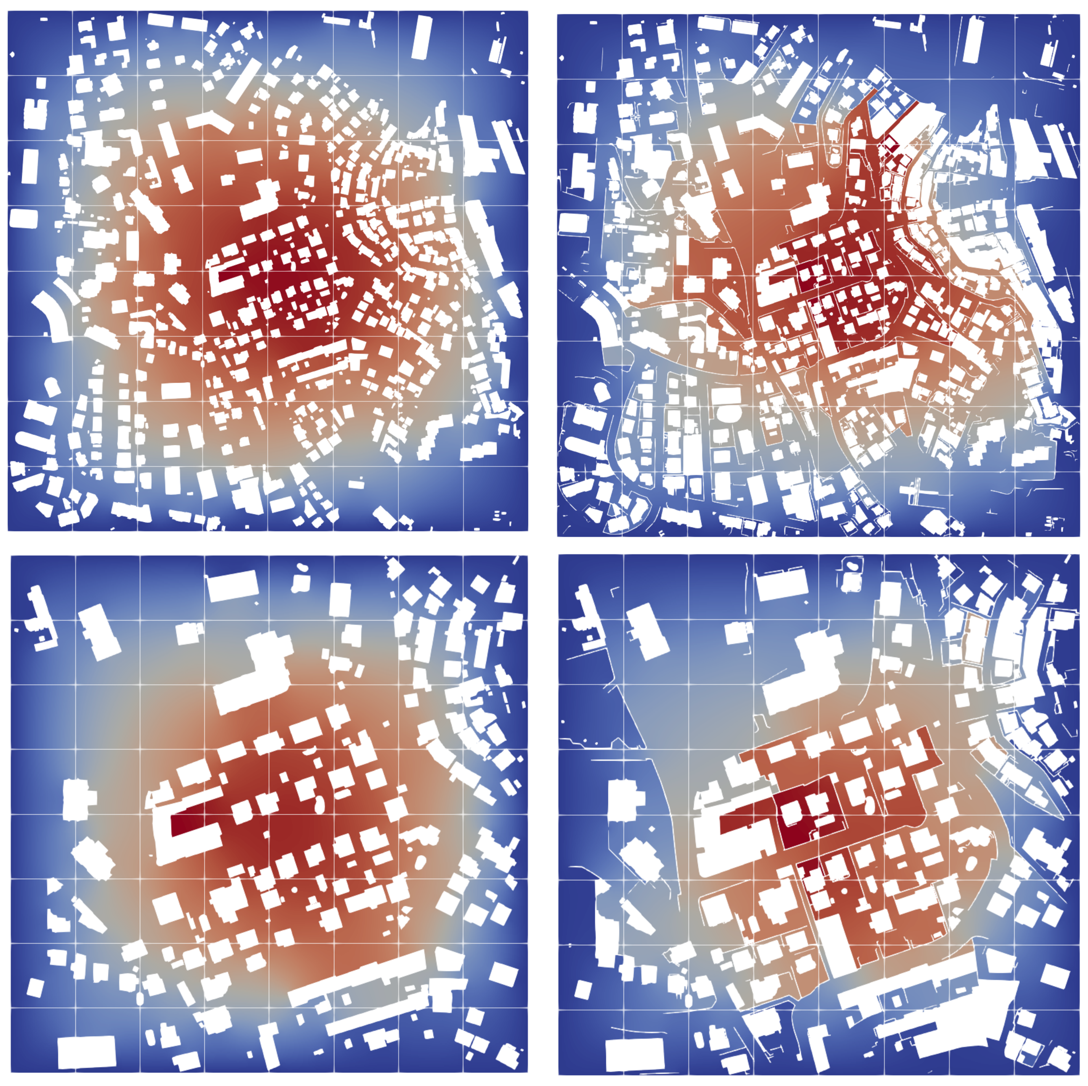
A Trefftz-like coarse space for the two-level Schwarz method on perforated domains

Miranda Boutilier*, Konstantin Brenner*† and Victorita Dolean*‡

*Université Côte d'Azur, CNRS † INRIA Team COFFEE ‡ University of Strathclyde

Model problem and perforated domain



Model geometry. Left to right: buildings no walls, buildings with walls. Top to bottom: larger data set, smaller data set. Both data sets share the same centre and are partitioned into 8×8 nonoverlapping subdomains.

Model geometry

- D: Open simply connected domain polygonal in $\mathbb{R}^2$;
- $(\Omega_{S,k})_k$: Finite family of perforations in D;
- $\Omega_S = \bigcup_k \Omega_{S,k}$ and $\Omega = D \setminus \overline{\Omega_S}$.

Model problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ -\dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_S, \\ u = 0 & \text{on} \quad \partial\Omega \setminus \partial\Omega_S. \end{cases} \qquad (1)$$

Finite element approximation

Notations:

- L^2 scalar product $(u, f) = \int_\Omega uf$;
- Bilinear form $a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in H^1(\Omega)$;
- $H^1_{\setminus \partial\Omega_S}(\Omega) = \{u \in H^1(\Omega) \mid u_{\partial\Omega \setminus \partial\Omega_S} = 0\}$;
- Finite element space $V_{h,0}(\Omega) \subset H^1_{\setminus \partial\Omega_S}(\Omega)$.

The finite element solution of (1) is given by: Find $u_h \in V_{h,0}(\Omega)$ s.t.

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_{h,0}(\Omega).$$

Additive Schwarz Framework

Notations

- $\left(\Omega'_j\right)_{j\in\{1,\dots,N\}}$: Nonoverlapping partitioning;
- $(\Omega_j)_{j\in\{1,\dots,N\}}$: Overlapping partitioning such that $\Omega'_j \subseteq \Omega_j$;
- $V_h(\Omega_j) = \{v|_{\Omega_j} : v \in V_h\}$: Space of restrictions to Ω_j;
- $V_{h,0}(\Omega_j) = \{v|_{\Omega_j} : v \in V_h, \text{supp}(v) \subset \Omega_j\}$: Space of finite element functions supported in Ω_j;
- $\mathcal{R}_j^T : V_{h,0}(\Omega_j) \to V_h$: Extension (by zero) operators.

Discrete ASM preconditioner [3]

$$M_{as,2}^{-1} = R_0^T(R_0AR_0^T)^{-1}R_0 + \sum_{j=1}^{N} R_j^T(R_jAR_j^T)^{-1}R_j.$$

- R_j and R_j^T: matrix representations of $\mathcal{R}_j$ and $\mathcal{R}_j^T$;
- R_0 corresponds to the coarse space.

Trefftz-like coarse space

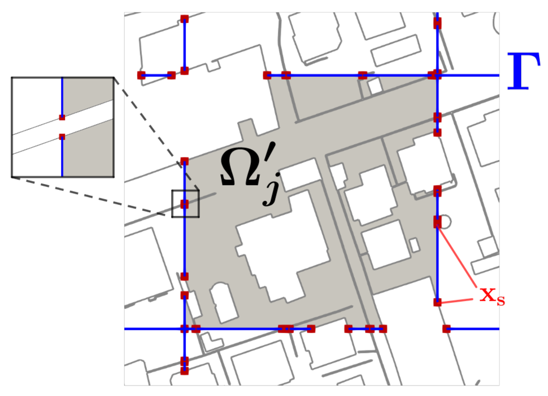
Notations

- Nonoverlapping skeleton: $\Gamma = \bigcup_{j\in\{1,\dots,N\}} \partial\Omega'_j$;
- $N_\mathbf{x}$ coarse grid nodes generated at $\Gamma \cap \Omega_S$: $(\mathbf{x}_s)_{s\in\{1,\dots,N_\mathbf{x}\}}$;
- Locally harmonic functions for each coarse grid node: $(\phi_s)_{s\in\{1,\dots,N_\mathbf{x}\}}$;

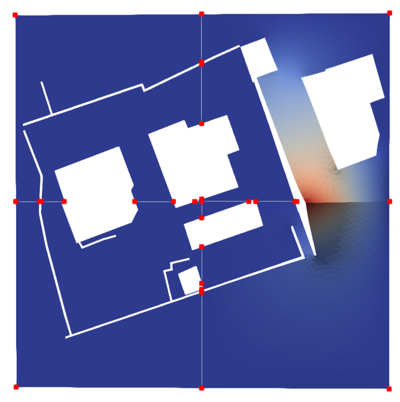
Continuously, the coarse space is given by

$$V_c = \text{span}\{\phi_s\}.$$

- $\text{supp}(\phi_s) = \{\bigcup_j \Omega'_j \mid \mathbf{x}_s \text{ is a coarse grid node belonging to } \partial\Omega'_j\}$.



Coarse grid nodes (large red dots).



Basis function at a given coarse grid node.

Basis functions

For each coarse grid node $\mathbf{x}_s$, define $g_s : \Gamma \to [0,1]$ as: for $i = 1, \dots N_\mathbf{x}$,

$$g_s(\mathbf{x}_i) = \begin{cases} 1, & s = i, \\ 0, & s \neq i, \end{cases}$$

- g_s is linearly extended on the remainder of Γ.

For all nonoverlapping $\left(\Omega'_j\right)_{j\in\{1,\dots,N\}}$ and $s = 1, \dots, N_\mathbf{x}$, to obtain $\phi_{s,j} = \phi_s|_{\Omega_j}$, solve

$$\begin{cases} -\Delta\phi_{s,j} = 0 & \text{in} \quad \Omega'_j, \\ -\dfrac{\partial\phi_{s,j}}{\partial n} = 0 & \text{on} \quad \partial\Omega'_j \cap \partial\Omega_S, \\ \phi_{s,j} = g_s & \text{on} \quad \partial\Omega'_j \setminus \partial\Omega_S. \end{cases}$$

Nicolaides by connected component

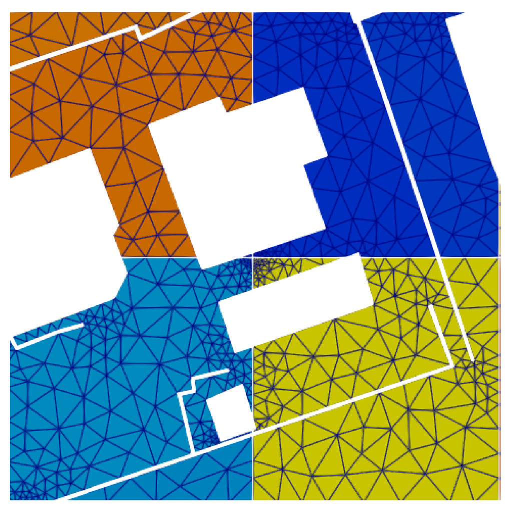
New partitioning

Given initial nonoverlapping partitioning; $\left(\Omega'_j\right)_{j\in\{1,\dots,N\}}$ and:

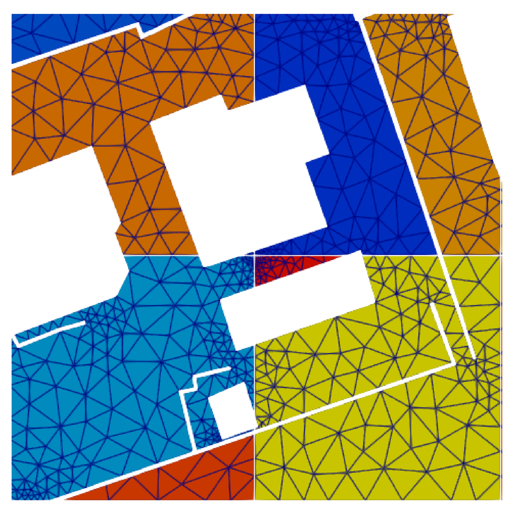
- m_j connected components in each Ω'_j: $\left(\Omega'_{j,l}\right)_{l\in\{1,\dots,m_j\}}$;
- $m = \sum_{j=1}^N m_j$.

The new subdomain partioning is given by

$$\left(\widehat{\Omega}_k\right)_{k\in\{1,\dots,m\}} = \left((\Omega'_{j,l})_{l\in\{1,\dots,m_j\}}\right)_{j\in\{1,\dots,N\}}$$



Original partitioning (Trad. Nic.)



New partitioning (Enhanced Nic.)

Coarse space

The coarse space is defined as a column space of the matrix Z, with columns given by

$$Z_k := \widehat{R}_k^T \widehat{D}_k \widehat{R}_k \mathbf{1}, \qquad k = 1, \dots, m.$$

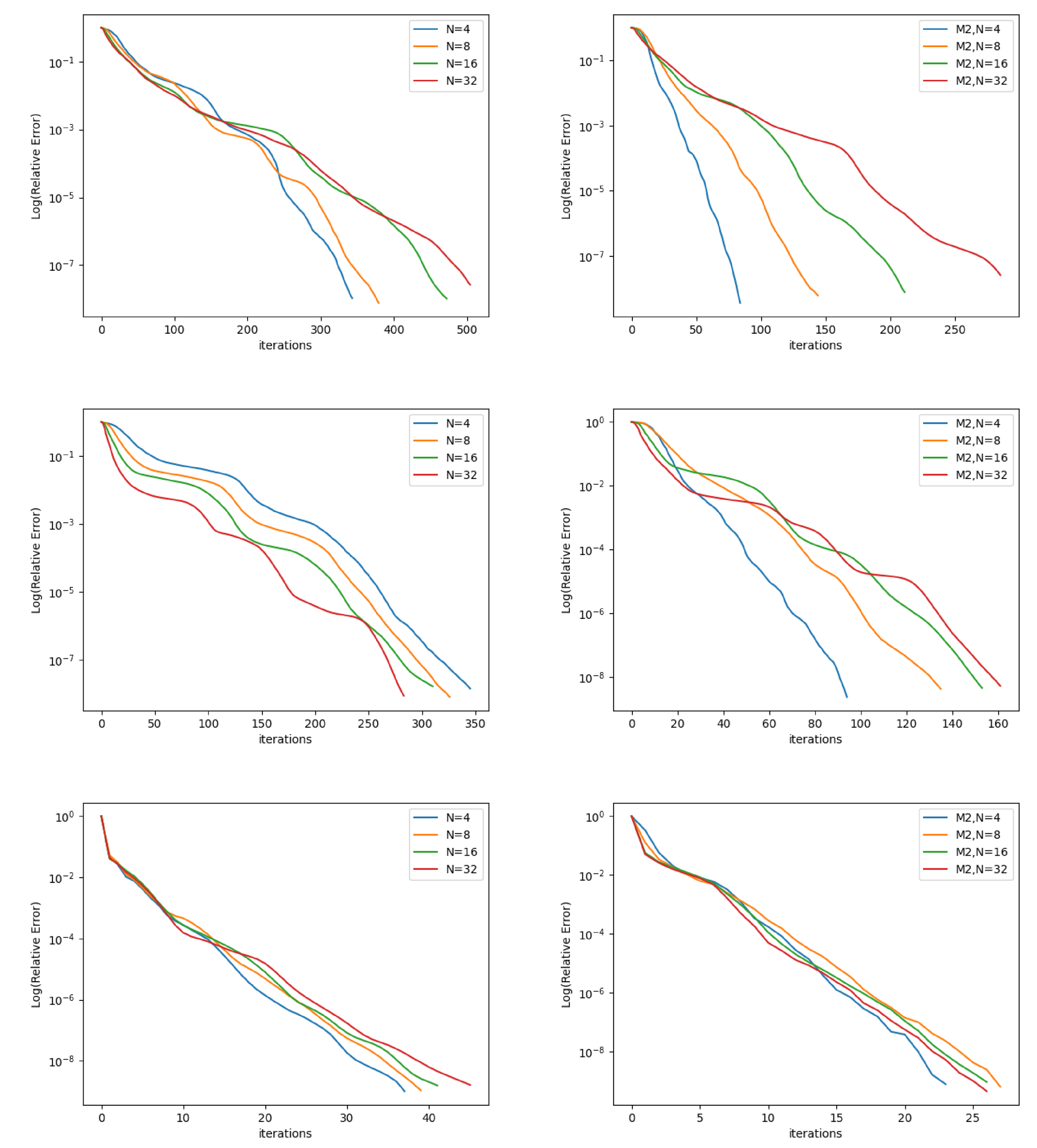
- $R_0 = Z^T$ gives the coarse matrix;
- $\widehat{R}_j$: The restriction matrices corresponding to $\left(\widehat{\Omega}_k\right)_{k\in\{1,\dots,m\}}$;
- $\widehat{D}_j$: The discrete partition of unity matrices satisfying

$$I = \sum_{j=1}^{m} \widehat{R}_j^T \widehat{D}_j \widehat{R}_j.$$

Numerical Results

N		Trad. Nic it.	Trad. Nic dim	Enr. Nic it.	Enr. Nic dim (rel)	Trefftz it.	Trefftz dim (rel)
4	no walls	48	16	48	17 (1.1)	20	192 (7.7)
	walls	85	16	95	50 (3.1)	24	408 (16)
8	no walls	73	64	73	68 (1.1)	20	443 (5.5)
	walls	145	64	136	174 (2.7)	28	868 (11)
16	no walls	77	256	73	289 (1.1)	20	1047 (3.6)
	walls	212	256	154	607 (2.4)	27	1918 (6.6)
32	no walls	76	1024	66	1139 (1.1)	21	2556 (2.3)
	walls	286	1024	162	1884 (1.8)	27	4262 (3.9)

GMRES iterations and corresponding (relative) dimensions of the coarse spaces shown for strong scalabiltiy experiments for $N \times N$ subdomains. The **total** *domain and its heterogeneities (buildings and walls) are fixed and N is varied, creating* **smaller** *subdomains as we increase N. Overlap is $\frac{1}{20}L$, where L is the nonoverlapping subdomain size. Average degrees of freedom ≈ 60k (no walls), 120k (walls). Relative dimension (rel) of the coarse space is computed as $\frac{\dim(R_0)}{N^2}$ for Nicolaides space and as $\frac{\dim(R_0)}{(N+1)^2}$ for a vertex-based Trefftz space.*



GMRES convergence curves shown for strong scalabiltiy experiments for $N \times N$ subdomains. The **total** *domain and its heterogeneities (buildings and walls) are fixed and N is varied, creating* **smaller** *subdomains as we increase N. Left to right: Minimal geometric overlap, overlap $\frac{1}{20}L$. Top to bottom: Trad. Nic, enhanced Nic, Trefftz.*

Conclusions and future work

- The enriched Nicolaides coarse space is **robust** with respect to the complexity of the data and the number of subdomains on a fixed total domain size.
- The Trefftz-like coarse space is also **robust**, and provides an additional **acceleration** in terms of Krylov iteration count.
- However, the dimention of the Trefftz-like coarse space is larger and controlled by the model geometry.
- Future work: Coarse approximation error estimates and extention to nonlinear parabolic problems. Nonlinear problems can be approached with a nonlinear preconditioning method such as RASPEN [2].

Motivations: The nonlinear problem

The Diffusive Wave model [1] is given by

$$\partial_t h(u, z_b(\mathbf{x})) - \text{div}\left(\kappa h(u, z_b(\mathbf{x}))^\alpha ||\nabla u||^{\gamma-1}\nabla u\right) = f,$$

- $z_b(\mathbf{x})$: Bathymetry;
- $h(u, z_b(\mathbf{x})) = \max(u - z_b(\mathbf{x}), 0)$: Water depth;
- κ: Friction coefficient;
- $\alpha > 1, 0 < \gamma \leq 1$: Depend on flow regimes and head-loss formula;

Ongoing work: Assuming laminar flow ($\gamma = 1$) and $z_b = 0, \kappa = 1$, the model can be simplified to the Porous Medium Equation

$$\partial_t u + \text{div}(u^\alpha \nabla u) = f.$$

References

[1] R Alonso, Mauricio Santillana, and Clint Dawson. On the diffusive wave approximation of the shallow water equations. *European Journal of Applied Mathematics*, 19(5):575–606, 2008.

[2] Victorita Dolean, Martin J Gander, Walid Kheriji, Felix Kwok, and Roland Masson. Nonlinear preconditioning: How to use a nonlinear schwarz method to precondition newton's method. *SIAM Journal on Scientific Computing*, 38(6):A3357–A3380, 2016.

[3] Victorita Dolean, Pierre Jolivet, and Frédéric Nataf. *An introduction to domain decomposition methods: algorithms, theory, and parallel implementation*. SIAM, 2015.