) \neq \emptyset \). Hence \( P \cap Q \neq \emptyset \), so by Lemma 4.8 \( P = Q \).

For surjectivity, suppose \( \mu \) is an ultrafilter of \( \mathfrak{B} \) and \( P \) is a prime filter of \( \mathfrak{A} \) with \( D[P] = f^{-1}(\mu) \). Choose some element \( a \in P \). Then \( D(a) \in f^{-1}(\mu) \), so \( f(D(a)) \in \mu \). Since \( f \) is a homomorphism, this gives \( D(f(a)) \in \mu \). Then \( (\mu; f(a))\uparrow \) is a prime filter of \( \mathfrak{B} \) that we will denote \( Q \). It is straightforward to show that \( D(Q) = \mu \). Then \( f^{-1}(Q) \) is nonempty (it contains \( a \)) and nondisjoint from \( P \) (for the same reason), and so by Lemma 4.8 equals \( P \). Since \( \mu \) was arbitrary, and \( P \) was arbitrary subject to \( D[P] = f^{-1}(\mu) \), we have star surjectivity.

For co-star surjectivity suppose \( \mu \) is an ultrafilter of \( \mathfrak{B} \) and \( P \) is a prime filter of \( \mathfrak{A} \) with \( R[P] = f^{-1}(\mu) \). Choose some element \( a \in P \). Then \( R(a) \in f^{-1}(\mu) \), so \( f(R(a)) \in \mu \). Since \( f \) is a homomorphism, this gives \( R(f(a)) \in \mu \). Then \( (f(a); \mu)\uparrow \) is a prime filter of \( \mathfrak{B} \). Extend it to a prime filter \( Q \), which necessarily satisfies \( R[Q] = \mu \). Then \( f^{-1}(Q) \) is nonempty (it contains \( a \)) and nondisjoint from \( P \) (for the same reason), and so by Lemma 4.8 equals \( P \). Since \( \mu \) was arbitrary, and \( P \) was arbitrary subject to \( R[P] = f^{-1}(\mu) \), we have co-star surjectivity. \( \blacksquare \)

5 From topological categories to algebras

In this section, we define the contravariant functor \( \text{SecCl}: C \to A \) used for the second half of the duality. The notation for the functor stands for ‘section on clopen’.

\[ \text{Definition 5.1. Let } \pi: E \to X \text{ be a local homeomorphism of topological spaces. A (local) section of } \pi \text{ is a continuous function } f: U \to E, \text{ for some open } U \subseteq X, \text{ with } \pi \circ f = \text{id}_U. \]

Since a local section is completely determined by its image, we will often identify it with this image, in which case an upper-case Roman letter will be used.

5.1 The functor \( \text{SecCl} \) on objects

Let \( \mathcal{C} \) be a Stone étale category in \( C \), with objects \( O \) and arrows \( M \), all of which are epimorphisms. Then we define \( \text{SecCl}(\mathcal{C}) \) to be the following \{; A, R, \cup\}-algebra.

- The universe of \( \text{SecCl}(\mathcal{C}) \) is the set of all local sections \( U \to M \) of \( \text{d}: M \to O \) with \( U \) clopen.
- The operation \( ; \) is defined by \( A; B := \{ a \cdot b \mid a \in A, \ b \in B \} \). The confirmation that \( A; B \) is a section on a clopen is Lemma 5.6.
- The operation \( \text{A} \) is defined by \( \text{A}(A) := \{ 1_x \mid x \in O \setminus \text{d}[A] \} \). The confirmation that \( \text{A}(A) \) is a section on a clopen is Lemma 5.3.
- The operation \( \text{R} \) is defined by \( \text{R}(A) := \{ 1_x \mid x \in \text{r}[A] \} \). The confirmation that \( \text{R}(A) \) is a section on a clopen is Lemma 5.5.
- The operation \( \uparrow \) is defined by \( \text{A} \uparrow B := \text{A} \cup (\text{A}(A); B) \). The confirmation that \( \text{A} \uparrow B \) is a section on a clopen is Lemma 5.7.

Proving that \( \text{SecCl}(\mathcal{C}) \) is representable by partial functions is achieved by verifying that it validates all the equations and quasiequations of Theorem 2.9; this is done in Lemma 5.8.

\[ \text{Lemma 5.2. In any topological category, the map } \iota: x \mapsto 1_x \text{ sending each object to its identity arrow is a homeomorphism onto its image } I \text{ (equipped with the subspace topology).} \]

Proof. It is an axiom of categories that \( \text{d} \circ \iota \) is the identity on objects, so \( \iota \) is a bijection onto \( I \). Let \( U \) be an open subset of \( I \), so \( U = I \cap V \) for some \( V \) open in \( M \). Then \( \iota^{-1}(U) = \iota^{-1}(V) \) is open, as \( \iota \) is a continuous as a map \( O \to M \). Conversely, let \( U \) be an open subset of \( O \).
Then as \( d : M \to O \) is continuous, \( d^{-1}(U) \) is open in \( M \), hence \( I \cap d^{-1}(U) \) is open in \( I \). But \( I \cap d^{-1}(U) \) is the image of \( U \) under \( t \), so \( t \) is an open map.

▶ **Lemma 5.3.** Let \( A \) be a section with clopen domain. Then \( A(A) := \{1_x \mid x \in O \setminus d[A]\} \) is a section with clopen domain.

**Proof.** By definition, \( d[A] \) is clopen in \( O \), hence \( O \setminus d[A] \) is clopen in \( O \). So \( A(A) \) defines a function \( f : x \mapsto 1_x \) with clopen domain, clearly with left inverse \( d \). It remains to show this function is continuous. But this is immediate, since \( f \) is a restriction—given by a restriction of the domain—of the (continuous) identity assigning map, and all such restrictions of continuous maps are continuous.

▶ **Lemma 5.4.** Let \( \pi : E \to X \) be a local homeomorphism and \( U \subseteq X \) be open, and suppose \( f : U \to E \) has left inverse \( \pi \). If \( f \) is continuous (that is, a local section) then \( f[U] \) is open in \( E \). Conversely, if \( f[U] \) is open, then \( f \) is a local section.

**Proof.** For the first part let \( e \in f[U] \). Then as \( \pi \) is a local homeomorphism, there is a \( V \) open in \( E \) and containing \( e \) such that \( \pi|_V \) is a homeomorphism onto its image \( \pi|_V[V] = \pi[V] \).

As \( f \) is continuous, the set \( f^{-1}(V) \), which equals \( f^{-1}(V \cap f[U]) \), is open in \( X \). But \( f^{-1}(V \cap f[U]) \subseteq \pi[V] \), so as \( \pi|_V \) is a homeomorphism the inverse image of \( f^{-1}(V \cap f[U]) \) under \( \pi|_V \), that is, \( V \cap f[U] \), is open in \( V \). Since \( V \) itself is open, \( V \cap f[U] \), which contains \( e \), is open in \( E \). Since \( e \) was an arbitrary element of \( f[U] \), the set \( f[U] \) is open in \( E \).

Conversely, suppose \( f[U] \) is open. Let \( V \) be an open subset of \( E \), hence an open subset of \( f[U] \). We want to show that \( f^{-1}(V) \) is open. It suffices to show that any \( x \in f^{-1}(V) \) is contained in an open neighbourhood included in \( f^{-1}(V) \). But this is clear since \( \pi \) is a local homeomorphism, hence maps some open \( W \) containing \( f(x) \), and contained in \( V \), to the open \( \pi[W] = f^{-1}(W) \), which of course contains \( x \) and is included in \( V \).

Note that the first part of Lemma 5.4 immediately implies that the identity arrows form an open set, since the identity assigning map is manifestly a section.

▶ **Lemma 5.5.** Let \( A \) be a section with clopen domain. Then \( R(A) := \{1_x \mid x \in r[A]\} \) is a section with clopen domain.

**Proof.** Let \( A \) correspond to the function \( f : U \to M \). Then \( R(A) \) is the identity assigning function restricted to \( (r \circ f)(U) \). As \( R(A) \) is a restriction of a continuous function it is continuous. By the first part of Lemma 5.4, the set \( f[U] \) is open, then since \( r \) is an open map, \( (r \circ f)(U) \) is open. It remains to show \( (r \circ f)(U) \) is closed. Now \( r \circ f \) is a continuous map from a compact space \( U \) to a Hausdorff space (the space of objects). It is a basic and easy-to-prove result of general topology that such a map is a closed map (the ‘closed map lemma’). Hence \( (r \circ f)(U) \) is indeed closed.

▶ **Lemma 5.6.** Let \( A \) and \( B \) be sections with clopen domains. Then \( A : B := \{a \cdot b \mid a \in A, b \in B, r(a) = d(b)\} \) is a section with clopen domain.

**Proof.** Let \( A \) correspond to the function \( f : U \to M \), and \( B \) to \( g : V \to M \). It is clear that \( A : B \) corresponds to a function \( h \) on a subset of \( O \), and that \( d \) is a left inverse for this function. Since \( h \) can be expressed as a composition \( g' \circ r' \circ f' \) of restrictions of the continuous functions \( f \), \( r \), and \( g \), we see that \( h \) is continuous. It remains to show that the domain of \( h \) is clopen. By Lemma 5.5, the set \( (r \circ f)(U) \) is clopen; hence \( (r \circ f)(U) \cap V \) is clopen. Now the domain of \( h \) is \( (r \circ f)^{-1}((r \circ f)(U) \cap V) \), so clopen by continuity of \( f \) and \( r \).

▶ **Lemma 5.7.** If \( A \) and \( B \) are sections on clopens, then so is \( A \sqcup B := A \cup (A(A) : B) \).
Proof. It is immediate that \( A \sqcup B \) defines a function with left inverse \( d \). By Lemma 5.6 and Lemma 5.3, we know \( A(A) ; B \) is a section on a clopen, and hence the domain of \( A \sqcup B \) is a clopen. It remains to argue that \( A \sqcup B \) is continuous. But this is a function given by the union of two continuous functions with open domains, which always yields a continuous function. ▷

Lemma 5.8. The \( \{; , A , R , \sqcup \} \)-algebra \( PF(\mathcal{C}) \) validates the axioms for representability by partial functions listed in Theorem 2.9.

Proof. We state each axiom anew before giving a justification for its validity.

\[
A ; (B ; C) = (A ; B) ; C
\]

Clear.

\[
A(A) ; A = A(B) ; B
\]

Both sides yield the empty set.

\[
A : A(B) = A(A ; B) ; A
\]

Suppose \( c \in A ; A(B) \). Then \( c \in A \) and there is no arrow in \( B \) whose source is \( r(c) \). Since \( c \) is the unique arrow in \( A \) with \( d(c) \), there is then no pair of composable arrows \( a \in A \) and \( b \in B \) with \( d(a) = d(c) \). Hence \( 1_{d(c)} \) is in \( A(A ; B) \), so \( c \) belongs to the right-hand side.

Conversely, suppose \( c \in A(A ; B) ; A \), then in particular there is no \( (A, B) \)-path from \( d(c) \) so in particular, no arrow in \( B \) whose source is \( r(c) \). Hence \( c \in A ; A(B) \).

\[
D(A) ; B = D(A) ; C \wedge A(A) ; B = A(A) ; C \quad \rightarrow \quad B = C
\]

For any partition \( I , J \) of the identity arrows into two parts, any set \( D \) of arrows is the union of \( I ; D \) and \( J \wedge D \). Since \( D(A), A(A) \) is such a partition, the validity of the quasiequation follows.

\[
1' ; A = A
\]

We noted that \( A(A) ; A \) yields the empty set. So \( 1' \) is by definition \( A(\emptyset) \)—precisely the identity arrows. The equation is then clear.

\[
0 ; A = A
\]

Clear.

\[
D(R(A)) = R(A)
\]

The set \( R(A) \) is a set of identity arrows, and \( D := A^2 \) is the identity operation on any such set.

\[
A ; R(A) = A
\]

By definition \( R(A) \) is all identities on objects that are the target of some arrow in \( A \). So the validity of the equation is clear.

\[
A ; B = A ; C \quad \rightarrow \quad R(A) ; B = R(A) ; C
\]
This is the only case of note, for we must use the fact that all arrows of \( \mathcal{C} \) are epimorphisms. Assume the antecedent holds, and suppose \( b \in R(A) \cup B \). Then \( b \in B \) and there is some \( a \in A \) whose target is \( d(d) \). So \( a \cdot b \in A \cup B \) and hence, by the supposition, \( a \cdot b \in A \cup C \). Hence \( a \cdot b = a' \cdot c \), for some \( a' \in A \) and \( c \in C \), though necessarily \( a = a' \). Since \( a \) is an epimorphism, we obtain \( b = c \), hence \( b \in C \), and therefore \( b \in R(A) \cup C \). The reverse inclusion is of course by a symmetric argument.

\[
D(A) \colon (A \sqcup B) = A
\]

If \( a \in D(A) \colon (A \sqcup B) \) then firstly there is an arrow in \( A \) with the same source as \( a \). Secondly, by the definition of \( A \sqcup B \) on local sections, \( a \) is in either \( A \) or \( A(A) \cup B \). But \( a \) cannot be in \( A(A) \cup B \), since that implies there is not an arrow in \( A \) with the same source as \( a \). Hence \( a \in A \). Conversely, if \( a \in A \) then \( a \in A \sqcup B \) and \( 1d(a) \in D(A) \), so \( 1d(a) \cdot a = a \in D(A) \colon (A \sqcup B) \).

\[
A(A) \colon (A \sqcup B) = A(A) \cup B
\]

If \( a \in A(A) \colon (A \sqcup B) \) then firstly there is no arrow in \( A \) with the same source as \( a \). Secondly, by the definition of \( A \sqcup B \) on local sections, \( a \) is in either \( A \) or \( A(A) \cup B \)—it must be \( A(A) \cup B \). Conversely, if \( a \in A(A) \cup B \) then immediately \( a \in A \sqcup B \), using the definition of \( \sqcup \) on local sections again. That \( a \in A(A) \cup B \) also implies \( 1d(a) \in A(A) \). Hence \( 1d(a) \cdot a = a \in A(A) \colon (A \sqcup B) \).

### 5.2 The functor \( \text{SecCl} \) on morphisms

The action of \( \text{SecCl} \) on morphisms is given by inverse image. That is, given \( \mathcal{C}, \mathcal{D} \) Stone étale categories whose arrows are epimorphisms, and a sesqui-star-bijection partial functor \( g \colon \mathcal{C} \rightarrow \mathcal{D} \) that is total on objects, we define

\[
\text{SecCl}(g) \colon \text{SecCl}(\mathcal{D}) \rightarrow \text{SecCl}(\mathcal{C})
\]

\[
A \mapsto g^{-1}(A).
\]

First it must be checked that \( g^{-1}(A) \) really is an element of \( \text{SecCl}(\mathcal{C}) \). If \( b_1, b_2 \in g^{-1}(A) \) have the same source, then as \( g \) is a partial functor, \( g(b_1) \) and \( g(b_2) \)—which are both defined—have the same source. As \( A \) is a section, \( g(b_1) \) and \( g(b_2) \) must then be equal. By star injectivity of \( g \), we get \( b_1 = b_2 \). Hence \( g^{-1}(A) \) defines a function on \( d[g^{-1}(A)] \). As \( g \) is by assumption continuous, and \( d[A] \) is clopen, \( g^{-1}(d[A]) = d[g^{-1}(A)] \) (star surjectivity) is clopen. Since \( g^{-1}(A) \) is open, the second part of Lemma 5.4 implies that it corresponds to a continuous function. That \( \text{SecCl}(g) \) is a homomorphism of \( \{; A, R, \sqcup\} \)-algebras is Lemma 5.9. It is immediate from its definition by inverse images that \( \text{SecCl} \) is functorial, that it respects compositions of sesqui-star-bijection continuous functors, and acts as the identity on all identity functors.

\[\blacktriangleleft\text{Lemma 5.9.}\] The map \( \text{SecCl}(g) \colon \text{SecCl}(\mathcal{D}) \rightarrow \text{SecCl}(\mathcal{C}) \) of \( \{; A, R, \sqcup\} \)-algebras is a homomorphism.

\[\text{Proof.}\] We start by showing that \( ; \) is preserved. Let \( A \) and \( B \) be sections of \( \mathcal{D} \). If \( c \in g^{-1}(A) \cup g^{-1}(B) \) then by functoriality of \( g \), it is clear that \( g \) maps \( c \) into \( A \cup B \). Hence \( g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B) \). Conversely, if \( c \in g^{-1}(A \cup B) \), with \( g(c) = a \cdot b \) say, then by star surjectivity applied at \( d(a) \) there is a \( c_1 \) with same source as \( c \) and with \( g(c_1) = a \). Applying star surjectivity again at \( r(a) = d(b) \) we obtain a \( c_2 \) with \( c_1 \cdot c_2 = c \) and \( g(c_2) = b \). Hence \( c \in g^{-1}(A) \cup g^{-1}(B) \), and we conclude that \( g^{-1}(A) \cup g^{-1}(B) = g^{-1}(A \cup B) \).
Next we show that $A$ is preserved. Let $B$ be a section of $\mathcal{D}$. First suppose $a \in g^{-1}(A(B))$, with $g(a) = 1_y \in A(B)$ say. Then by functoriality of $g$, the identity arrow of $d(a)$ is mapped to $1_y$. By star injectivity of $g$, we find $a = 1_{d(a)}$. Since $1_y \in A(B)$ there is no member of $B$ with domain $y$, hence there is no member of $g^{-1}(B)$ with domain $d(a)$. So by definition, $1_{d(a)} \in A(g^{-1}(B))$, that is, $a \in A(g^{-1}(B))$. We have our first inclusion: $g^{-1}(A(B)) \subseteq A(g^{-1}(B))$. Conversely, suppose $1_x \in A(g^{-1}(B))$. We need to argue that $1_{g(x)}$—which we know $g(x)$ is defined, because $g$ is total on objects—is in $A(B)$. But if there were a member $b$ of $B$ with source $g(x)$, then by star surjectivity of $g$, there would be an arrow with source $x$ mapped by $g$ to $b$, contradicting the fact that $1_x \in A(g^{-1}(B))$. Hence $1_{g(x)}$ is indeed in $A(B)$, and we conclude that $g^{-1}(A(B)) = A(g^{-1}(B))$.

Showing that $R$ is preserved is similar, but simpler. Again, let $B$ be a section of $\mathcal{D}$. First suppose we have an element in $g^{-1}(R(B))$. By functoriality and star injectivity of $g$, our element of $g^{-1}(R(B))$ is an identity element, $1_x$ say. By definition of $R$, there is some $b \in B$ with target $g(x)$. By co-star surjectivity of $g$ there is an $a$ with target $x$ such that $g(a) = b$. So $a \in g^{-1}(B)$, hence $1_x \in R(g^{-1}(B))$. We conclude that $g^{-1}(R(B)) \subseteq R(g^{-1}(B))$. Conversely, suppose $1_x \in R(g^{-1}(B))$. So there is an $a$ with target $x$ such that $g(a) \in B$. Then $1_{g(x)} \in R(B)$, so $1_x \in g^{-1}(R(B))$. We conclude that $g^{-1}(R(B)) = R(g^{-1}(B))$.

Finally we note that $\sqcup$ is preserved, because of the definition $A \sqcup B := A \cup A(A) \cup B$, and the elementary fact that inverse images (even of partial functions) preserve unions.

The next lemma relates to the restricted duality of Theorem 3.7.

\begin{lemma}
If $g$ is a total functor, the map $\text{SecCl}(g) : \text{SecCl}(\mathcal{D}) \to \text{SecCl}(\mathcal{C})$ of \{$; A, R, \sqcup$\}-algebras is locally proper.
\end{lemma}

\begin{proof}
Take $P$ a prime filter in $\text{SecCl}(\mathcal{C})$. Then by Lemma 4.3, we know $D[P]$ is an ultrafilter of $D[\text{SecCl}(\mathcal{C})]$.
\end{proof}

\begin{example}
Returning again to our running example (Example 2.10 and Example 4.14), the reader may examine the (discrete) category $PF(\mathfrak{A})$ described in Example 4.14 and calculate its dual. Sections must contain at most one arrow with source ‘1, 2’ (there are three such arrows, so four possible choices), and at most one arrow with source ‘3’ (one arrow, so two choices). In total there are eight sections, so eight element of the dual of $PF(\mathfrak{A})$. One can verify that this ‘double dual’ is isomorphic to the original algebra $\mathfrak{A}$.
\end{example}

\section{The functors form a duality}

In this section we will first show that the double dual functor on the category $\mathcal{A}$ of representable \{$; A, R, \sqcup$\}-algebras is naturally isomorphic to the identity functor. Then we will do the same for the double dual functor on the category $\mathcal{C}$ of Stone étale categories all of whose arrows are epimorphisms. This will complete the proof that we have given a duality between the categories $\mathcal{A}$ and $\mathcal{C}$.

\subsection{The double dual on algebras}

First we describe an isomorphism from $\mathfrak{A}$ to $\text{SecCl}(PF(\mathfrak{A}))$ for an arbitrary representable \{$; A, R, \sqcup$\}-algebra. Then we will show this construction is natural.

In fact, our isomorphism is already hidden in notation we have defined. Recall that for $a \in \mathfrak{A}$, the set $a^0$ is defined to be $\{P \in pf(\mathfrak{A}) \mid a \in P\}$. We define $\theta : \mathfrak{A} \to \text{SecCl}(PF(\mathfrak{A}))$ by $a \mapsto a^0$. Note that $a^0$ is indeed an element of the algebra $\text{SecCl}(PF(\mathfrak{A}))$, for it is clearly
a section, and its domain is the set \( \widehat{D(a)} := \{ \mu \in \text{uf}(D[A]) \mid D(a) \in \mu \} \), which is open by definition and closed because \( \Lambda(\widehat{a}) \) is open.

To see that \( \theta \) is injective, it suffices to show that when \( a \not\leq b \), there exists a prime filter containing \( a \) but not \( b \). This argument can be found in the proof of Lemma 4.9 in [12]. To see that \( \theta \) is surjective, we need to argue that all sections on clopens of \( PF(\mathfrak{A}) \) are of the form \( a^\theta \). Let \( A \) be a section on a clopen of \( PF(\mathfrak{A}) \). By a similar argument to that in the proof of Lemma 4.17, for each \( P \in A \) there is an \( a_P \) with \( P \in a_P^\theta \subseteq A \). By compactness of the space of objects of \( PF(\mathfrak{A}) \), the domain of \( A \) can be covered by \( \widehat{D(a_P^1)}, \ldots, \widehat{D(a_P^n)} \) for some finite \( n \). Then \( A = (a_P \sqcup \cdots \sqcup a_P) \).

**Lemma 6.1.** The map \( \theta : \mathfrak{A} \rightarrow \text{SecCl}(PF(\mathfrak{A})) \) given by \( a \mapsto a^\theta \) is a homomorphism of \( \{; A, R, |\} \)-algebras

**Proof.** To appear shortly! ▷

We now show that our isomorphisms together give a natural transformation from the identity functor to the double dual. For each \( \mathfrak{A} \in \mathfrak{A} \), denote now the isomorphism just described by \( \theta_\mathfrak{A} \). Then given \( \mathfrak{A}, \mathfrak{B} \in \mathfrak{A} \) and a homomorphism \( f : \mathfrak{A} \rightarrow \mathfrak{B} \), we are required to show that \( \text{SecCl}(PF(f)) \circ \theta_\mathfrak{A} = \theta_\mathfrak{B} \circ f \). The right-hand side sends an element \( a \in \mathfrak{A} \) to the set \( f(a)^\theta \) of prime filters of \( \mathfrak{B} \). Seeing that the left-hand side has the same effect just involves unravelling the definitions. The element \( a \) is sent first to \( a^\theta \), then \( \text{SecCl}(PF(f)) \) sends this to

\[
\{ P \in PF(\mathfrak{B}) \mid PF(f)(P) \in a^\theta \} = \{ P \in PF(\mathfrak{B}) \mid f^{-1}(P) \in a \}
\]

\[
= \{ P \in PF(\mathfrak{B}) \mid a \in f^{-1}(P) \}
\]

\[
= \{ P \in PF(\mathfrak{B}) \mid f(a) \in P \}
\]

\[
= f(a)^\theta
\]

as required.

### 6.2 The double dual on categories

Let \( \mathcal{C} \in \mathcal{C} \). Define \( \varphi : \mathcal{C} \rightarrow PF(\text{SecCl}(\mathcal{C})) \) by \( c \mapsto c^\varphi := \{ A \in \text{SecCl}(\mathcal{C}) \mid c \in A \} \). We will first verify that \( c^\varphi \) is a prime filter in \( \text{SecCl}(\mathcal{C}) \), so \( \varphi \) indeed has codomain \( PF(\text{SecCl}(\mathcal{C})) \) (and is everywhere defined). As \( d \) is a local homeomorphism from the arrows to the objects of \( \mathcal{C} \), there exists some section on an open \( s \) that has \( c \) in its image. Since the space of objects of \( \mathcal{C} \) has a basis of clopens, we can restrict \( s \) to a clopen still with \( c \) in its image. Hence \( c^\varphi \) is nonempty. By a similar argument, \( c^\varphi \) is down directed. It is trivial that \( c^\varphi \) is upwards closed, and straightforward that it satisfies the primality condition.

**Lemma 6.2.** The map \( \varphi \) is a (total) functor between the categories \( \mathcal{C} \) and \( PF(\text{SecCl}(\mathcal{C})) \).

**Proof.** To appear shortly! ▷

To see that \( \varphi \) is injective let \( c, d \in \mathcal{C} \). Choose any section on a clopen \( A \) that contains \( c \). If \( d(c) = d(d) \), then \( A \) cannot contain \( d \), so \( c^\varphi \) and \( d^\varphi \) are not equal. If \( d(c) \neq d(d) \), then we can find a clopen set \( U \) of objects that contains \( d(c) \) but not \( d(d) \). By restricting \( A \) to \( U \), we obtain a section on a clopen containing \( c \) but not \( d \), so again \( c^\varphi \) and \( d^\varphi \) are not equal.

To see that \( \varphi \) is surjective, take a prime filter \( P \) of sections on clopens of \( \mathcal{C} \). Let \( S = \cap P \). If \( c \in S \) then \( P \subseteq c^\varphi \) and so \( P = c^\varphi \). Hence we only need to show \( S \) cannot be empty. Let \( A \in P \). As \( P \) is a filter \( \cap P = \emptyset \) implies \( \cap \{ B \in P \mid B \subseteq A \} = \emptyset \). As \( A \) is compact, this
implies $B_1 \cap \cdots \cap B_n = \emptyset$ for some $B_1, \ldots, B_n \in \{B \in P \mid B \subseteq A\}$. As $P$ is a filter, this implies $\emptyset \in P$—the required contradiction.

Since $\varphi$ is bijective, it is certainly sesqui-star-bijective. To show $\varphi$ is an isomorphism in $C$ it remains to show that $\varphi$ and its inverse are continuous. First we need a lemma.

**Lemma 6.3.** In any Stone étale category, the sections on clopens provide a basis for the topology on the arrows.

**Proof.** Let $U$ be an open set of arrows, and $c \in U$. As $d$ is a local homeomorphism, $c$ has an open neighbourhood $V$, which we may assume is a subset of $U$, such that $d|_V$ provides a homeomorphism onto its image, which is also open. That is, $d|_V^{-1}$ is a section on an open.

As the set of objects has a clopen basis, we may restrict $d|_V^{-1}$ to a clopen containing $d(c)$, giving the section on a clopen that we seek. ◀

By the lemma, to show that $\varphi$ is a clopen map, it suffices to consider an arbitrary section on a clopen $A$ of $\mathcal{C}$. We claim that $\varphi[A]$ equals $A^0$, and is therefore in particular open. If $c^\varphi \in \varphi[A]$ (for $c \in A$), then as $A$ is a section on a clopen, $A \in c^\varphi$, so $c^\varphi \in A^0$. That is $\varphi[A] \subseteq A^0$. Now let $P \in A^0$, and suppose for a contradiction that $P \not\in \varphi[A]$. Since for prime filters inclusion implies equality, $P \not\in \varphi[A]$ implies we can find, for each $c \in A$, a section on a clopen $B_c$ that contains $c$, but is not in $P$. We may assume (by the same reasoning as in the proof of Lemma 6.3) that each $B_c$ is a subset of $A$. Then as $A$ is compact, some finite collection $B_{c_1}, \ldots, B_{c_n}$ cover $A$. That is, $A = B_{c_1} \cup \cdots \cup B_{c_n}$ in SecCl$(\mathcal{C})$. As $P$ is prime and contains $A$, it must contain some $B_{c_i}$—the required contradiction.

Continuity of $\varphi$ now follows straightforwardly. That $\varphi[A] = A^0$ and $\varphi$ is injective implies $\varphi^{-1}(A^0) = A$, which is open if $A$ is a section on a clopen. The $A^0$’s, for such $A$’s provide a basis for $\text{PF}(\text{SecCl}(\mathcal{C}))$, so we are done.

We now show that our isomorphisms together give a natural transformation from the identity functor to the double dual. For each $\mathcal{C} \in \mathcal{C}$, denote now the isomorphism just described by $\varphi_\mathcal{C}$. Then given $\mathcal{C}, \mathcal{D} \in \mathcal{C}$ and a sesqui-star-bijective partial functor $g: \mathcal{C} \to \mathcal{D}$, we are required to show that $\text{PF}(\text{SecCl}(g)) \circ \varphi_\mathcal{C} = \varphi_\mathcal{D} \circ g$ (as partial functors). The right-hand side sends an arrow $c \in \mathcal{C}$ to $g(c)^\varphi$ if $g(c)$ is defined, and is undefined otherwise. On the left-hand side, $c$ is first sent to $c^\varphi$, then $\text{PF}(\text{SecCl}(g))$ sends this to

$$\{A \in \text{SecCl}(\mathcal{D}) \mid \text{SecCl}(g)(A) \in c^\varphi\} = \{A \in \text{SecCl}(\mathcal{D}) \mid g^{-1}(A) \in c^\varphi\}$$

$$= \{A \in \text{SecCl}(\mathcal{D}) \mid c \in g^{-1}(A)\}$$

$$= \{A \in \text{SecCl}(\mathcal{D}) \mid g(c) \text{ is defined and } g(c) \in A\}$$

if this set is nonempty; otherwise $\text{PF}(\text{SecCl}(g))(c^\varphi)$ is undefined. But the set is nonempty precisely in the case that $g(c)$ is defined, and then it equals $g(c)^\varphi$, exactly as required. This completes the proof of Theorem 3.6.

### 7 Word-to-word functions

Let $\Sigma$ be a finite alphabet. Recall that the **rational** functions over $\Sigma$, which we denote $\text{Rat}_t(\Sigma)$, are the partial functions from $\Sigma^*$ to $\Sigma^*$ realisable by a one-way transducer. The **regular** functions over $\Sigma$, which we denote $\text{Reg}_t(\Sigma)$, are the partial functions from $\Sigma^*$ to $\Sigma^*$ realisable by a two-way transducer. The rational and the regular functions are both closed under $\cdot$, $\Lambda$, $R$, and $\sqcup$, and hence are both $\{\cdot, \Lambda, R, \sqcup\}$-algebras of partial functions with base $\Sigma^*$. Clearly

$$\text{Rat}_t(\Sigma) \subseteq \text{Reg}_t(\Sigma),$$
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and so \( \text{Rat}_\Sigma(\Sigma) \) is a subalgebra of \( \text{Reg}_\Sigma(\Sigma) \). For both \( \text{Rat}_\Sigma(\Sigma) \) and \( \text{Reg}_\Sigma(\Sigma) \), the subalgebra of subidentity functions is the set of identity functions on regular languages (by a simple ‘forgetting the output’ argument). The Stone dual of the regular languages over \( \Sigma \) is known to be (the underlying space of) the profinite completion \( \hat{\Sigma}^* \) of the monoid \( \Sigma^* \) [16]. Hence both \( \text{Rat}_\Sigma(\Sigma) \) and \( \text{Reg}_\Sigma(\Sigma) \) have duals whose space of objects is \( \hat{\Sigma}^* \).

Since the category of representable \( \{; \cdot, \Lambda, R, \sqcup\} \)-algebras with (arbitrary) homomorphisms is a concrete category, the embedding \( \text{Rat}_\Sigma(\Sigma) \hookrightarrow \text{Reg}_\Sigma(\Sigma) \) is a monomorphism in that category.

\begin{itemize}
  \item \textbf{Problem 7.1. Is the embedding} \( \text{Rat}_\Sigma(\Sigma) \hookrightarrow \text{Reg}_\Sigma(\Sigma) \) \textit{locally proper}?
\end{itemize}

The answer is no, by the following general result.

\begin{itemize}
  \item \textbf{Proposition 7.2. Let} \( f : \mathfrak{A} \rightarrow \mathfrak{B} \) \textit{be a locally proper homomorphism of} \( \{; \cdot, \Lambda, R, \sqcup\} \)-\textit{algebras such that the induced map} \( D[\mathfrak{A}] \rightarrow D[\mathfrak{B}] \) \textit{is an isomorphism}. \textit{Then} \( f \) \textit{is an isomorphism}.
\end{itemize}

\textbf{Proof.} If the hypotheses hold, then the dual \( g : \text{PF}(\mathfrak{B}) \rightarrow \text{PF}(\mathfrak{A}) \) of \( f \) is a total functor and a bijection on objects. But any star-bijective total functor that is bijective on arrows, and thus \( g \) is an algebraic isomorphism of categories. It follows in particular (surjectivity of \( g \)) that \( f \) is injective, so we may assume \( f \) is an inclusion of a subalgebra and the induced \( D[\mathfrak{A}] \rightarrow D[\mathfrak{B}] \) is the identity.

We know that \( g \) is continuous, so to show that \( g \) is an isomorphism of topological categories it only remains to show \( g \) is an open map. From there the conclusion that \( f \) is an isomorphism is immediate, by duality.

The topology on \( \text{PF}(\mathfrak{A}) \) is generated by sets of the form \( a^\theta \) where \( a \in \mathfrak{A} \), and the topology on \( \text{PF}(\mathfrak{B}) \) is generated by \( b^\theta \) for \( b \in \mathfrak{B} \). We must show that each \( b^\theta \) is open in the topology of \( \text{PF}(\mathfrak{B}) \). For \( b \in \mathfrak{B} \), let \( P \) be a prime filter of \( \mathfrak{B} \) containing \( b \). Then \( \mu = D[P] \) is an ultrafilter on \( D[\mathfrak{A}] = D[\mathfrak{B}] \) containing \( D(b) \). Then we know there is some \( a \in \mathfrak{A} \) such that...That is, \( \alpha; a = \alpha; b \) for some \( \alpha \in \mu \). So \( b^\theta \supseteq (\alpha; b)^\theta \), with \( (\alpha; a) \in \mathfrak{A} \). Since \( P \) was an arbitrary point in \( b^\theta \), we conclude that \( b^\theta \) is open in the topology of \( \text{PF}(\mathfrak{A}) \). \hfill \blacktriangleleft

Since \( \text{Rat}_\Sigma(\Sigma) \) and \( \text{Reg}_\Sigma(\Sigma) \) have the same Boolean subalgebra of domain elements—the regular languages encoded as subidentity functions—if the embedding \( \text{Rat}_\Sigma(\Sigma) \hookrightarrow \text{Reg}_\Sigma(\Sigma) \) were locally proper, Proposition 7.2 would apply. But \( \text{Rat}_\Sigma(\Sigma) \) is strictly included in \( \text{Reg}_\Sigma(\Sigma) \), hence the embedding cannot be locally proper.

We conclude with a problem.

\begin{itemize}
  \item \textbf{Problem 7.3. Give descriptions of the duals of} \( \text{Rat}_\Sigma(\Sigma) \) \textit{and} \( \text{Reg}_\Sigma(\Sigma) \), \textit{and of the dual of the embedding} \( \text{Rat}_\Sigma(\Sigma) \hookrightarrow \text{Reg}_\Sigma(\Sigma) \).
\end{itemize}

\section*{8 Appendix: examples}

In this appendix, we give some examples of algebras and their duals, starting with the some of the simplest \( \{; \cdot, \Lambda, R, \sqcup\} \)-algebras that are representable by partial functions. Later, we also give some examples of locally proper and non locally proper morphisms of representable \( \{; \cdot, \Lambda, R, \sqcup\} \)-algebras—see Definition 2.11. Similarly, we give some examples of allowed and disallowed morphisms on the dual side.

Recall that in the signature \( \{; \cdot, \Lambda, R, \sqcup\} \) the constants 0 and 1’, and the unary operation \( D \) are definable and must have in any representation their intended interpretations: empty function, identity function, and domain, respectively. In a finite representable \( \{; \cdot, \Lambda, R, \sqcup\} \)-algebra every prime filter must be principal, and so the topology on the dual is the discrete...
topology. In fact this observation applies more generally to any occasion when the Boolean subalgebra of domain elements is finite.

► Example 8.1 (the trivial \{; A, R, \sqcup\}-algebra). If the Boolean subalgebra of domain elements is trivial, then \( 0 = 1' \), and hence the algebra itself is trivial, for then \( a = 1' ; a = 0 ; a = 0 \) for any \( a \). This algebra can be represented by partial functions as the empty function on the empty set. The trivial algebra has no prime filters. Its dual is the empty category—no objects and no morphisms.

► Example 8.2 (two-element Boolean algebra of domain elements). Next we consider \{; A, R, \sqcup\}-algebras whose Boolean subalgebra of domain elements is the two-element Boolean algebra \( 2 \). We can calculate which algebras are possible by considering their duals.

The dual of \( 2 \) is a space with a single point. Thus the duals of the \{; A, R, \sqcup\}-algebras are precisely the categories with one object all of whose arrows are epimorphisms. Since in such categories all pairs of arrows are composable, the sets of arrows form \textit{left-cancellative monoids}, and all such monoids are obtainable. (Recall that we have reversed the usual convention for composition; thus an epimorphism \( a \) validates \( a \cdot b = a \cdot c \rightarrow b = c \).

The \{; A, R, \sqcup\}-algebra obtained from a left-cancellative monoid \((M, \cdot)\) will be as follows.

- The \( ; \)-reduct will be a left-cancellative monoid with zero (with \( ; \) agreeing with \( \cdot \) on nonzero elements).
- \( A(0) = 1' \) and for all other \( a \), we have \( A(a) = 0 \).
- \( R(0) = 0 \) and for all other \( a \), we have \( R(a) = 1' \).
- The \( \sqcup \)-reduct will be a left-zero band on nonzero elements, that is \( a \sqcup b = a \), for nonzero \( a, b \). For 0 we have \( 0 \sqcup a = a \sqcup 0 = a \).

The algebra is finite if and only if the category/monoid is finite. Thus our duality restricts to a one-to-one correspondence between such finite \{; A, R, \sqcup\}-algebras and the finite left-cancellative monoids, that is to say, the finite groups.

► Example 8.3 (four-element Boolean algebra of domain elements). If the Boolean subalgebra of domain elements is the next smallest Boolean algebra, \( 2^2 \), then the dual is a category with two objects. There are three distinct cases to consider.

1. The source of each arrow equals its target. In this case we just have two disjoint instances \( M, N \) of the categories in Example 8.2 (the coproduct of these categories in \( \text{Cat} \)). The corresponding \{; A, R, \sqcup\}-algebra is the product \( M_0 \times N_0 \) of the algebras dual to \( M \) and \( N \).

2. There exist arrows in one direction, from object 1 to object 2 say, but none in the opposing direction. The submonoid \( \text{Hom}(2, 2) \) can be any left-cancellative monoid. Any arrow \( a : 1 \rightarrow 2 \), being an epimorphism, induces an injection \( \text{Hom}(2, 2) \rightarrow \text{Hom}(1, 2) \), so necessarily the cardinality of \( \text{Hom}(1, 2) \) is at least that of \( \text{Hom}(2, 2) \). It is possible for the cardinality of \( \text{Hom}(1, 1) \) to be strictly greater or strictly less than that of \( \text{Hom}(1, 2) \) and \( \text{Hom}(2, 2) \), as we will see in later examples.

Elements of the corresponding \{; A, R, \sqcup\}-algebra can be partitioned into \( 3 \times 2 = 6 \) sets according to which types of arrows their double dual contains.

3. There exist arrows from object 1 to object 2, and arrows from object 2 to object 1. These provide injections \( \text{Hom}(1, 1) \rightarrow \text{Hom}(2, 1) \) and \( \text{Hom}(1, 1) \rightarrow \text{Hom}(1, 1) \) so by Schröder–Bernstein these hom-sets have the same cardinality. Similarly \( \text{Hom}(2, 2) \) and \( \text{Hom}(1, 2) \)
have the same cardinality. In the finite case, all arrows then induce bijections on hom-sets.
So given \( a: 1 \to 2 \) there exists \( b: 2 \to 1 \) such that \( a \cdot b \) is the identity \( e \) of the group \( \text{Hom}(1,1) \). Then \( a \cdot (b \cdot a) = (a \cdot b) \cdot a = e \cdot a = a = a \cdot e \), so as \( a \) is an epimorphism, \( b \cdot a = e \), thus \( a \) and \( b \) are two-sided inverses of one another. As \( a \) was arbitrary, every arrow in \( \text{Hom}(1,2) \) is an isomorphism. It follows that the category is a groupoid (every arrow is an isomorphism).

Elements of the corresponding \( \{; A, R, \cup\} \)-algebra can be partitioned into \( 3 \times 3 = 9 \) sets according to which types of arrows their double dual contains.

It is clear from the discussion in Example 8.3, Item 3, that in any finite dual category, given any strongly connected set of objects, the induced subcategory will form a groupoid.

We now examine algebras representable as partial functions on small base sets. These are mostly special cases of the examples that have gone before. We start at base size \( 2 \).

**Example 8.4 (algebras representable on a base of size 2).** Let \( A \) be an algebra of the signature \( \{; A, R, \cup\} \) that is representable by partial functions on the base \( X = \{1, 2\} \). The subalgebra of domain elements can be either \( 2 \) or \( 2^2 \). In the first case, two algebras are possible. Either only \( 0 \) and \( 1' \) are present, in which case the dual is the discrete category on one element, or a third element is present: the function that swaps \( 1 \) and \( 2 \). The dual of the three-element algebra is the one-object category with group of arrows isomorphic to \( C_2 \), the cyclic group of order \( 2 \).

When the subalgebra of domain elements is \( 2^2 \), up to isomorphism, three algebras are possible. They are depicted using their duals in the following diagrams. Since there is at most one arrow in each hom-set, the double duals provide representations of the algebras.

The second part of Example 8.4 illustrates a basic principal: if the domain subalgebra of a (finite) algebra \( A \) has as many atoms as the cardinality of the smallest base set on which \( A \) is representable, then the dual of \( A \) will be a preorder, and the double dual will provide a representation. Conversely, any preorder on a finite set is the dual of some such algebra.

**Example 8.5 (algebras representable on a base of size 3).** Let \( A \) be an algebra of the signature \( \{; A, R, \cup\} \) that is representable by partial functions on the base \( X = \{1, 2, 3\} \). The subalgebra of domain elements can be \( 2, 2^2, \) or \( 2^3 \). In the first case, again two algebras are possible. Either only \( 0 \) and \( 1' \) are present, or two further elements are present: the two derangements of \( \{1, 2, 3\} \). The dual of this four-element algebra is the one-object category with group of arrows isomorphic to \( C_3 \).

If the subalgebra of domain elements is \( 2^2 \), we may assume its two atoms are the identities on \( \{1, 2\} \) and \( \{3\} \). To complete the set of atoms of \( A \), the functions \( \{1 \mapsto 3, \ 2 \mapsto 3\} \) and the swap of \( \{1, 2\} \) may (independently) optionally be present. If both these atoms are present, then \( A \) contains \( 4 \times 2 = 8 \) elements. This is an example where the dual looks like in Example 8.3, Item 2 and the upstream hom-set is larger than those downstream.
If the subalgebra of domain elements is $2^3$, then the possible duals are all the possible preorders on $\{1, 2, 3\}$.

Example 8.6 (algebras representable on a base of size 4). We give an example where the dual looks like in Example 8.3, Item 2 and the upstream hom-set is smaller than those downstream. Let $\mathfrak{A}$ be the \{; A, R, ∪\}-algebra of partial functions on base $\{1, 2, 3, 4\}$ with the following five atoms: the identity on $\{1, 2\}$, the identity on $\{3, 4\}$, the swap on $\{3, 4\}$, the function $\{1 \mapsto 3, 2 \mapsto 4\}$, and the function $\{1 \mapsto 4, 2 \mapsto 3\}$. These are the arrows of the dual of $\mathfrak{A}$. The dual is depicted in the following diagram.

References

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