Numerical modeling of two-phase flow in fractured porous media

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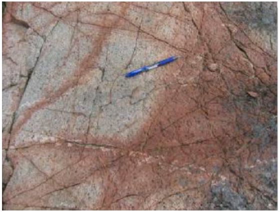
Computational Techniques and Applications Conference 2020

September 2, 2020

Motivation

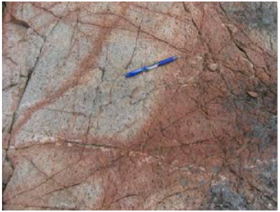
Why fractures?

- Omnipresent: almost any geological formation is naturally fractured
- May control flow pattern

Characteristics

- Very large $\frac{\text{length}}{\text{width}}$ ratio
- Extreme contrast in hydrodynamical properties: fracture/matrix
- Exist at many scales: from few cm to km (faults)

Motivation



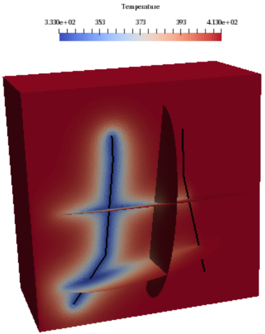
Multiple industrial applications

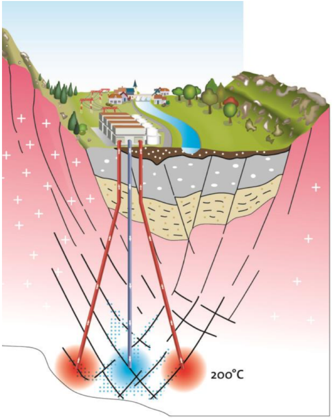
- Geothermal energy production
- Tight gas and oil extraction
- Nuclear safety

Application: High temperature geothermal energy

Geothermal energy extraction

- Flow manly through fracture network
- Heat exchange with matrix



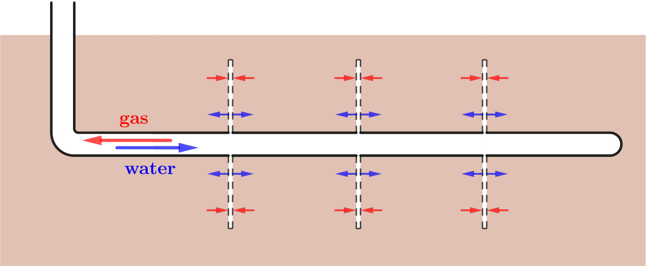


Gas extraction from low permeability reservoir

- create new or activate existing fractures

Features:

- Strong capillary forces at matrix-fracture interface
- Mass exchange between matrix and fracture



Storage project Cigéo:

- High- and intermediate-level waste
- $500m$ below the ground
- Very low permeability clay:

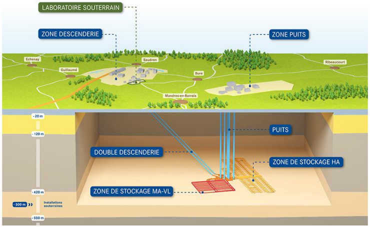
$$K_m \approx 10^{-20} m^2$$

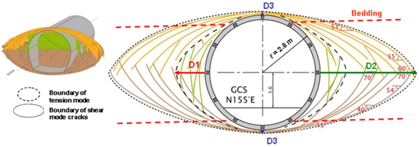
Small fractures: $d_f = 10\mu m$-$1mm$

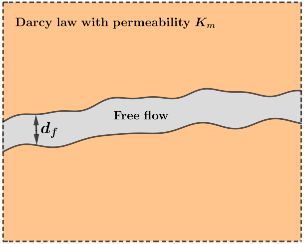
- originated during excavation phase

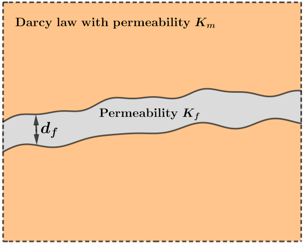
Fracture permeability

$$K_f = \frac{d_f^2}{12} = 10^{-11} - 10^{-7} m^2 \gg K_m$$









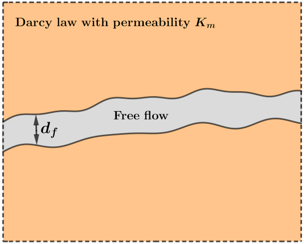
Kinds of fractures

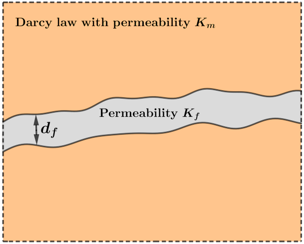
- Open or filled with porous material
- Drains or barriers

Characterization

- Apertures d_f distribution
- Permeability and porosity

Drains vs. barriers





Kinds of fractures

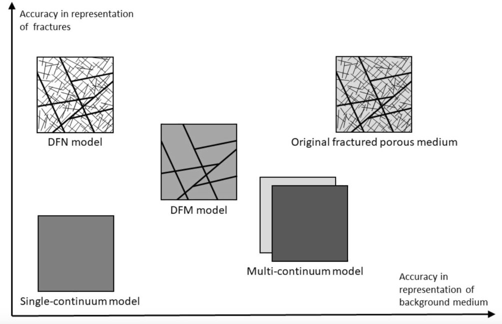
- Open or filled with porous material
- Drains or barriers

Highly conductive fractures are the most studied

Characterization

- Apertures d_f distribution
- Permeability and porosity

Discrete Fractures Matrix model



I. Berre *et al.*, 2018

Fracture modeling

- Upscaling: one or multiple (overlapping) equivalent media
- Discrete Fracture models:
  - DFN - only fractures
  - DFM - fractures and matrix

Outline

Single-phase DFM

- Drains and barriers
- Numerical modeling

Two-phase DFM

- Two-phase flow in homogeneous and heterogeneous media
- Drains are barriers?!

Acceleration of Newton's method

- Variable switching
- Nonlinear Jacobi preconditioning

Outline

Single-phase DFM

- **Drains and barriers**
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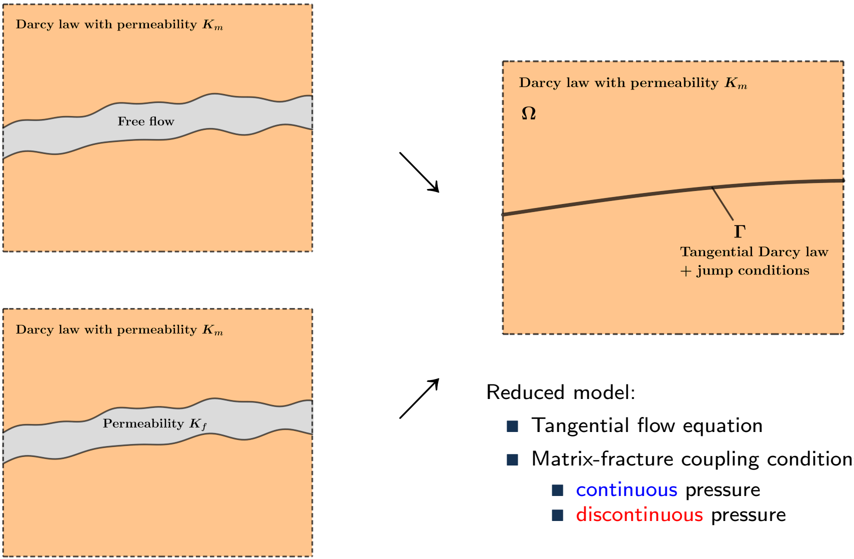
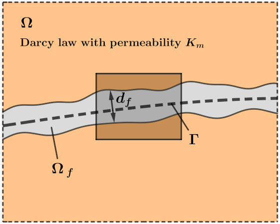
Dimension reduction



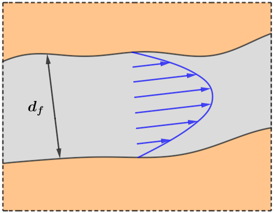
Reduced model:

- Tangential flow equation
- Matrix-fracture coupling conditions
  - continuous pressure
  - discontinuous pressure

Equi-dimensional representation



Zoom to selected area



Assumptions:

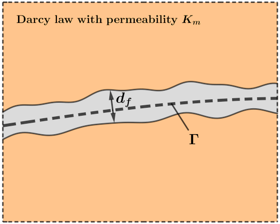
- Laminar flow parabolic velocity profile along Γ with an average velocity

$$\mathbf{v}_f = -\frac{d_f^2}{12\mu}\left(\nabla_\tau p_f + \rho \mathbf{g}_\tau\right)$$

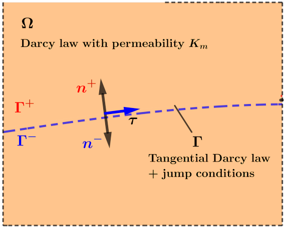
- No pressure drop across Γ

Open fractures/continuous pressure model

Equi-dimensional representation



Hybrid-dimensional model



Matrix equations:

$$\mathrm{div}\mathbf{q}_m = 0, \qquad \mathbf{q}_m = -\frac{K_m}{\mu}\left(\nabla p_m + \rho\mathbf{g}\right)$$

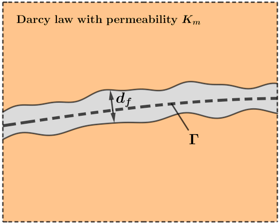
Fracture equations:

$$\mathrm{div}_\tau \mathbf{q}_f = \underbrace{\mathbf{q}_m|_{\Gamma+} \cdot \mathbf{n}^+ + \mathbf{q}_m|_{\Gamma-} \cdot \mathbf{n}^-}_{\text{jump of the normal trace across } \Gamma}, \qquad \mathbf{q}_f = d_f \mathbf{v}_f = -\frac{d_f^3}{12\mu}\left(\nabla_\tau p_f + \rho\mathbf{g}_\tau\right)$$

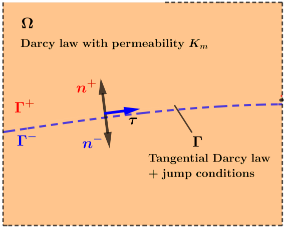
No pressure drop: $p_m|_{\Gamma+} = p_m|_{\Gamma-} = p_f$.

Open fractures/continuous pressure model

Equi-dimensional representation



Hybrid-dimensional model



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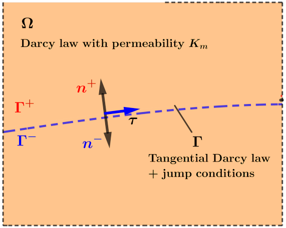
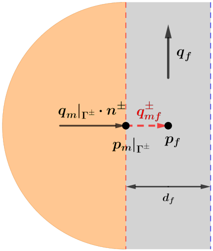
No pressure drop: $p_m|_{\Gamma+} = p_m|_{\Gamma-} = p_f$.

- Remark: $p_m \in H^1(\Omega) \cap H^1(\Gamma)$

Equi-dimensional representation



Transmission condition



Fracture mass balance:

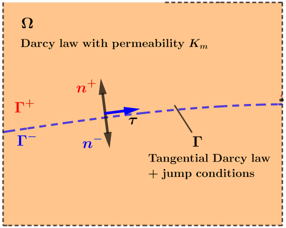
$$\mathrm{div}_\tau \mathbf{q}_f = \mathbf{q}_m|_{\Gamma+} \cdot \mathbf{n}^+ + \mathbf{q}_m|_{\Gamma-} \cdot \mathbf{n}^-$$

Width-averaged Darcy law: $\mathbf{q}_f = -d_f \frac{K_f}{\mu} (\nabla_\tau p_f + \rho \mathbf{g}_\tau)$

Matrix-fracture flow: $\mathbf{q}_m|_{\Gamma\pm} \cdot \mathbf{n}^\pm = \mathbf{q}_{mf}^\pm := -\frac{K_f}{\mu}\left(\frac{p_m|_{\Gamma\pm} - p_f}{d_f/2} - \rho \mathbf{g} \cdot \mathbf{n}^\pm\right)$

Filled fractures/discontinuous pressure model

Equi-dimensional representation



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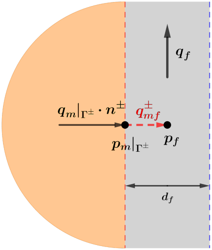

Fracture mass balance:

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Remarks:

- Pressure jumps across Γ: $p_m \in H^1(\Omega \setminus \Gamma)$ and $p_f \in H^1(\Gamma)$
- Extension: Rigorous derivation leads to a family of coupling conditions

Similarities: Coupling of Darcy flow in matrix (3D) and fracture (2D)

$$\begin{cases} \mathrm{div}\mathbf{q}_m = 0 \\ \mathrm{div}_\tau \mathbf{q}_f = \mathbf{q}_m|_{\Gamma^+} \cdot \mathbf{n}^+ + \mathbf{q}_m|_{\Gamma^-} \cdot \mathbf{n}^- \end{cases}$$

Continuous pressure model

- No pressure jump across Γ

$$p_m|_{\Gamma^+} = p_m|_{\Gamma^-} = p_f.$$

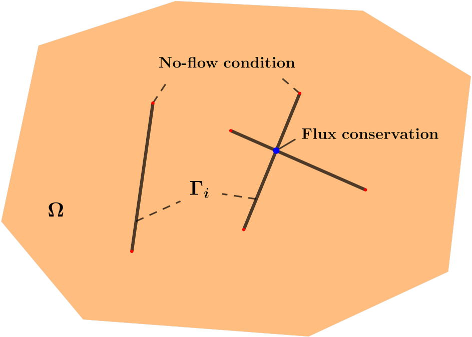
- Extra regularity: $p_m \in H^1(\Omega) \cap H^1(\Gamma)$

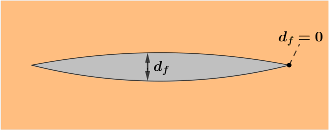
Discontinuous pressure model

- Pressure jump-flux relation on $\Gamma^\pm$

$$\mathbf{q}_m|_{\Gamma^\pm} \cdot \mathbf{n}^\pm = -\frac{K_f}{\mu}\left(\frac{p_m|_{\Gamma^\pm} - p_f}{d_f/2} - \rho\mathbf{g} \cdot \mathbf{n}^\pm\right)$$

- Broken Sobolev space: $p_m \in H^1(\Omega \backslash \Gamma)$





- Flux conservation at fracture intersections
- No-flow condition at fracture tips

Outline

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- Drains and barriers
- **Numerical modeling**

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Acceleration of Newton's method

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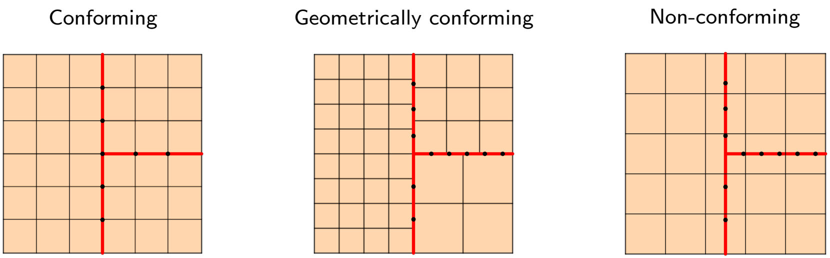
Computational challenges

- Geometric complexity: dense networks, acute angles
  - Efficient mesh generators
  - Nonconforming methods

- Linear solvers: high contrasts (barriers, drains), large correlation length
  - Direct solvers
  - Domain decomposition with an adequate coarse space

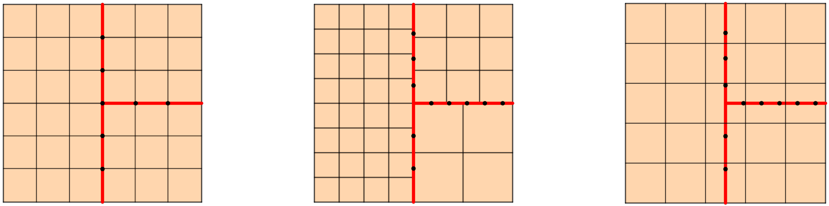


Usual assumption: planar fractures

- Conforming methods: Mesh on Γ is maid of faces of the mesh on Ω_m
  - FVM, FEM, DG, VEM, HHO, ...
- Geometrically conforming methods: computational mesh resolves $\Omega\backslash\Gamma$
  - Domain decomposition, mortar methods
- Nonconforming: Ω and Γ and meshed independently
  - X-FEM, E-FEM

Discretization methods

Conforming | Geometrically conforming | Non-conforming



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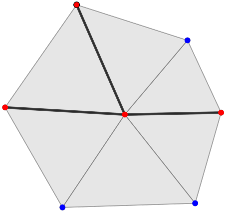
Nodal discretzations

Motivation: much fewer DOF at tetrahedral meshes

P_1 FEM discretization

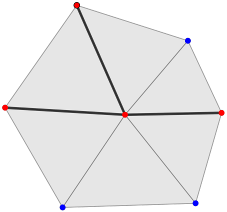
- Conforming mesh $\mathcal{T}_h$
- A broken $\mathbb{P}_1(\mathcal{T}_h)$ space of element-wise affine functions

Conforming mesh



Nodal discretzations: cont. pressure

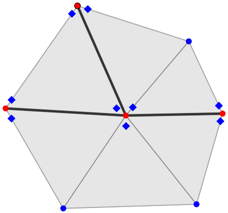
No extra DOF



Continuous pressure model

- Discrete functional space $V_h = \mathbb{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$
- Weak formulation (modulo B.C. on $\partial\Omega$): Find $p \in V_h$

$$\int_\Omega K_m \nabla p \cdot \nabla v \mathrm{d}\mathbf{x} + \int_\Gamma K_f \nabla_\tau p|_\Gamma \cdot \nabla_\tau v|_\Gamma \mathrm{d}\sigma(\mathbf{x}) = 0 \quad \forall v \in V_h$$



Discontinuous pressure model

- Discrete functional spaces

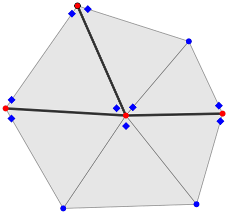
$$V_{m,h} = \mathbb{P}_1(\mathcal{T}_h) \cap H^1(\Omega\backslash\Gamma) \quad \text{and} \quad V_{f,h} = \mathbb{P}_1(\mathcal{T}_h)|_\Gamma \cap H^1(\Gamma)$$

- Weak formulation (modulo B.C. on $\partial\Omega$): Find p_m and p_f

$$\int_\Omega K_m \nabla p_m \cdot \nabla v_m \mathrm{d}\mathbf{x} + \int_\Gamma K_f \nabla_\tau p_f \cdot \nabla_\tau v_f \mathrm{d}\sigma(\mathbf{x}) + \sum_{\pm} \int_\Gamma \frac{2K_f}{d_f} [\![p]\!]_h^\pm [\![v]\!]_h^\pm \mathrm{d}\sigma(\mathbf{x}) = 0$$

for all $v_m \in V_h(\mathcal{T}_h), v_f \in V_h(\Gamma)$.

Standard jump operator defined as $[\![u]\!]_h^\pm = u_m|_{\Gamma^\pm} - u_f$



Discontinuous pressure model

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for all $v_m \in V_h(\mathcal{T}_h), v_f \in V_h(\Gamma)$.

Jump operator defined as $[\![u]\!]_h^\pm = \pi_h u_m|_{\Gamma^\pm} - \pi_h u_f$ using mass lumping

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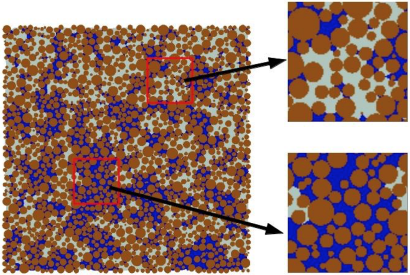
Two-phase DFM

- **Two-phase flow in homogeneous and heterogeneous media**
- Drains are barriers?!

Acceleration of Newton's method

- Variable switching
- Nonlinear Jacobi preconditioning

Two fluids shares the pore space

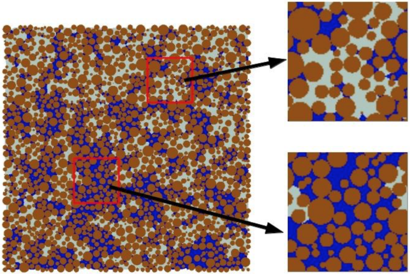


Assumptions

- Two immiscible phases: sharp interfaces at pore scale
- Wetting (say liquid) and non-wetting (say gas) phases

Two-phase flow in porous media

Two fluids shares the pore space



Assumptions

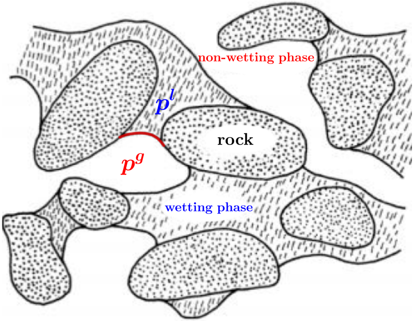
- Two immiscible phases: sharp interfaces at pore scale (essential)
- Phases are: liquid wetting and gas non-wetting

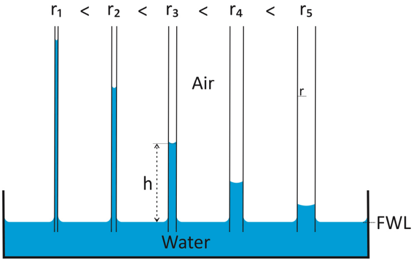
Saturation of phase α

$$s^\alpha = \frac{\text{volume of phase } \alpha \text{ in REV}}{\text{volume of void space in REV}}$$

Volume conservation $\sum_\alpha s^\alpha = 1$

Capillary pressure





At the pore scale: pressure jump across free surface

- Laplace capillary pressure law:

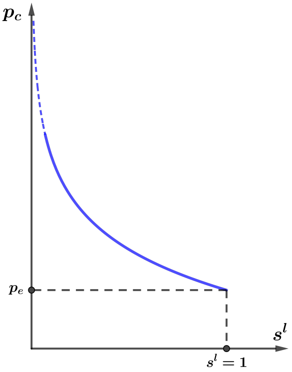
$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

Observations:

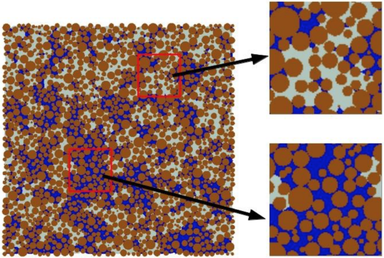
- Small pore size $\Rightarrow$ large pressure jump (if shared by both phases!)
- Wetting phase "prefers" small pores

Capillary pressure at Darcy scale

$$p^g - p^l = p_c(s^l)$$

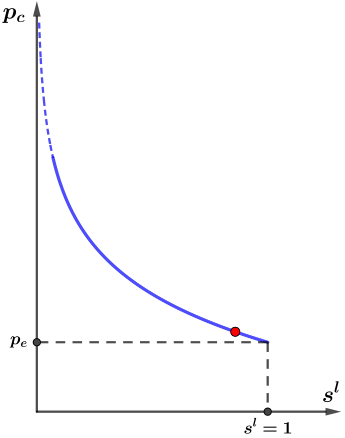


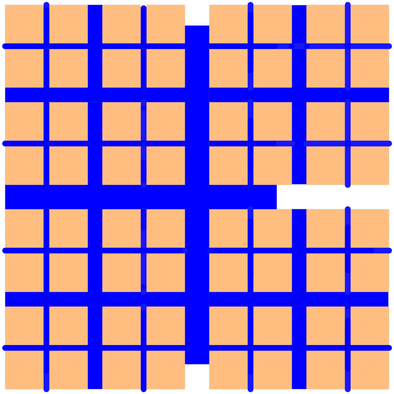
- Entry pressure p_e



- Capillary pressure law depends on pore-size distribution

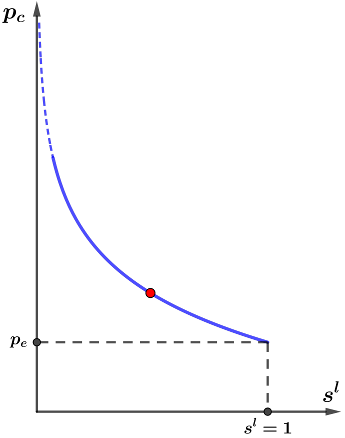
Macroscopic capillary pressure

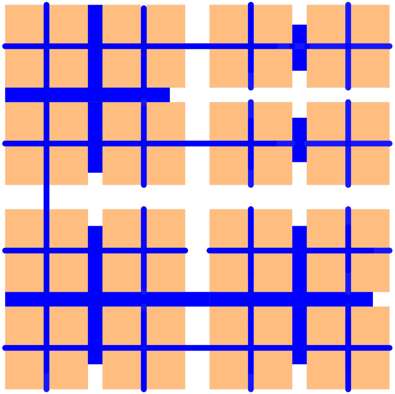




As the rock dries

- Interface between the phases moves to smaller pores $\Rightarrow$ pressure jump increases





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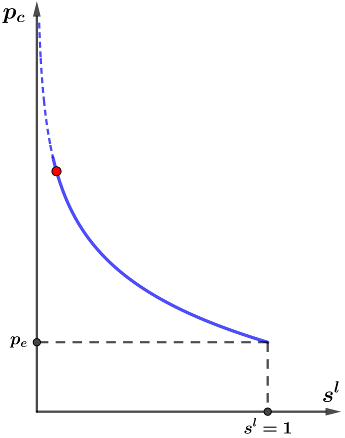
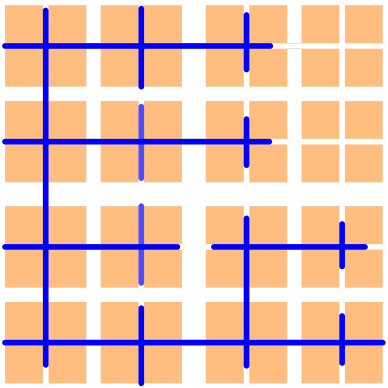
Macroscopic capillary pressure

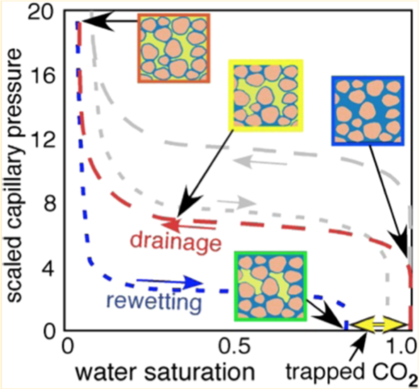




As the rock dries

- Interface between the phases moves to smaller pores $\Rightarrow$ pressure jump increases

Capillary hysteresis (not covered)



Capillary pressure depends on the rate of wetting (or drying):

$$p_c = p_c\left(s^l, \partial_t s^l\right)$$

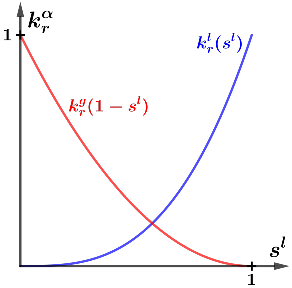
Conservation of each phase $\alpha = l, g$

$$\phi \partial_t s^\alpha + \mathrm{div} \mathbf{q}^\alpha = 0, \qquad \mathbf{q}^\alpha = -\frac{k_r^\alpha(s^\alpha) K}{\mu^\alpha} (\nabla p^\alpha + \rho^\alpha \mathbf{g})$$

Closure laws

$$\sum_\alpha s^\alpha = 1 \qquad \text{and} \qquad p^g - p^l = p_c(s^l)$$

Relative permeability $k_r^\alpha : [0,1] \to [0,1]$

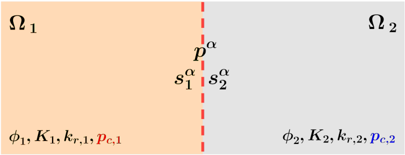


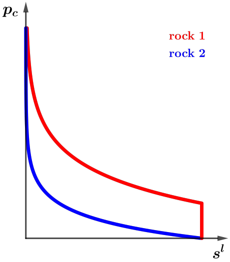
Natural energy estimate:

$$\sum_\alpha \int_0^T \int_\Omega \frac{k_r^\alpha(s^\alpha) K}{\mu^\alpha} |\nabla p^\alpha|^2 \leqslant C^{te}$$

Loss of control on $\|\nabla p^\alpha\|_{L^2(L^2)}$ as $s^\alpha \to 0$

Piece-wise constant rock properties





Flow equations

$$\phi_i \partial_t s^\alpha - \operatorname{div}\left(\left(\frac{k_{r,i}^\alpha(s^\alpha) K_i}{\mu^\alpha} \nabla p^\alpha - \rho^\alpha \mathbf{g}_\tau\right)\right) = 0 \qquad \text{in} \quad \Omega_i$$

Interface conditions:

- Flux continuity
- Continuity of some variable?
  - Pressure continuity (in some sens)
  - Saturation is not continuous

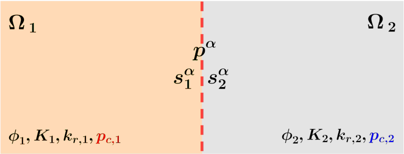
If $s_1^\alpha, s_2^\alpha > 0$ then p^α is cont. and

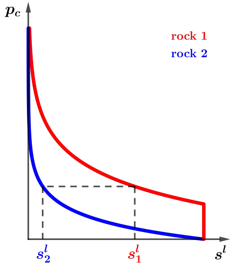
$$p_{c,1}(s_1^l) = p_{c,2}(s_2^l)$$

If $s_1^l = 0$ then

$$p_{c,1}(s_1^l) \cap p_{c,2}(s_2^l) \neq \emptyset$$

Piece-wise constant rock properties





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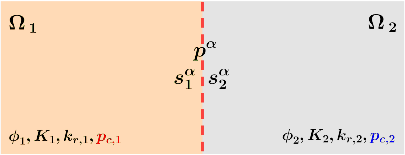
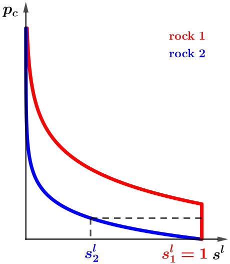
Flow in heterogeneous porous medium

Piece-wise constant rock properties





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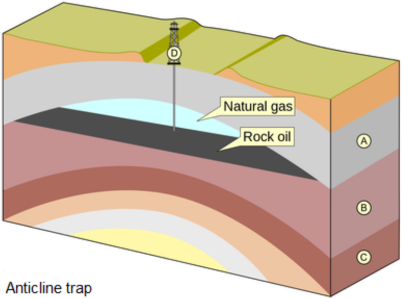
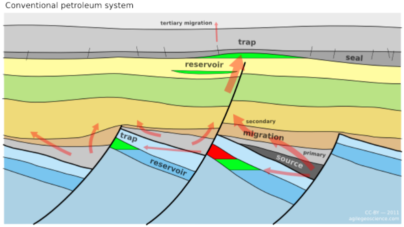
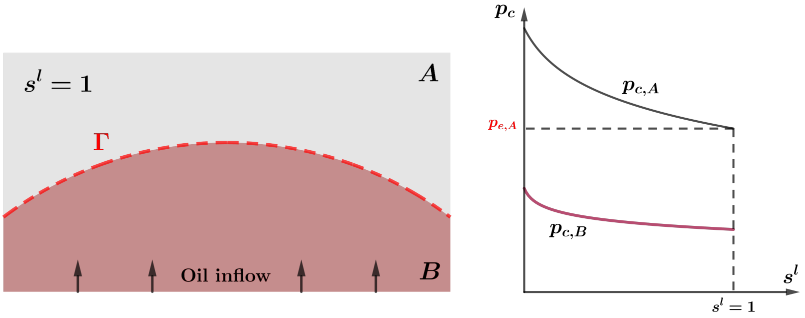
Oil and gas reservoir formation:

- Generation of the hydrocarbons in a deep formations
- Upward migration due to buoyancy
- Why hydrocarbons don't reach the ground surface?



Anticline trap



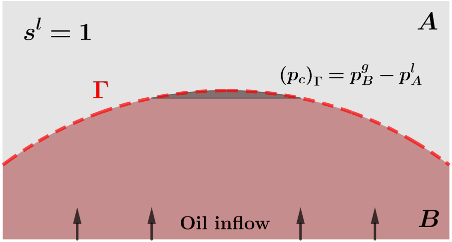


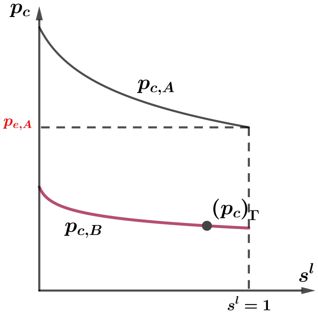
- Capillary pressure at the interface Γ

$$(p_c)_\Gamma = \underbrace{p_B^g}_{\text{oil pressure below}} - \underbrace{p_A^l}_{\text{water pressure above}}$$

- Extra oil pressure due to buoyancy $\Delta p \approx \Delta\rho g \times \text{depth}$
- No oil flow trough Γ as long as $p_B^g \leqslant p_A^l + p_{e,A}$

Interface capillary processes: Oil trapping





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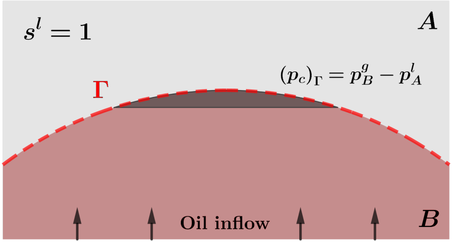

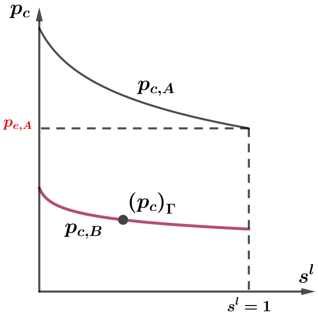

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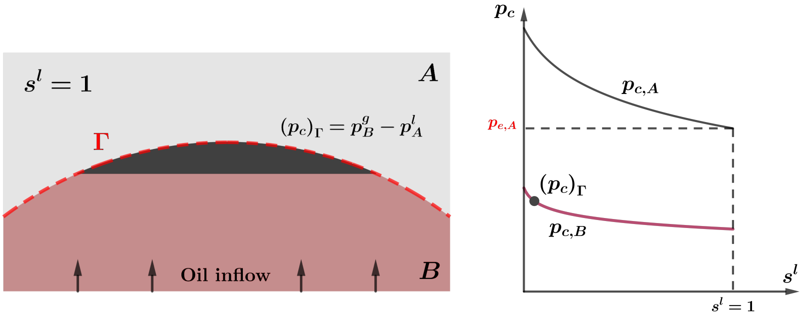

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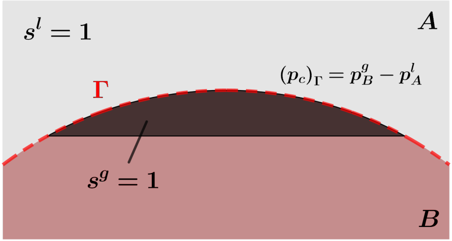

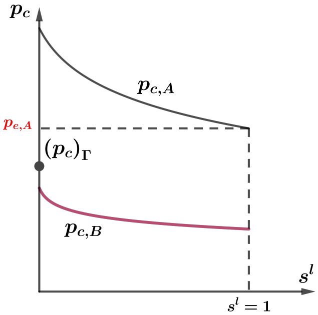

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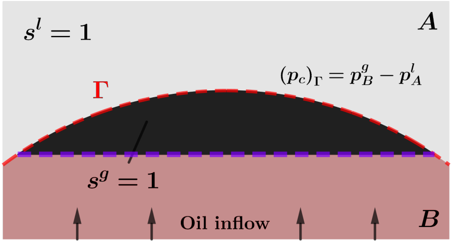

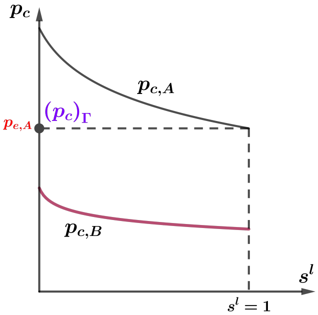

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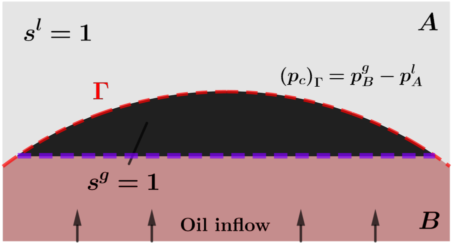

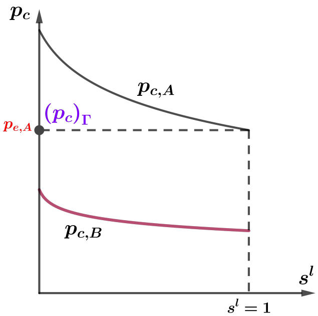

- Capillary trapping controls reservoir depth
- Similar process can be used for geological CO_2 sequestration

Outline

Single-phase DFM

- Drains and barriers
- Numerical modeling

Two-phase DFM

- Two-phase flow in homogeneous and heterogeneous media
- **Drains are barriers?!**

Acceleration of Newton's method

- Variable switching
- Nonlinear Jacobi preconditioning

Coupled flow in matrix and fracture network

$$\begin{cases} \mathrm{div}\mathbf{q}_m &= 0 \\ \mathrm{div}_\tau \mathbf{q}_f &= \mathbf{q}_m|_{\Gamma^+} \cdot \mathbf{n}^+ + \mathbf{q}_m|_{\Gamma^-} \cdot \mathbf{n}^- \end{cases}$$

Continuous pressure model

- No pressure jump across Γ

$$p_m|_{\Gamma^+} = p_m|_{\Gamma^-} = p_f.$$

Discontinuous pressure model

- Pressure jump-flux relation on $\Gamma^\pm$

$$\mathbf{q}_m|_{\Gamma^\pm} \cdot \mathbf{n}^\pm = -\frac{K_f}{\mu}\left(\frac{p_m|_{\Gamma^\pm} - p_f}{d_f/2}\right)$$

Refresher on the single-phase flow models

Coupled flow in matrix and fracture network

$$\begin{cases} \phi_m \partial_t s_m^\alpha + \mathrm{div}\mathbf{q}_m^\alpha & = & 0 \\ \phi_f d_f \partial_t s_f^\alpha + \mathrm{div}_\tau \mathbf{q}_f^\alpha & = & \mathbf{q}_m^\alpha|_{\Gamma^+} \cdot \mathbf{n}^+ + \mathbf{q}_m^\alpha|_{\Gamma^-} \cdot \mathbf{n}^- \end{cases}$$

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Relative permeability upwinding

Discontinuous pressure model

Matrix-fracture interface

- Interface capillary pressure

$$p_{c,m}|_{\Gamma^\pm} = \left(p_m^g - p_m^l\right)|_{\Gamma^\pm}$$

- Saturation jump at the interface

$S_m^\alpha(p_{c,m}|_{\Gamma^\pm})$ and $S_f^\alpha(p_{c,m}|_{\Gamma^\pm})$

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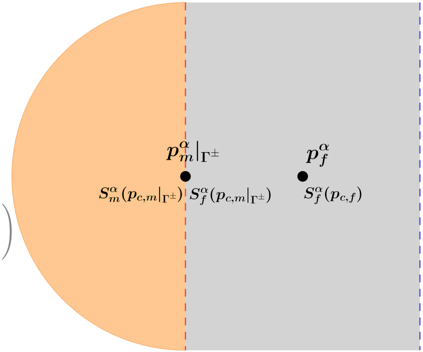

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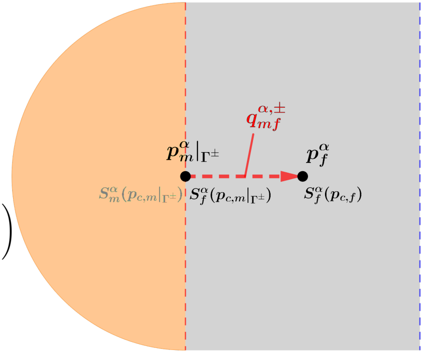
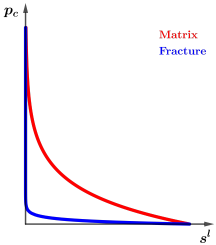


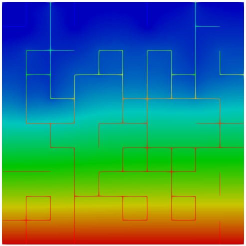
Test case: Drying of a damaged zone

- Domain $\Omega = (0, 10m)^2$
- Fracture width $d = 1mm$
- Permeability contrast $K_f / K_m = 10^4$
- Capillary pressure contrast
- Boundary conditions

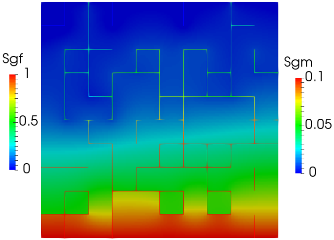
$$\begin{aligned} \text{Saturated top:} \quad & s_m^l = 1, \quad && p^g = 1atm \\ \text{Dry bottom:} \quad & s_m^l = 0.9, \quad && p^g = 1atm \end{aligned}$$



Continuous pressure

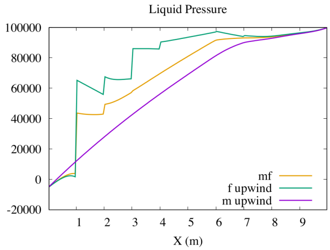


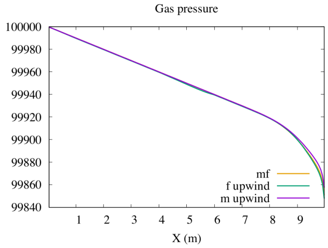
Discontinuous pressure



- Disc. pressure model: some fractures acts as barriers. Why?

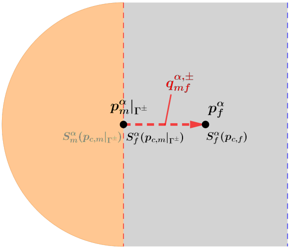
Continuous pressure models: validity

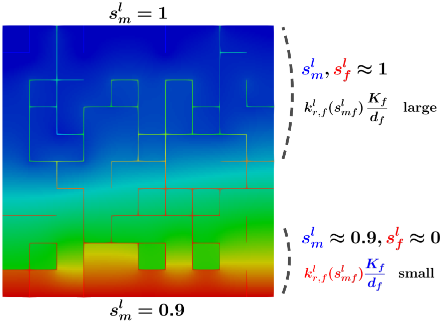




p^l and p^g plot over a vertical line

- disc. pressure
- cont. pressure

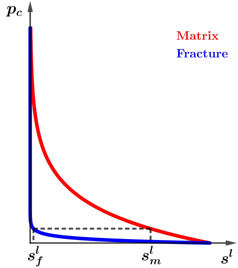




Pressure jump-flux relation for liquid

$$\mathbf{q}_m^l|_{\Gamma\pm} \cdot \mathbf{n}^\pm = -k_{r,f}^l(s_{mf}^l)\frac{K_f}{\mu^l d_f/2}\left(p_m^l|_{\Gamma\pm} - p_f^l\right)$$

$$s_{mf}^\alpha = \begin{cases} S_f^\alpha(p_{c,m}|_{\Gamma\pm}), & p_m^\alpha|_{\Gamma\pm} - p_f^\alpha \geqslant 0 \\ S_f^\alpha(p_{c,f}), & p_m^\alpha|_{\Gamma\pm} - p_f^\alpha < 0 \end{cases}$$



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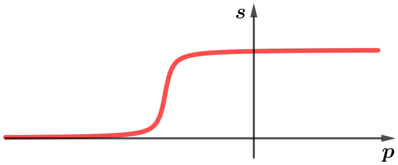
Acceleration of Newton's method

- **Variable switching**
- Nonlinear Jacobi preconditioning

Model problem

$$\partial_t S(p) - \Delta p = 0$$

- Homogeneous medium



Discrete problem: $\mathcal{F}(S(\mathbf{p}), \mathbf{p}) = 0$

Apply Newton's method to

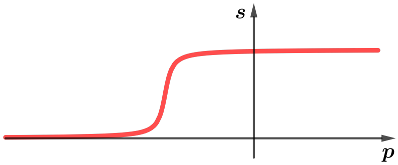
- $\mathcal{F}(S(\mathbf{p}), \mathbf{p}) = 0$: p-formulation
- $\mathcal{F}(\mathbf{s}, S^{-1}(\mathbf{s})) = 0$: s-formulation

Primary variable selection

Model problem

$$\partial_t S(p) - \Delta p = 0$$

- Homogeneous medium



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Questions:

- Which formulation to chose?
- Does the choice matters?

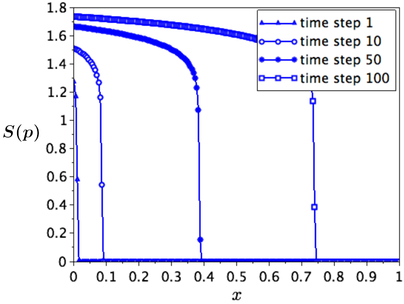
Example: 1D porous media equation

Porous media equation on $(0,1) \times (0,T)$

$$\partial_t S(p) - \partial^2_{xx} p = 0, \qquad S(p) = p^{1/m}, \quad m > 1$$

with Neumann boundary conditions

- Inflow at $x = 0$: $-\partial_x p(0,t) = q \geqslant 0$
- No-flow at $x = 1$
- Almost "dry" initial condition: $S(p(x,0)) = 10^{-10}$



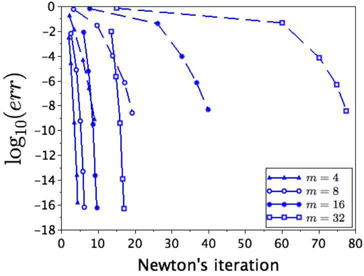
Typical solution profile

Original p-formulation:

$$\partial_t S(p) - \partial^2_{xx} p = 0,$$

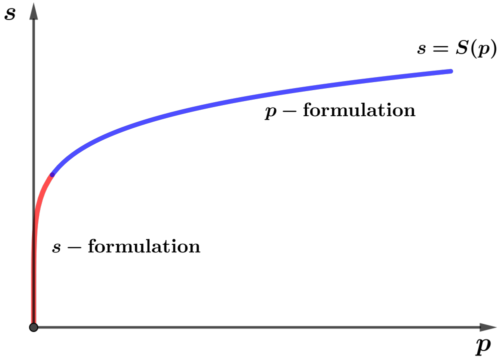
Alternative s-formulation:

$$\partial_t s - \partial^2_{xx} S^{-1}(s) = 0$$



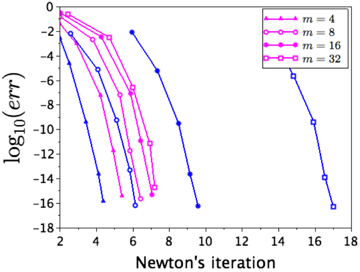
- s-formulation (solid) is much more efficient
- Can we find an even better primary variable?

Variable switching



- Switching between s and p may be a good idea
- Well-known for Richards' equation

- s-formulation: $\partial_t s - \triangle S^{-1}(s) = 0$
- variable switching: PDE?



- Variable switching: is more **efficient** and is **robust** w.r.t. m
- Drawback: implementation using **if/else** conditions

Graph parametrization (B. & Cancès '17)

Parametrization of the graph $s = S(p)$:

Let $\overline{p}, \overline{s} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

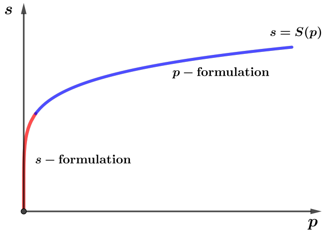
$$\overline{s}(\tau) = S(\overline{p}(\tau)) \qquad \forall \tau \in \mathbb{R}^+$$

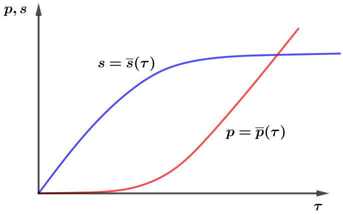
PDE in terms of the new variable τ

$$\partial_t \overline{s}(\tau) - \Delta \overline{p}(\tau) = 0$$

Variable switching:

$$\max(\overline{s}'(\tau), \overline{p}'(\tau)) = 1$$

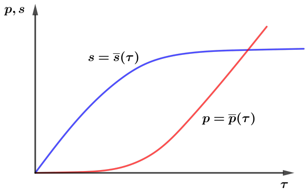




Estimates on $F_\tau(\boldsymbol{\tau}) = \mathcal{F}\left(\overline{s}(\boldsymbol{\tau}), \overline{p}(\boldsymbol{\tau})\right)$

$$\|F'_\tau(\boldsymbol{\tau})\|, \|F'_\tau(\boldsymbol{\tau})\|^{-1} < C$$

uniformly w.r.t. τ and the form of S.



Corollaries:

- Control of $\mathrm{cond}(F'_\tau)$
- Stopping criterion:

$$\|F_\tau(\boldsymbol{\tau})\| < \epsilon \Rightarrow \|\boldsymbol{\tau} - \boldsymbol{\tau}_\star\| < C\epsilon \Rightarrow \begin{cases} \|\overline{s}(\boldsymbol{\tau}) - \mathbf{s}_\star\| < C\epsilon, \\ \|\overline{p}(\boldsymbol{\tau}) - \mathbf{p}_\star\| < C\epsilon \end{cases}$$

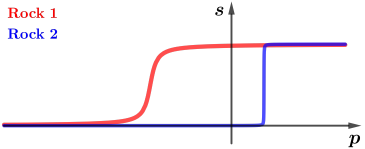
Application to the flow in heterogeneous porous medium

Heterogeneous model PDE

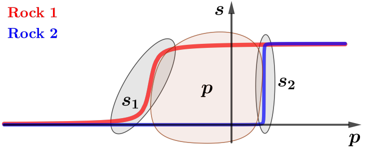
$$\partial_t S(p, \mathbf{x}) - \triangle p = 0$$

Piece-wise constant $S(\cdot, \mathbf{x})$

- $S(p, x)|_{\Omega_i} = S_i(p), \quad i = 1, 2$



Multiple variable switching



via simultaneous parametrization of $S_1(p)$ and $S_2(p)$

Primary variable selection in two-phase flow

Using notations $\Lambda^\alpha = \frac{k_r^\alpha(s^\alpha)K}{\mu^\alpha}$ and neglecting gravity

$$\left\{\begin{array}{rcll} \phi\partial_t s^l & - & \operatorname{div}\left(\Lambda^l \nabla p^l\right) & = & 0 \\ \phi\partial_t s^g & - & \operatorname{div}\left(\Lambda^g \nabla p^g\right) & = & 0 \end{array}\right.$$

Use s^l and p^g to eliminate dependent variables with $p^l = p^g - p_c(s^l)$ and $\sum_\alpha s^\alpha = 1$

$$\left\{\begin{array}{rcll} \phi\partial_t s^l & - & \operatorname{div}\left(\Lambda^l \nabla p^g - \Lambda^l \nabla p_c(s^l)\right) & = & 0 \\ -\phi\partial_t s^l & - & \operatorname{div}\left(\Lambda^g \nabla p^g\right) & = & 0 \end{array}\right.$$

Using notations $\Lambda^\alpha = \frac{k_r^\alpha(s^\alpha)K}{\mu^\alpha}$ and neglecting gravity

$$\begin{cases} \phi\partial_t s^l & - & \operatorname{div}\left(\Lambda^l \nabla p^l\right) & = & 0 \\ \phi\partial_t s^g & - & \operatorname{div}\left(\Lambda^g \nabla p^g\right) & = & 0 \end{cases}$$

Use s^l and p^g to eliminate dependent variables with $p^l = p^g - p_c(s^l)$ and $\sum_\alpha s^\alpha = 1$

$$\begin{cases} \phi\partial_t s^l & - & \operatorname{div}\left(\Lambda^l \nabla p^g - \Lambda^l \nabla p_c(s^l)\right) & = & 0 \\ & - & \operatorname{div}\left((\Lambda^g + \Lambda^l) \nabla p^g - \Lambda^l \nabla p_c(s^l)\right) & = & 0 \end{cases}$$

Primary variable selection in two-phase flow

Using notations $\Lambda^\alpha = \frac{k_r^\alpha(s^\alpha)K}{\mu^\alpha}$ and neglecting gravity

$$\begin{cases} \phi\partial_t s^l & - & \operatorname{div}\left(\Lambda^l \nabla p^l\right) & = & 0 \\ \phi\partial_t s^g & - & \operatorname{div}\left(\Lambda^g \nabla p^g\right) & = & 0 \end{cases}$$

Use s^l and p^g to eliminate dependent variables with $p^l = p^g - p_c(s^l)$ and $\sum_\alpha s^\alpha = 1$

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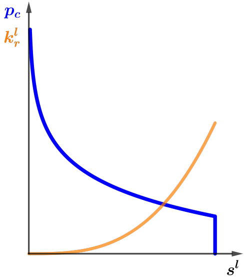
Use s^l and p^g to eliminate dependent variables with $p^l = p^g - p_c(s^l)$ and $\sum_\alpha s^\alpha = 1$

$$\begin{cases} \phi\partial_t s^l & - & \operatorname{div}\Big(\overbrace{\Lambda^l \nabla p^g}^{\text{convection}} - \overbrace{\Lambda^l \nabla p_c(s^l)}^{\text{diffusion}} \Big) & = & 0 \\ & - & \operatorname{div}\Big(\underbrace{(\Lambda^g + \Lambda^l)}_{\geqslant cK \ \text{ with } c>0} \nabla p^g - \Lambda^l \nabla p_c(s^l)\Big) & = & 0 \end{cases}$$

- Elliptic equation for p^g
- Degenerate parabolic equation for s^l

$$\begin{cases} \phi\partial_t s^l & - \quad \mathrm{div} K\left(\Lambda^l \nabla p^g - \Lambda^l \nabla p_c(s^l)\right) & = \quad 0 \\ & - \quad \mathrm{div} K\left((\Lambda^g + \Lambda^l)\nabla p^g - \Lambda^l \nabla p_c(s^l)\right) & = \quad 0 \end{cases}$$



Primary variable selection

- Good choice (p^g, s^l) : $\Lambda^l p_c'(s^l) < \infty$
- Bad choice (p^g, p_c) : $\dfrac{\partial s^l}{\partial p_c}$ and Λ^l vanish at dry regions: equation gives $0 \approx 0$

$$\begin{cases} \phi\partial_t s^l & - & \operatorname{div} K\left(\Lambda^l \nabla p^g - \Lambda^l p_c'(s^l)\nabla s^l\right) & = & 0 \\ & - & \operatorname{div} K\left((\Lambda^g + \Lambda^l)\nabla p^g - \Lambda^l \nabla p_c(s^l)\right) & = & 0 \end{cases}$$

Primary variable selection

- Good choice (p^g, s^l) : $\Lambda^l p_c'(s^l) < \infty$
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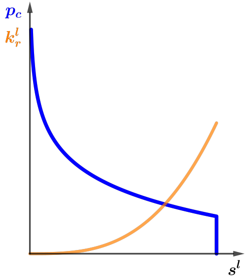

$$\begin{cases} \phi\left(p_c^{-1}\right)'\partial_t p_c & - \quad \mathrm{div} K\left(\Lambda^l \nabla p^g - \Lambda^l \nabla p_c\right) & = \quad 0 \\ & - \quad \mathrm{div} K\left((\Lambda^g + \Lambda^l)\nabla p^g - \Lambda^l \nabla p_c(s^l)\right) & = \quad 0 \end{cases}$$

Primary variable selection

- Good choice (p^g, s^l) : $\Lambda^l p_c'(s^l) < \infty$
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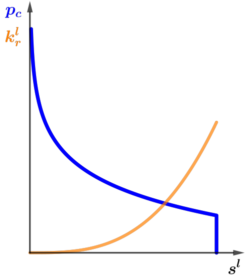
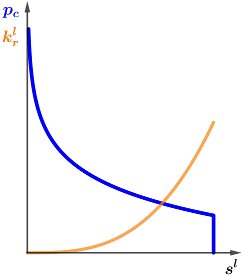


$$\begin{cases} \phi\left(p_c^{-1}\right)'\partial_t p_c & - & \mathrm{div} K\left(\Lambda^l \nabla p^g - \Lambda^l \nabla p_c\right) & = & 0 \\ & - & \mathrm{div} K\left((\Lambda^g + \Lambda^l)\nabla p^g - \Lambda^l \nabla p_c(s^l)\right) & = & 0 \end{cases}$$

Primary variable selection

- Good choice (p^g, s^l) : $\Lambda^l p_c'(s^l) < \infty$
- Bad choice (p^g, p_c) : $\dfrac{\partial s^l}{\partial p_c}$ and Λ^l vanish at dry regions: equation gives $0 \approx 0$

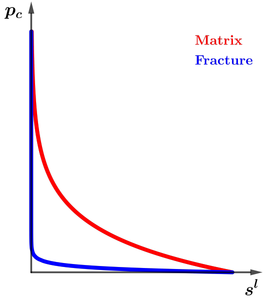
PV selection is more tricky in heterogeneous setting



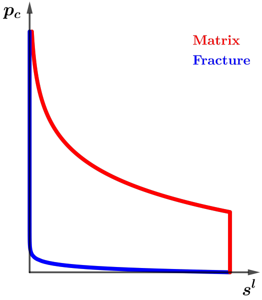
Primary variable selection in two-phase flow

Heterogeneous two-phase flow problem

- Multiple switching of the "second" primary variable



Without entry pressure:
$(p^g, s^l_f - s^l_m)$

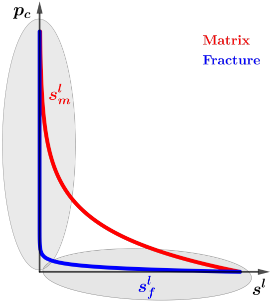


With entry pressure:
$(p^g, s^l_f - p_c - s^l_m)$

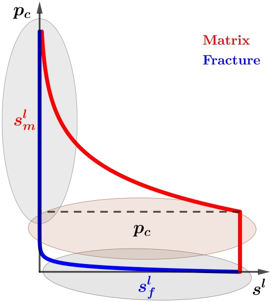
Primary variable selection in two-phase flow

Heterogeneous two-phase flow problem

- Multiple switching of the "second" primary variable

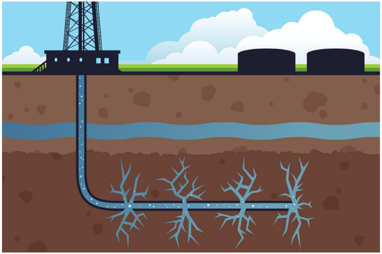


Without entry pressure:
$(p^g, s_f^l - s_m^l)$



With entry pressure:
$(p^g, s_f^l - p_c - s_m^l)$

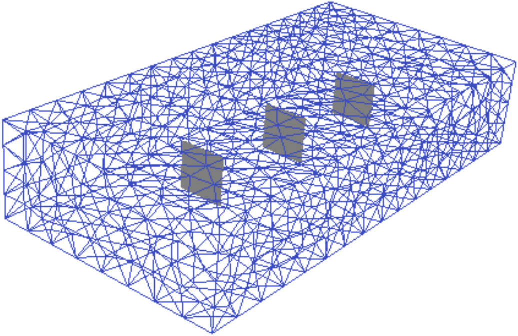
Tight gas recovery test case: configuration



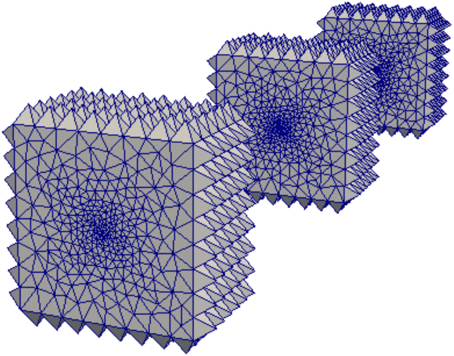
Test case scenario

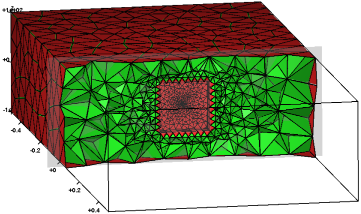
- Water injected at $100\ 10^6$ Pa for one day
- Wells are closed for 3 days
- Production of gas

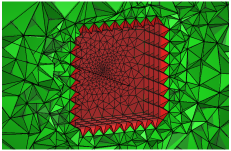
Continuous pressure model



Hybrid mesh: prismatic, pyramidal and tetrahedral elements using TetGen







Nodal space discretization

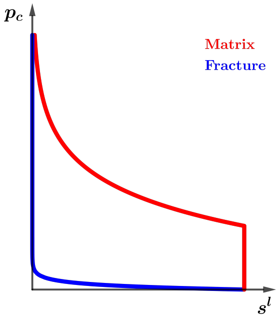
Nb_{cells}	Nb_{nodes}	Nb_{FracF}	linear system d.o.f.
232 920	45 193	1 634	46 827

Bentsen-Anli model

$$P_{c,i}(s^l) = \begin{cases} [-\infty, p_{ent,i}], & s^l = 1 \\ p_{ent,i} - b_i \log(s^l), & \text{else} \end{cases}$$

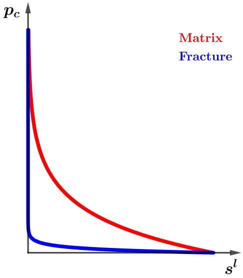
Parameters

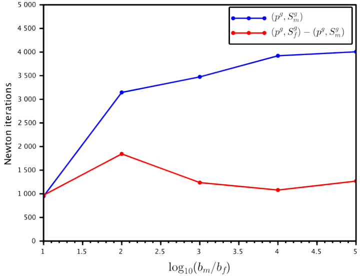
- Entry pressure p_{ent}
- Shape parameter b



Performance: (p^g, s_m^l) vs. $(p^g, s_f^l - s_m^l)$

No entry pressure: $p_{c,i} = -b_i \log(s^l), \quad b_m = 10^5$

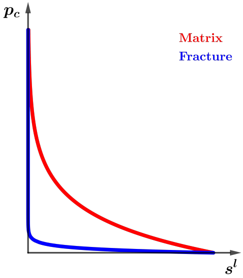


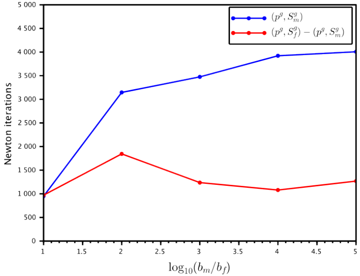


	(p^g, s_m^l)					$(p^g, s_f^l - s_m^l)$				
$\frac{b_m}{b_f}$	$\mathbf{N}_{dt}$	$\mathbf{N}_{Chop}$	$\mathbf{N}_{Newton}$	$\mathbf{N}_{GMRes}$	CPU(s)	$\mathbf{N}_{dt}$	$\mathbf{N}_{Chop}$	$\mathbf{N}_{Newton}$	$\mathbf{N}_{GMRes}$	CPU(s)
10	226	2	4.2	25.9	4 638	226	2	4.3	26.2	5 523
10^2	294	21	10.7	20.1	14 557	246	8	7.5	22.2	9 016
10^3	297	22	11.7	19.7	16 183	225	1	5.5	24.2	6 245
10^4	304	24	12.9	19.8	17 742	225	1	4.8	25.1	5 492
10^5	313	26	12.8	19.6	18 346	235	4	5.4	23.9	6 260
∞	n/a	n/a	n/a	n/a	n/a	235	4	5.3	23.9	6 448

Performance: (p^g, s^l_m) vs. $(p^g, s^l_f - s^l_m)$

No entry pressure: $p_{c,i} = -b_i \log(s^l), \quad b_m = 10^5$





	(p^g, s^l_m)					$(p^g, s^l_f - s^l_m)$				
$\frac{b_m}{b_f}$	N_{dt}	N_{Chop}	N_{Newton}	N_{GMRes}	CPU(s)	N_{dt}	N_{Chop}	N_{Newton}	N_{GMRes}	CPU(s)
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∞	n/a	n/a	n/a	n/a	n/a	235	4	5.3	23.9	6 448

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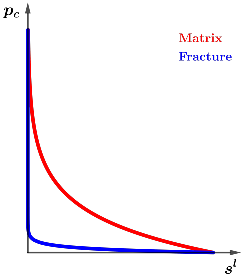

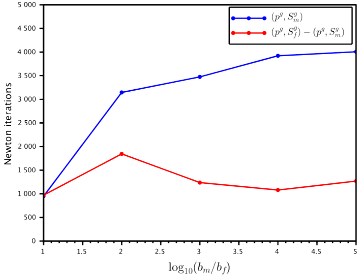

	(p^g, s^l_m)					$(p^g, s^l_f - s^l_m)$				
$\frac{b_m}{b_f}$	N_{dt}	N_{Chop}	N_{Newton}	N_{GMRes}	CPU(s)	N_{dt}	N_{Chop}	N_{Newton}	N_{GMRes}	CPU(s)
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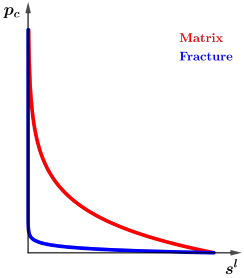
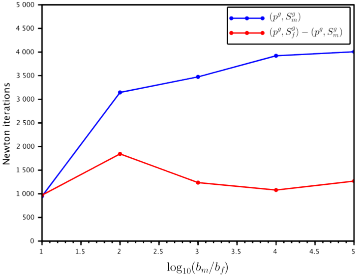
∞	n/a	n/a	n/a	n/a	n/a	235	4	5.3	23.9	6 448

Performance: (p^g, s^l_m) vs. $(p^g, s^l_f - s^l_m)$

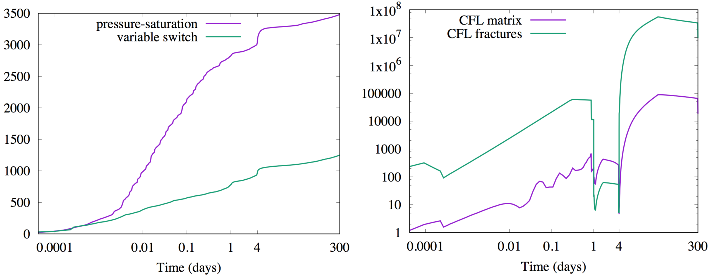
No entry pressure: $p_{c,i} = -b_i \log(s^l), \quad b_m = 10^5$



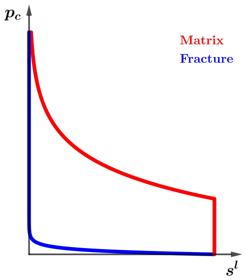


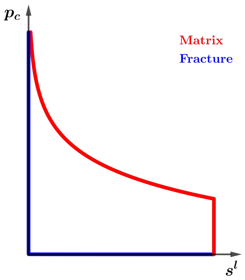
	(p^g, s^l_m)					$(p^g, s^l_f - s^l_m)$				
$\frac{b_m}{b_f}$	N_{dt}	N_{Chop}	N_{Newton}	N_{GMRes}	CPU(s)	N_{dt}	N_{Chop}	N_{Newton}	N_{GMRes}	CPU(s)
10	226	2	4.2	25.9	4 638	226	2	4.3	26.2	5 523
10^2	294	21	10.7	20.1	14 557	246	8	7.5	22.2	9 016
10^3	297	22	11.7	19.7	16 183	225	1	5.5	24.2	6 245
10^4	304	24	12.9	19.8	17 742	225	1	4.8	25.1	5 492
10^5	313	26	12.8	19.6	18 346	235	4	5.4	23.9	6 260
∞	n/a	n/a	n/a	n/a	n/a	235	4	5.3	23.9	6 448

Performance: (p^g, s^l_m) vs. $(p^g, s^l_f - s^l_m)$



For $\frac{b_m}{b_f} = 1000$: Cumulated number of Newton iterations and CFL numbers





	(p^g, s_f^l)					$(p^g, s_f^l - p_c - s_m^l)$				
$\frac{b_m}{b_f}$	$\mathbf{N}_{dt}$	$\mathbf{N}_{Chop}$	$\mathbf{N}_{Newton}$	$\mathbf{N}_{GMRes}$	**CPU(s)**	$\mathbf{N}_{dt}$	$\mathbf{N}_{Chop}$	$\mathbf{N}_{Newton}$	$\mathbf{N}_{GMRes}$	**CPU(s)**
2	221	0	3	29.2	3 937	221	0	3.1	28.9	4 479
10	398	52	9.9	20.2	23 400	262	13	6.8	22.7	10 378
10^2	n/c	n/c	n/c	n/c	n/c	269	14	9.9	20.8	14 185
10^3	n/c	n/c	n/c	n/c	n/c	285	18	8.9	20.1	13 740
10^4	n/c	n/c	n/c	n/c	n/c	242	6	6.9	22.8	9 067
10^5	n/c	n/c	n/c	n/c	n/c	276	16	7.5	21.3	11 516
∞	n/a	n/a	n/a	n/a	n/a	299	22	8.1	19.1	10 770

Outline

Single-phase DFM

- Drains and barriers
- Numerical modeling

Two-phase DFM

- Two-phase flow in homogeneous and heterogeneous media
- Drains are barriers?!

Acceleration of Newton's method

- Variable switching
- **Nonlinear Jacobi preconditioning**

Monotone but slow convergence

Model problem

$$\partial_t S(p) - \Delta p = 0$$

Discretized algebraic problem at each time step

$$S(\mathbf{p}) + L\mathbf{p} = \mathbf{b}, \qquad \mathbf{b} \geqslant 0$$

Assumptions:

- $S : \mathbb{R}^+ \to \mathbb{R}^+$ increasing and concave, $S'(0) \leqslant +\infty$
- $S'(\mathbf{p}) + L$ is M-matrix

Monotone Newton's method

Let

$$F(\mathbf{p}) = S(\mathbf{p}) + L\mathbf{p} - \mathbf{b}$$

Newton's method:

$$\mathbf{p}_{k+1} = \mathbf{p}_k - F'(\mathbf{p}_k)^{-1}F(\mathbf{p}_k), \qquad k \geqslant 0$$

Theorem (Monotone Newton Theorem (Baluev '52; Ortega & Rheinboldt '70))

Let $\mathbf{p}_0$ *satisfy* $F(\mathbf{p}_0) \leqslant 0$*, then*

- $\mathbf{p}_k$ *converges to the unique solution* $\mathbf{p}_\star$
- $\mathbf{p}_k \leqslant \mathbf{p}_{k+1} \leqslant \mathbf{p}_\star$ *for all* $k \geqslant 0$

Illustration (N = 1)

The method is semi-globally convergent, but is very slow!

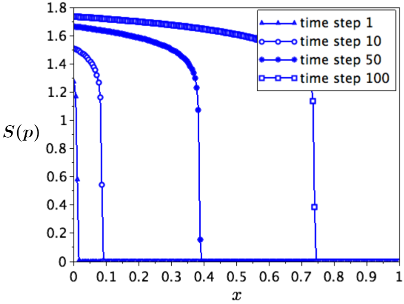
Example: 1D porous media equation

Porous media equation on $(0,1) \times (0,T)$

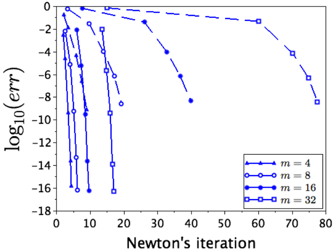
$$\partial_t S(p) - \partial^2_{xx} p = 0, \qquad S(p) = p^{1/m}, \quad m > 1$$

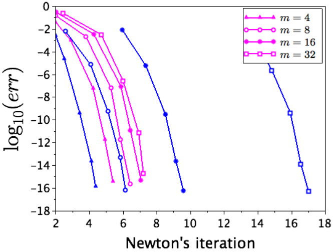
with Neumann boundary conditions

- Inflow at $x = 0$: $-\partial_x p(0,t) = q \geqslant 0$
- No-flow at $x = 1$
- Almost "dry" initial condition: $S(p(x,0)) = 10^{-10}$



Typical solution profile





$$
\begin{aligned}
u-\text{formulation}: &\quad S(\mathbf{p}) + L\mathbf{p} - \mathbf{b}_n &= 0\\
v-\text{formulation}: &\quad \mathbf{s} + LS^{-1}(\mathbf{s}) - \mathbf{b}_n &= 0\\
\tau-\text{formulation}: &\quad \overline{v}(\boldsymbol{\tau}) + L\overline{p}(\boldsymbol{\tau}) - \mathbf{b}_n &= 0
\end{aligned}
$$

- $p-$formulation is the worst!
- $v-$ and $\tau-$ formulations provide better performance, but no convergence theorem
- Is it possible to have both?

Nonlinear Jacobi preconditioner

Nonlinear Jacobi method:

- Separate diagonal and off-diagonal terms

$$\underbrace{S(\mathbf{p}) + \mathrm{diag}(L)\mathbf{p}}_{f(\mathbf{p})} + \underbrace{(L - \mathrm{diag}(L))\mathbf{p}}_{A\mathbf{p}} = \mathbf{b}$$

- Use fixed-point iterations

$$\mathbf{p}_{k+1} = g(\mathbf{b} - A\mathbf{p}_k), \qquad g = f^{-1}$$

Idea: use Jacobi method as preconditioner not a solver

- Left preconditioned method: apply Newton to

$$\mathbf{p} - g(\mathbf{b} - A\mathbf{p}) = 0$$

- Right preconditioned method: apply Newton to

$$\mathbf{p} + Ag(\mathbf{p}) - \mathbf{b} = 0$$

Preconditioned methods satisfy MNT.

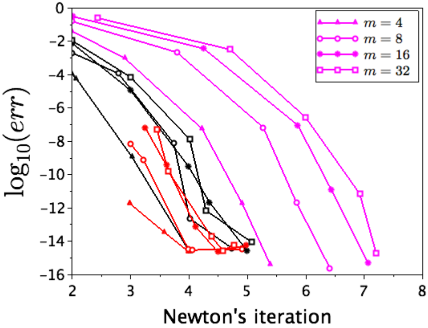
Efficiency of the preconditioned methods

Left-preconditioned:

$$\mathbf{p} - g(\mathbf{b} - A\mathbf{p}) = 0$$

Right-preconditioned:

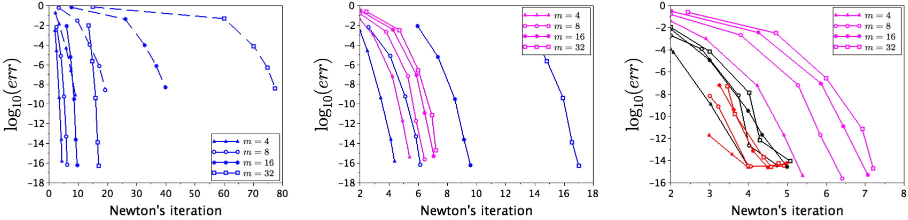
$$\mathbf{p} + Ag(\mathbf{p}) - \mathbf{b} = 0$$



- **Left** and **right** preconditioned methods beat $\tau-$ formulation!

Nonlinear Jacobi preconditioning

- accelerates convergence of Newton's method,
- while preserving monotone convergence



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\text{Left-preconditioned}: &\quad \mathbf{p} - g(\mathbf{b} - A\mathbf{p}) &= 0\\
\text{Right-preconditioned}: &\quad \mathbf{p} + Ag(\mathbf{p}) - \mathbf{b} &= 0
\end{aligned}
$$

Conclusion

Single-phase DFM

- Two kinds of models
- Large spectrum of numerical methods

Two-phase DFM

- Capillary effects are crustal
- Validity of models is less clear
- Numerical analysis is sparser

Acceleration of Newton's method

- Variable switching is extended to heterogeneous problems
- Nonlinear Jacobi preconditioning is under investigation

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Appendix I: M-matrix

Definition

We say that A is an M-matrix if

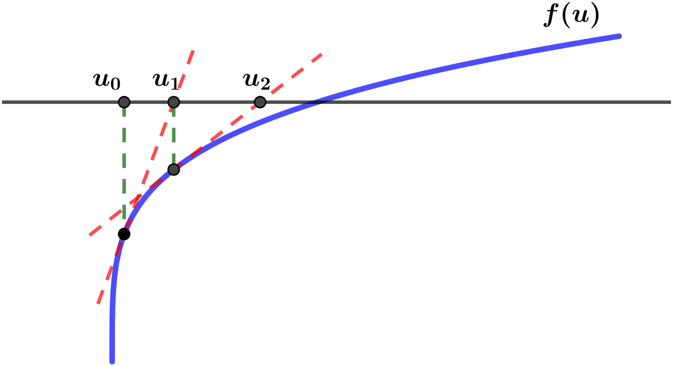
- A is invertible and $A^{-1} \geqslant 0$;
- Off-diagonal elements of A are nonpositives.

Go back

Newton's method for

$$f(p) = 0, \qquad p \in \mathbb{R}$$

- f concave and increasing



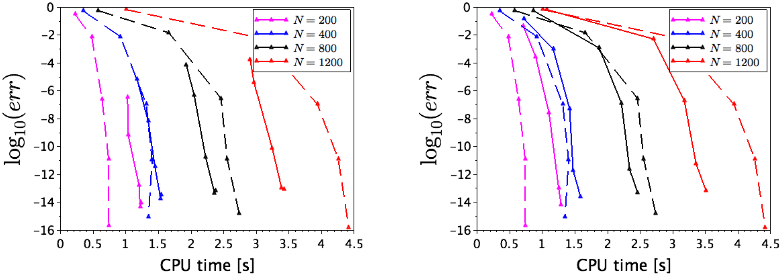
Go back

CPU time efficiency

Preconditioned methods have to evaluate $g = f^{-1}$.

- At each Newton's iteration one solves a system of N uncoupled equations

How expensive is that?



Relative error versus CPU time for different grid sizes.
$\tau-$formulation $=$ dashed lines
preconditioned method $=$ solid lines

- Preconditioned methods are more efficient for large problems ($N \gtrsim 400$) because they require less linear solves

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