Autour de l'équation de Richards et de sa résolution numérique

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Séminaire de Modélisation et Calcul Scientifique

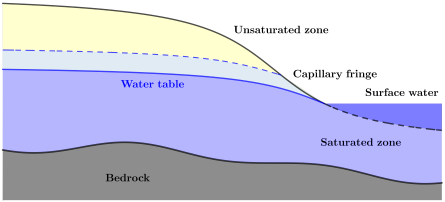
LAGA, Paris 13

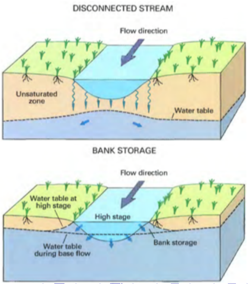
25 juin, 2021

Applications of Richards' equation

Unsaturated/saturated groundwater flow

- pores occupied by water + air





Applications:

- Water resource estimation
- Irrigation
- Contaminant transport
- Interaction with surface water

Why Richards' equation?

French Wikipedia: La résolution numérique de l'équation de Richards demeure l'un des problèmes d'analyse numérique les plus difficiles pour les sciences naturelles[1].

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[1] Farthing and Ogden, Numerical solution of Richards' Equation: a review of advances and challenges, 2017

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Major numerical challenge: Robustness and efficiency of the nonlinear solvers.

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Outline

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Introduction to Richards' equation

- **From saturated to unsaturated flow**
- Capillary pressure
- Analysis
- Simulation pause
- Numerical solution

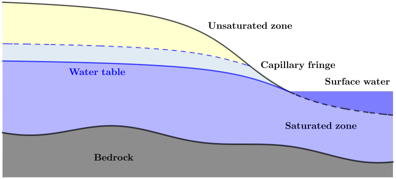
Improving convergence of Newton's method

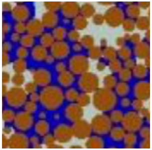
- Monotone Newton Theorem
- Primary variable switching
- Jacobi-Newton method

Simplified models

- Groundwater table movement: Dupuit model
- Infiltration: Green-Ampt model
- Bridging water and infiltration

Saturated vs. unsaturated incompressible groundwater flow





$$s = \frac{|\text{water}|}{|\text{void}|} \text{ in a REV}$$

Saturated flow

Continuity equation:

$$\operatorname{div} \boldsymbol{v} = 0$$

Darcy law:

$$\boldsymbol{v} = -\frac{\mathbb{K}}{\mu}(\nabla p - \rho \boldsymbol{g})$$

Find p satisfying

$$-\operatorname{div}\left(\frac{\mathbb{K}}{\mu}(\nabla p - \rho \boldsymbol{g})\right) = 0$$

Unsaturated flow

Continuity equation:

$$\phi \partial_t s + \operatorname{div} \boldsymbol{v} = 0$$

Darcy-Buckingham law:

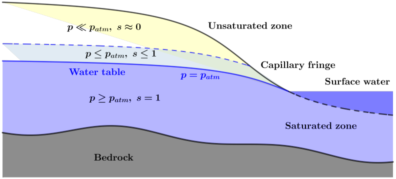
$$\boldsymbol{v} = -\frac{\mathbb{K} k(s)}{\mu}(\nabla p - \rho \boldsymbol{g})$$

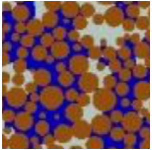
Find p and s satisfying

$$\phi \partial_t s - \operatorname{div}\left(\frac{\mathbb{K} k(s)}{\mu}(\nabla p - \rho \boldsymbol{g})\right) = 0$$

together with $s = S(p)$

Saturated vs. unsaturated incompressible groundwater flow





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Hydrodynamical properties

Find p and s satisfying

$$\phi \partial_t s - \operatorname{div}\left(\frac{\mathbb{K} k(s)}{\mu}(\nabla p - \rho \mathbf{g})\right) = 0$$

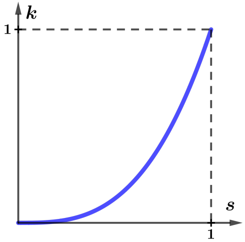
together with $s = S(p)$

Parameters:

- Porosity $\phi \approx 0.01 - 1$
- Permeability $\mathbb{K} \approx 10^{-7} - 10^{-20} m^2$, possibly a full tensor

Closure laws:

- Relative permeability $k(s) \approx s^m$, $m \approx 3$
- Retention curve S: increasing, $S(-\infty) = 0$ and $S(p \geq p_e) = 1$



Relative permeability k as a function of saturation

Hydrodynamical properties

Find p and s satisfying

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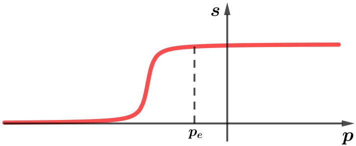
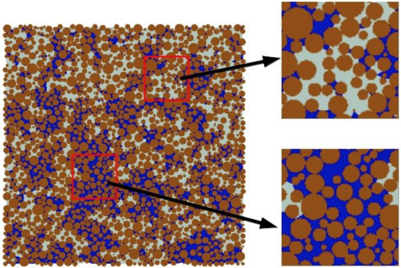


Retention curve $s = S(p)$

Outline

Two-phase flow in porous media

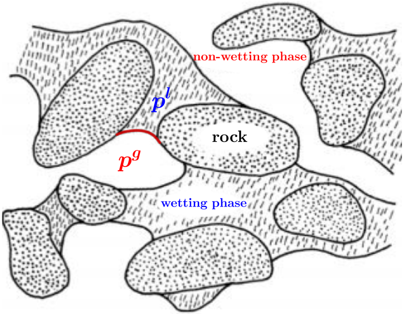
Two fluids shares the pore space

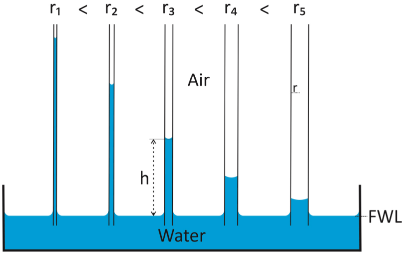


Assumptions

- Two immiscible phases: sharp interfaces at pore scale
- Wetting (water) and non-wetting (air) phases

Capillary pressure at the pore-scale





At the pore scale: pressure jump across free surface

- Young–Laplace law

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

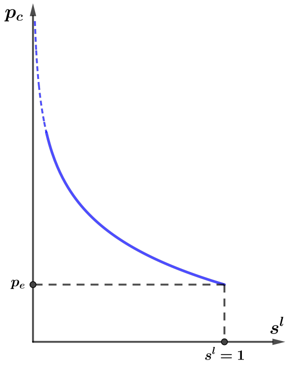
Observations:

- Small pore size $\Rightarrow$ large pressure jump (if shared by both phases!)
- Wetting phase “prefers” small pores

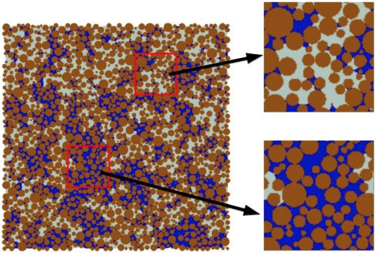
Macroscopic capillary pressure

Capillary pressure at Darcy (macroscopic) scale

$$p^g - p^l = p_c(s^l)$$

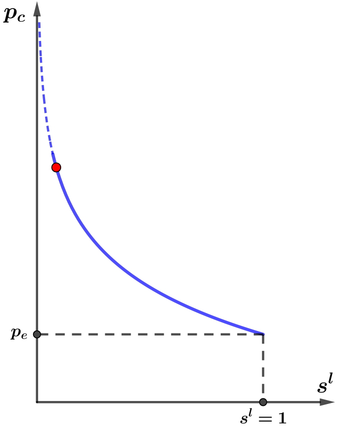


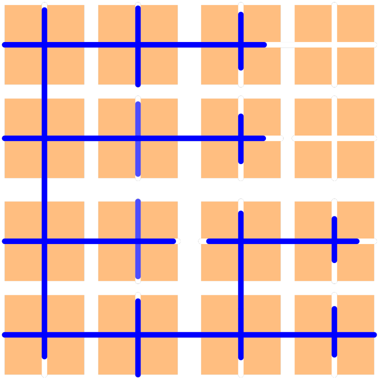
- Entry pressure p_e



- Capillary pressure law depends on pore-size distribution

Macroscopic capillary pressure

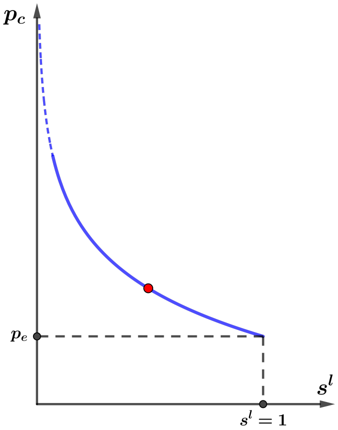


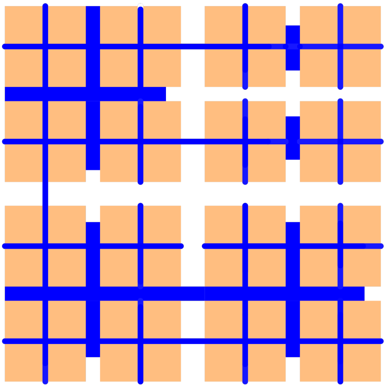


Wetting of the rock:

- Interface between the phases moves to larger pores $\Rightarrow$ pressure jump decreases

Macroscopic capillary pressure





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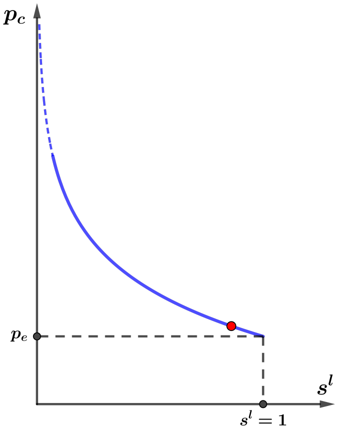

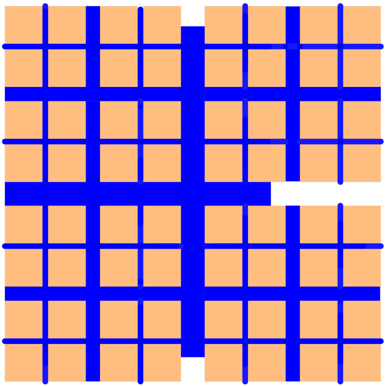

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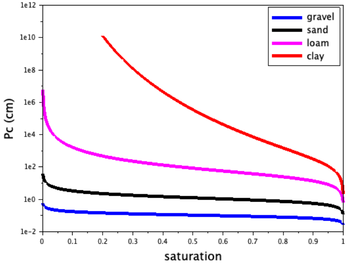
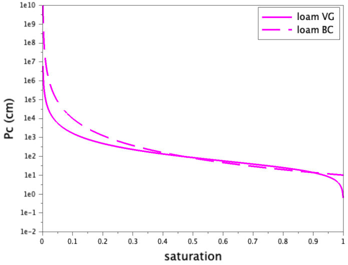
Capillary pressure curves





Macroscopic capillary pressure law

- depends on pore-size distribution;
- may be neglected/assumed constant for some soils.

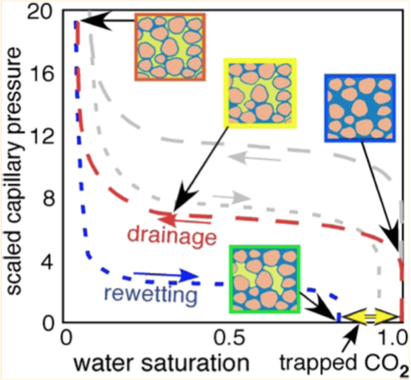
Van Genuchten model

$$p_c(s) = p_\alpha \left(s^{-\frac{1}{m}} - 1 \right)^{\frac{1}{n}}$$

Brooks-Corey model

$$p_c(s) = p_e s^{-\lambda}$$

Capillary hysteresis (not covered)



Capillary pressure depends on the rate of wetting (or drying):

$$p_c = p_c\left(s^l, \partial_t s^l\right)$$

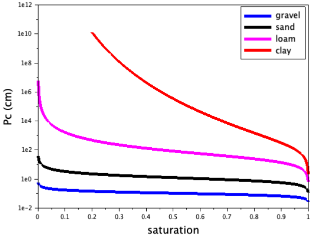
Retention curve

Set $p_g = p_{atm}$, from capillary pressure law:

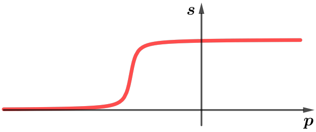
$$p_l = p_{atm} - p_c(s) \qquad \text{for} \quad 0 < s < 1$$

Let $p_{atm} = 0$, we have

$$\begin{cases} (p_l + p_c(s))(1 - s) = 0, \\ p_l + p_c(s) \geq 0, \quad 1 - s \geq 0. \end{cases}$$



Functional closure $s = S(p)$



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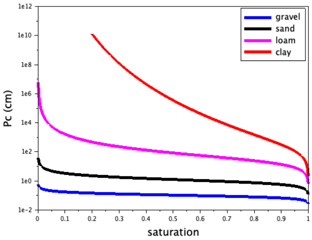

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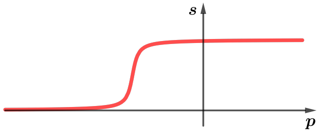
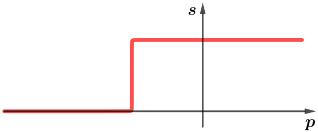


Graphical closure $s \in S(p)$



Outline

Formulation using Kirchhoff transform

Richards' equation

$$\partial_t s - \operatorname{div} k(s)\,(\nabla p - \boldsymbol{g}) = 0$$

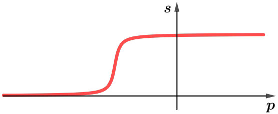
Natural energy estimate

$$\sup_{t \leq T} \int_\Omega \Psi(\mathrm{x})\,\mathrm{dx} + \int_0^T \int_\Omega k(s)|\nabla p|^2\,\mathrm{dx} < +\infty, \qquad \Psi(p) = S(p)p - \int^p S(\pi)\,\mathrm{d}\pi$$

does not provide control on $\|\nabla p\|_{L^2}$

Kirchhoff transform

$$U(p) = \int^p k(\pi)\,\mathrm{d}\pi$$

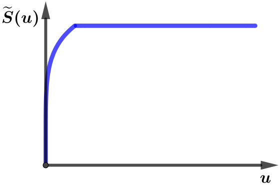


Reformulated equation

$$\partial_t s - \operatorname{div}\left(\nabla u - k(s)\boldsymbol{g}\right) = 0$$

with $s = \widetilde{S}(u) := S(U^{-1}(u))$

Equation is linear w.r.t. "generalized pressure" u.



Formulation using Kirchhoff transform

Richards' equation using Kirchhoff transform

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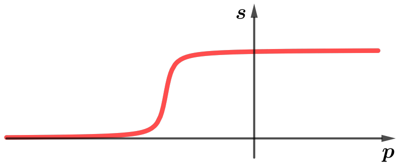
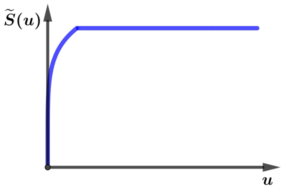


$s = \widetilde{S}(u)$



Typically $\widetilde{S}(u) \sim u^{1/m}$, $m > 1$ near $u = 0$,

- connections to porous media equation: $\partial_t u^{1/m} = \Delta u$
- almost hyperbolic behavior near $u = 0$: $\partial_t s + \mathrm{div} k(s)\boldsymbol{g} = 0$.
  Set of $1d$ problems!

Formulation using Kirchhoff transform

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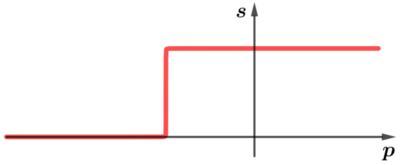

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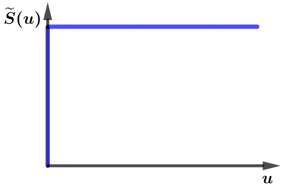

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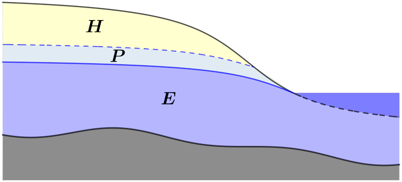

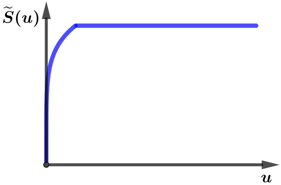

Coexistence of multiple "regimes"

Existence and uniqueness

- Parabolic-elliptic: Van Duyn & Peletier '82, Alt & Luckhaus '83
- Hyperbolic-elliptic: Carrillo '94
- Hyperbolic-parabolic-elliptic: Carrillo '99

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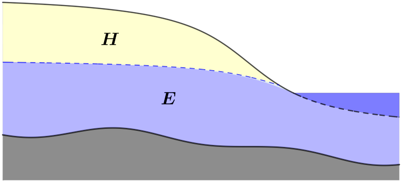

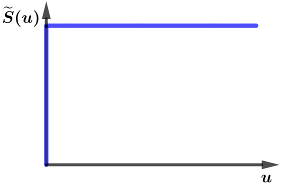

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- Hyperbolic-elliptic (Dam problem): Visintin '80, Carrillo '94
- Hyperbolic-parabolic-elliptic: Carrillo '99

Formulation using Kirchhoff transform

Richards' equation using Kirchhoff transform

$$\partial_t s - \operatorname{div}\left(\nabla u - k(s)\boldsymbol{g}\right) = 0$$

Pros & cons of f Kirchhoff formulation

Makes mathematicians 😀

- Good for the analysis of PDE and numerical schemes
- Easier to solve the discrete problem

Makes engineers 😕

- No analytical expression of $U(p)$ for some closure laws
- Harder to incorporate additional physics

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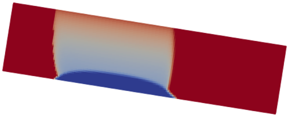
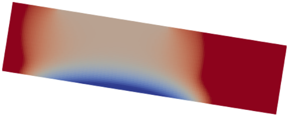
Simulation pause

Infiltration into the dry soil

Sand



Clay



Infiltration during a flood event

Simulation pause

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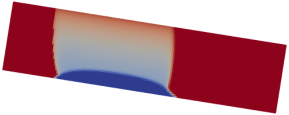

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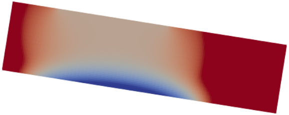

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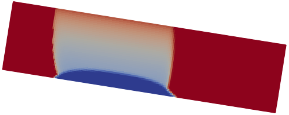

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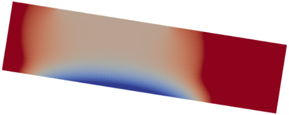

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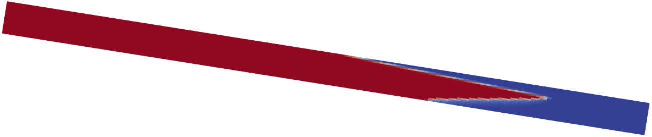

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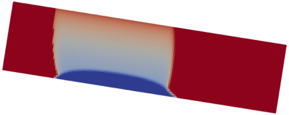
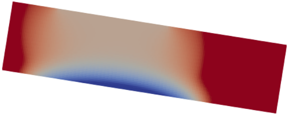
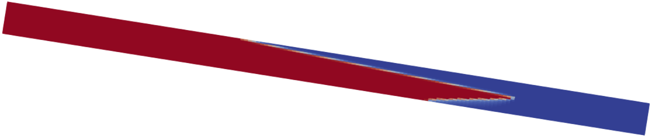


Clay



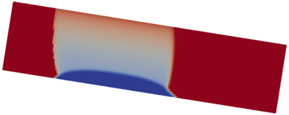
Infiltration during a flood event



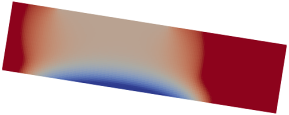
Simulation pause

Infiltration into the dry soil

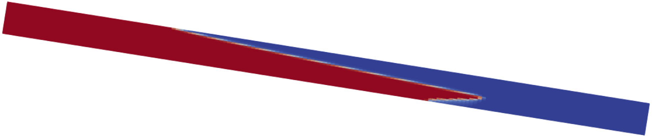
Sand



Clay



Infiltration during a flood event



Simulation pause

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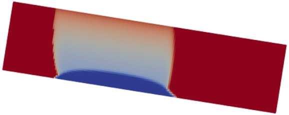

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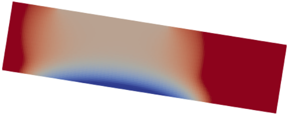

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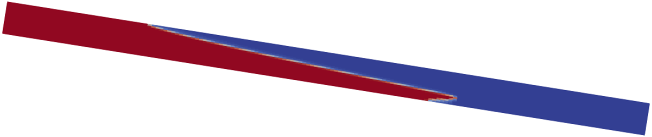

Outline

Introduction to Richards' equation

- From saturated to unsaturated flow
- Capillary pressure
- Analysis
- Simulation pause
- **Numerical solution**

Improving convergence of Newton's method

- Monotone Newton Theorem
- Primary variable switching
- Jacobi-Newton method

Simplified models

- Groundwater table movement: Dupuit model
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Numerical solution of Richards' equation

Richards' equation

$$\partial_t s - \mathrm{div}\, k(s)\,(\nabla p - \boldsymbol{g}) = 0, \qquad s = S(p)$$

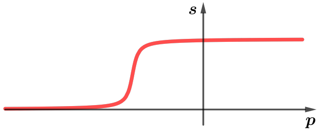
Implicit discretization

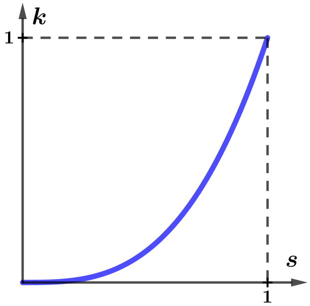
$$\begin{cases} \boldsymbol{F}(\boldsymbol{s}, \boldsymbol{p}) = 0 \\ \boldsymbol{s} = S(\boldsymbol{p}) \end{cases}$$

Closure law elimination gives

- either $\boldsymbol{F}(S(\boldsymbol{p}), \boldsymbol{p}) = 0$
- either $\boldsymbol{F}(\boldsymbol{s}, S^{-1}(\boldsymbol{s})) = 0$

Primary variable selection





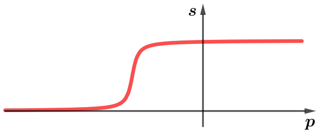
$p-$formulation

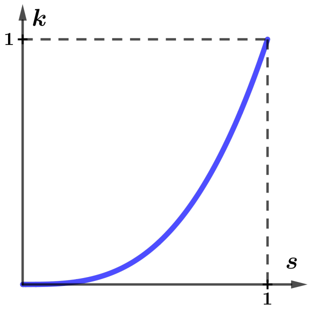
$$S'(p)\partial_t p - \mathrm{div} k(s) (\nabla p - \boldsymbol{g}) = 0$$

$s-$formulation

$$\partial_t s - \mathrm{div} \left((S^{-1})'(s)k(s)\nabla s - k(s)\boldsymbol{g}\right) = 0$$

Primary variable selection





$p-$formulation

$$S'(p)\partial_t p - \mathrm{div} k(s)\left(\nabla p - \boldsymbol{g}\right) = 0$$

Pros:

- Both sat. and unsat. regimes

Cons:

- Dry soil $s \approx 0$: gives $0 \approx 0$
- Does not cover the case $s \in S(p)$

$s-$formulation

$$\partial_t s - \mathrm{div}\left((S^{-1})'(s)k(s)\nabla s - k(s)\boldsymbol{g}\right) = 0$$

Cons:

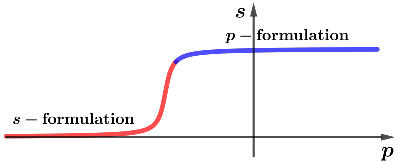
- Does not cover the saturated regime

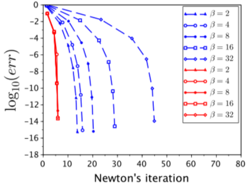
Pros:

- Dry soil $s \approx 0$: $k(s)(S^{-1})'(p) < +\infty$
- Covers the case $s \in S(p)$

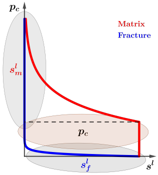
Primary variable selection

Solution: Variable switching[1,2,3,4]





Numerical efficiency



Heterogeneous multi-phase flow

[1] Forsyth, Wu, Pruess, 1995
[2] Diersch, Perrochet, 1999
[3] Brenner, Groza, Jeannin, Masson, Pellerin, 2017
[4] Brenner, Canceès, 2017

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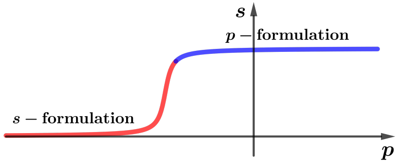

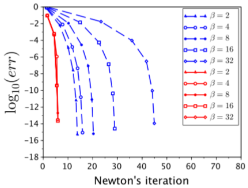

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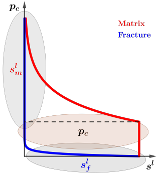

Heterogeneous multi-phase flow

Similar ideas applies to $\partial_t s - \operatorname{div}(\nabla u - k(s)\boldsymbol{g}) = 0$

[1] Forsyth, Wu, Pruess, 1995
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Classical discretizations: Kirchhoff pressure

Implicit finite volume scheme

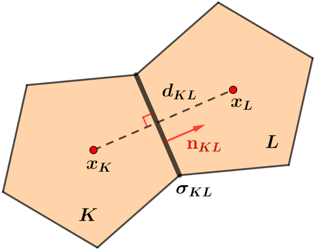
$$\frac{s_K^n - s_K^{n-1}}{\Delta t_n} + \sum_L q_{KL}^n = 0$$

Flux discretiaztion

$$\frac{1}{|\sigma_{KL}|}\int_{\sigma_{KL}} \nabla u - k(s)\boldsymbol{g}\,\mathrm{d}\sigma \approx \frac{u_K - u_L}{d_{KL}} - k(s_{KL})\,\boldsymbol{g}\cdot \mathrm{n}_{KL}$$

with upwinding

$$s_{KL} = \begin{cases} s_K, & -\boldsymbol{g}\cdot \mathrm{n}_{KL} \geq 0 \\ s_L, & \text{else} \end{cases}$$



Admissible mesh

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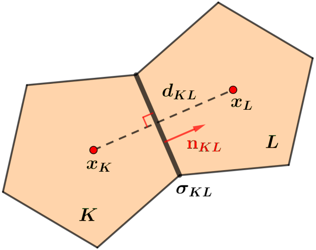

Admissible mesh

Algebraic system:

$$\boldsymbol{s} + A\boldsymbol{u} + Bk(\boldsymbol{s}) = \boldsymbol{s}^{n-1}, \quad \boldsymbol{s} = \widetilde{S}(\boldsymbol{u})$$

Structural properties:

- A and $Bk'(\boldsymbol{s})$ are M-matrices
- $\widetilde{S}$ is diagonal and concave

Classical discretizations: Kirchhoff pressure

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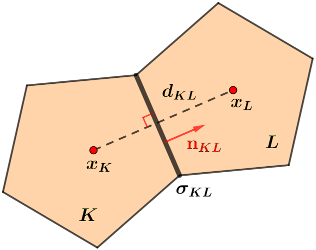


Admissible mesh

Semi-implicit discretization:

$$\mathbf{s} + Au + Bk(\mathbf{s}^{n-1}) = \mathbf{s}^{n-1}, \quad \mathbf{s} = \widetilde{S}(\mathbf{u})$$

Structural properties:

- A and $Bk'(\mathbf{s})$ are M-matrices
- $\widetilde{S}$ is diagonal and concave

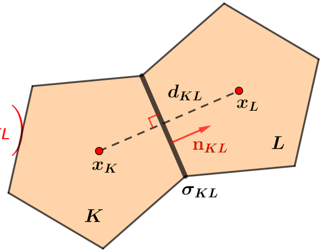
Classical discretizations: physical pressure

Implicit finite volume scheme

$$\frac{s_K^n - s_K^{n-1}}{\Delta t_n} + \sum_L q_{KL}^n = 0$$

Flux discretiaztion

$$\frac{1}{|\sigma_{KL}|}\int_{\sigma_{KL}} k(s)\,(\nabla p - \boldsymbol{g})\,\mathrm{d}\sigma \approx k(s_{KL})\left(\frac{p_K - p_L}{d_{KL}} - \boldsymbol{g}\cdot \mathrm{n}_{KL}\right)$$



with upwinding

$$s_{KL} = \begin{cases} s_K, & \dfrac{p_K - p_L}{d_{KL}} - \boldsymbol{g}\cdot \mathrm{n}_{KL} \geq 0 \\ s_L, & \text{else} \end{cases}$$

Remarks:

- Upwinding handles degeneracy $k(0) = 0$
- Semi-implicit schemes

$$q_{KL}^n = |\sigma| k(s_{KL}^{n-1}) \left(\frac{p_K^n - p_L^n}{d_{KL}} - \boldsymbol{g}\cdot \mathrm{n}_{KL}\right)$$

are problematic since $\nabla p \cdot \mathrm{n}_{KL}$ may be very large.

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Simplified models

- Groundwater table movement: Dupuit model
- Infiltration: Green-Ampt model
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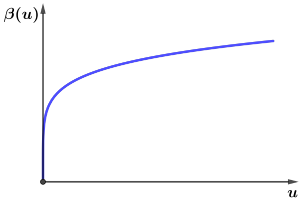
Objectives

Model problem: Find $\boldsymbol{u} \in \mathbb{R}^N$

$$\beta(\mathrm{u}) + A\mathrm{u} = \boldsymbol{b}, \qquad \boldsymbol{b} \geq 0$$

Objective: Newton-like iterative method

- efficient and robust w.r.t. to the shape of β
- with guarantied (semi-)global convergence



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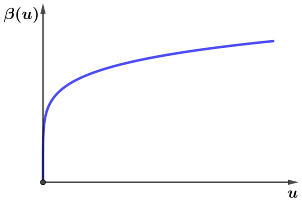

Objective: Newton-like iterative method

- efficient and robust w.r.t. to the shape of β
- with guarantied (semi-)global convergence

Assumptions:

- $J(\boldsymbol{u}) = \beta'(\boldsymbol{u}) + A$ is M-matrix:
  $J(\boldsymbol{u})^{-1} \geq 0$ and $(J(\boldsymbol{u}))_{ij} \leq 0, i \neq j$
- $\beta_i : \mathbb{R}^+ \to \mathbb{R}^+$ diagonal, increasing and concave, $\beta_i'(0) \leq +\infty$

Outline

Monotone Newton's method

Notations

$$F(\boldsymbol{u}) = \beta(\boldsymbol{u}) + A\boldsymbol{u} - \boldsymbol{b}$$

Newton's method

$$F'(\boldsymbol{u}_k)\,(\boldsymbol{u}_{k+1} - \boldsymbol{u}_k) + F(\boldsymbol{u}_k) = 0$$

Monotone Newton Theorem (Baluev '52; Ortega & Rheinboldt '70)

Let $\boldsymbol{u}_0$ satisfy $F(\boldsymbol{u}_0) \leq 0$, then

- $\boldsymbol{u}_k$ converges to the unique solution $\boldsymbol{u}_\star$
- $\boldsymbol{u}_k \leq \boldsymbol{u}_{k+1} \leq \boldsymbol{u}_\star$ for all $k \geq 0$

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Main ingredients:

- F is concave (or convex)
- $F'(\boldsymbol{u})$ is an M-matrix

Illustration (N = 1)

Monotone Newton's method

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Main ingredients:

- F is concave (or convex)
- $F'(\boldsymbol{u})$ is an M-matrix

Illustration (N = 1)

The method is semi-globally convergent. Is it efficient?

Monotone Newton's method

Notations

$$F(\boldsymbol{u}) = \beta(\boldsymbol{u}) + A\boldsymbol{u} - \boldsymbol{b}$$

Newton's method

$$F'(\boldsymbol{u}_k)\,(\boldsymbol{u}_{k+1} - \boldsymbol{u}_k) + F(\boldsymbol{u}_k) = 0$$

Removing concavity (convexity) assumption for problems with diagonal nonlinearities:

- Accelerated monotone iterations:[1] compute the sequence of lower/upper solutions

$$\left(\max_{\underline{\boldsymbol{u}}_k \leq \xi \leq \overline{\boldsymbol{u}}_k} F'(\xi)\right)(\boldsymbol{v}_{k+1} - \boldsymbol{v}_k) + F(v_k) = 0, \qquad \boldsymbol{v}_k = \underline{\boldsymbol{u}}_k, \overline{\boldsymbol{u}}_k$$

- Nested Newton's method[2]: $F(\boldsymbol{u}) = F_1(\boldsymbol{u}) - F_2(\boldsymbol{u})$

  Outer iteration loop:

$$F_1'(\boldsymbol{u}_k)(\boldsymbol{u}_{k+1} - \boldsymbol{u}_k) + F_1(\boldsymbol{u}_k) - F_2(\boldsymbol{u}_{k+1}) = 0$$

[1] Ortega & Rheinboldt '70, Pao '98, '03

[2] Brugnano & Casulli '09, Casulli & Zanolli '12

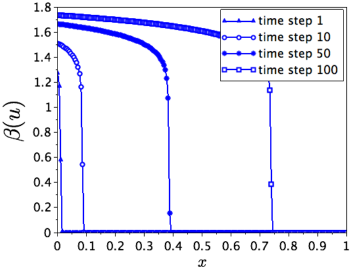
1D numerical experiment

Porous media equation on $(0,1) \times (0,T)$

$$\partial_t \beta(u) - \partial^2_{xx} u = 0, \qquad \beta(u) = u^{1/m}$$

with Neumann boundary conditions

- Inflow at $x = 0$: $-\partial_x u(0,t) = q > 0$
- No-flow at $x = 1$
- Almost "dry" initial condition: $\beta(u(x,0)) = 10^{-10}$



Solution profile at different time steps

Performance assessment: $u-$ and $v-$formulations

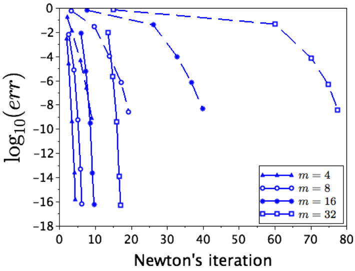
Original u-formulation:

$$\beta(\boldsymbol{u}) + A\boldsymbol{u} - \boldsymbol{b} = 0$$

Alternative v-formulation:

$$\boldsymbol{v} + A\beta^{-1}(\boldsymbol{v}) - \boldsymbol{b} = 0$$

Different values of $m > 1$ in $\beta(u) = u^{1/m}$



- Dashed: Original formulation is inefficient, manly because $\beta'(0) = +\infty$.
- Solid: Alternative formulation is more efficient, but concavity is lost: note that $(A)_{ii}(A)_{ij} \leq 0, i \neq j$

Performance assessment: $u-$ and $v-$formulations

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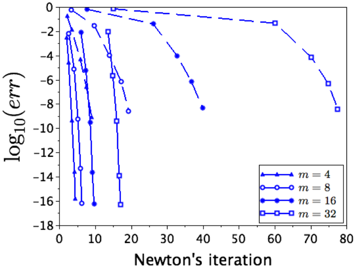

- Performance of both formulations depends on m
- Can we find an even more efficient primary variable?

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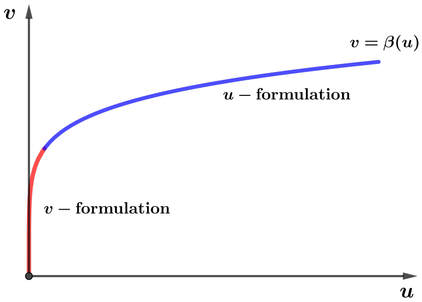
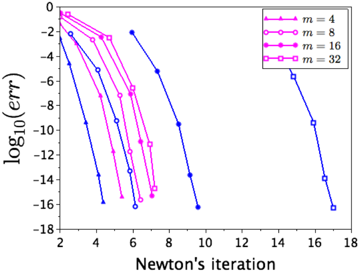
Adaptive choice of the variable



- Switching between v and u may be a good idea
- Well known for Richards' equation

Efficiency of variable switching

- v-formulation: $\partial_t v - \triangle\beta^{-1}(v) = 0$
- variable switching: PDE?



- Variable switching: is more **efficient** and is **robust** w.r.t. m
- Drawback: implementation using **if/else** conditions

Graph parametrization

Parametrization of the graph $v = \beta(u)$:

Let $\overline{u}, \overline{v} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

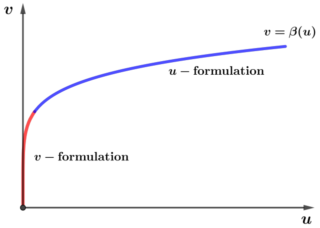
$$\overline{v}(\tau) = \beta(\overline{u}(\tau)) \qquad \forall \tau \in \mathbb{R}^+$$

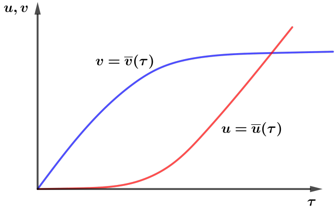
PDE in terms of the new variable τ

$$\partial_t \overline{v}(\tau) - \Delta \overline{u}(\tau) = 0$$

Variable switching:

$$\max(\overline{v}'(\tau), \overline{u}'(\tau)) = 1$$





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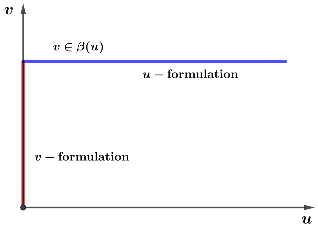
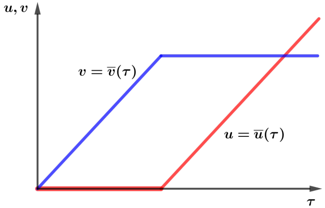




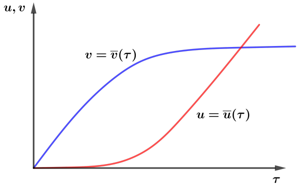
Estimates (B. & Cancès '17)

Define $F_\tau(\boldsymbol{\tau}) = \overline{v}(\boldsymbol{\tau}) + A\overline{u}(\boldsymbol{\tau}) - b$

Estimates on $F_\tau'(\boldsymbol{\tau})$

$$\|F_\tau'(\boldsymbol{\tau})\|, \|F_\tau'(\boldsymbol{\tau})\|^{-1} < C$$

uniformly w.r.t. $\boldsymbol{\tau}$ and the shape of β.



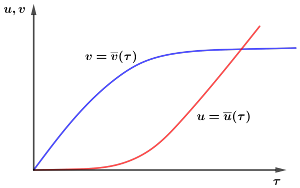
Estimates (B. & Cancès '17)

Define $F_\tau(\boldsymbol{\tau}) = \overline{v}(\boldsymbol{\tau}) + A\overline{u}(\boldsymbol{\tau}) - b$

Estimates on $F'_\tau(\boldsymbol{\tau})$

$$\|F'_\tau(\boldsymbol{\tau})\|, \|F'_\tau(\boldsymbol{\tau})\|^{-1} < C$$

uniformly w.r.t. $\boldsymbol{\tau}$ and the shape of β.



Corollaries:

- Control of $\mathrm{cond}(F'_\tau)$
- Justified stopping criterion:

$$\|F_\tau(\boldsymbol{\tau})\| < \epsilon \Rightarrow \|\boldsymbol{\tau} - \boldsymbol{\tau}_\star\| < C\epsilon \Rightarrow \begin{cases} \|\overline{v}(\boldsymbol{\tau}) - v_\star\| < C\epsilon, \\ \|\overline{u}(\boldsymbol{\tau}) - u_\star\| < C\epsilon \end{cases}$$

Outline

Introduction to Richards' equation

- From saturated to unsaturated flow
- Capillary pressure
- Analysis
- Simulation pause
- Numerical solution

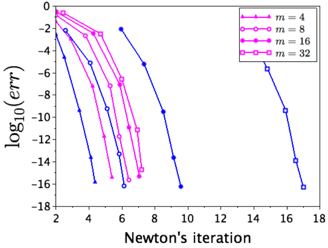
Improving convergence of Newton's method

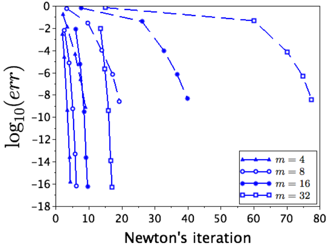
- Monotone Newton Theorem
- Primary variable switching
- **Jacobi-Newton method**

Simplified models

- Groundwater table movement: Dupuit model
- Infiltration: Green-Ampt model
- Bridging water and infiltration

Recap on various formulations





$$
\begin{aligned}
u-\text{formulation}: &\quad \beta(\boldsymbol{u}) + A\boldsymbol{u} - \boldsymbol{b} &= 0\\
v-\text{formulation}: &\quad \boldsymbol{v} + A\beta^{-1}(\boldsymbol{v}) - \boldsymbol{b} &= 0\\
\tau-\text{formulation}: &\quad \overline{v}(\boldsymbol{\tau}) + A\overline{u}(\boldsymbol{\tau}) - \boldsymbol{b} &= 0
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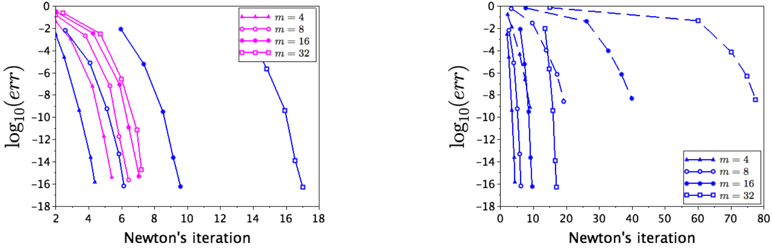
$$

- u−formulations: catastrophic performance, but convergence theorem
- τ−formulations: excellent performance, but no convergence theorem

Recap on various formulations



$$\begin{array}{lrcl} u-\text{formulation}: & \beta(\boldsymbol{u})+A\boldsymbol{u}-\boldsymbol{b} & = & 0 \\ v-\text{formulation}: & \boldsymbol{v}+A\beta^{-1}(\boldsymbol{v})-\boldsymbol{b} & = & 0 \\ \tau-\text{formulation}: & \overline{v}(\boldsymbol{\tau})+A\overline{u}(\boldsymbol{\tau})-\boldsymbol{b} & = & 0 \end{array}$$

- $u-$formulations: catastrophic performance, but convergence theorem
- $\tau-$formulations: excellent performance, but no convergence theorem

Can we have both performance and convergence result?

Jacobi-Newton method

Nonlinear Jacobi method:

- Separate diagonal and off-diagonal terms

$$\underbrace{\beta(\boldsymbol{u}) + \mathrm{diag}(A)\boldsymbol{u}}_{f(\boldsymbol{u})} + \underbrace{(A - \mathrm{diag}(A))\boldsymbol{u}}_{B\boldsymbol{u}} = \boldsymbol{b}$$

- Use fixed-point iterations

$$\boldsymbol{u}_{k+1} = g(\boldsymbol{b} - B\boldsymbol{u}_k), \qquad g = f^{-1}$$

(the method is linearly convergent)

Jacobi-Newton method

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Idea: Use Jacobi method as preconditioner not as a solver

- Left preconditioned method: apply Newton to

$$\boldsymbol{u} - g(\boldsymbol{b} - B\boldsymbol{u}) = 0$$

Jacobi-Newton method

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with $\boldsymbol{\xi} = f(\boldsymbol{u})$

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Preconditioned methods satisfy MNT: note that $B \leq 0$.

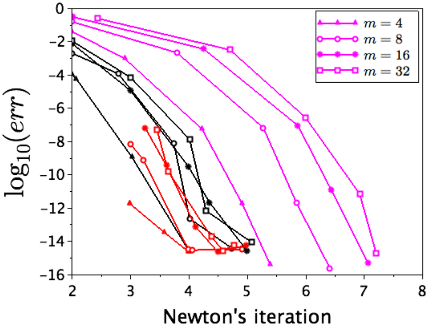
Efficiency of the preconditioned methods

Left-preconditioned:

$$\boldsymbol{u} - g(\boldsymbol{b} - A\boldsymbol{u}) = 0$$

Right-preconditioned:

$$\boldsymbol{\xi} + Ag(\boldsymbol{\xi}) - \boldsymbol{b} = 0$$



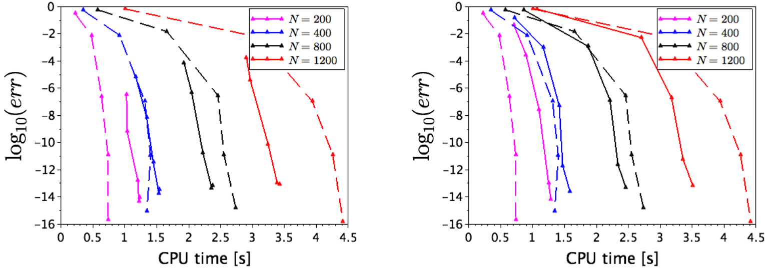
- **Left** and **right** preconditioned methods beat $\tau-$ formulation!

CPU time efficiency

Preconditioned methods have to evaluate $g = f^{-1}$:

- At each Newton's iteration one solves N uncoupled equations

  How expensive is that?



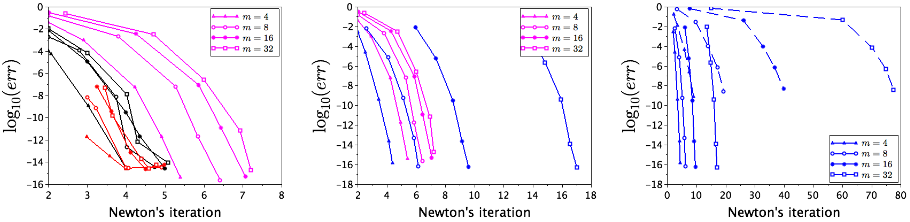
Relative error versus CPU time for different grid sizes:
$\tau-$formulation = dashed lines
preconditioned method = solid lines

- Efficient for all except very small problems ($N \gtrsim 400$) because less linear solves

Conclusion

Nonlinear Jacobi preconditioning

- accelerates convergence of Newton's method,
- while preserving monotone convergence.
- Approximate evaluation of g is Ok.



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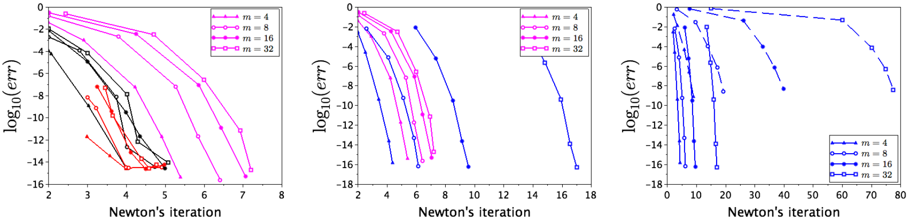

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$$

Extensions and perspectives

Industrial problems:

Non diagonal nonlinearities and non monotone discretizations
Richards' equation, two-phase flow, heterogeneous media, etc, ...

- Works extremely well with parametrization
- Ongoing work on Jacobi-Newton
  - Difficulty: $\partial_{ij} \boldsymbol{F}_k \neq 0$

Toy problems:

- Non-convex diagonal nonlinearities: Jacobi-Newton + Pao's or Casulli's method
- Analysis of (Block-Jacobi, Gauss-Seidel, DD)-Newton method

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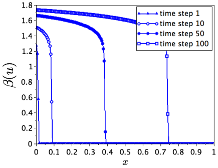
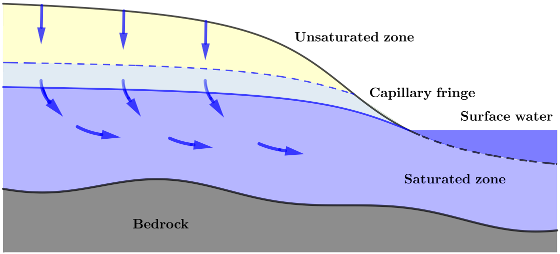


Outline

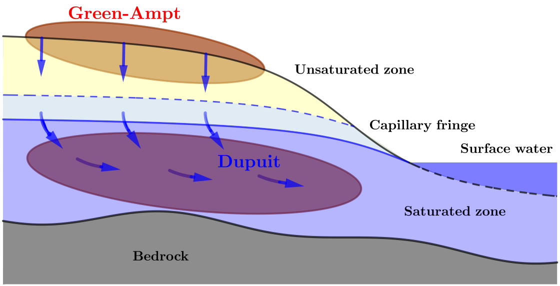
Simplified models



General considerations:

- Below water table: flow is mostly horizontal
- In the unsaturated zone: flow is mostly vertical

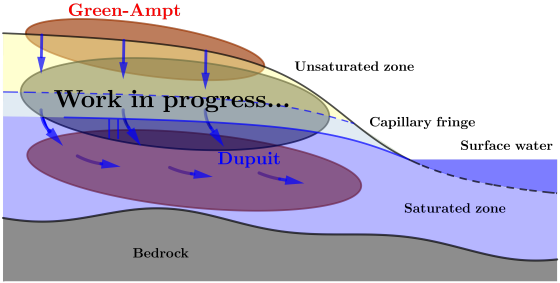
Simplified models



Available models:

- Dupuit for groundwater table evolution: $2D$
- Green-Ampt for infiltration: $0D \times N_x \times N_y$

Simplified models



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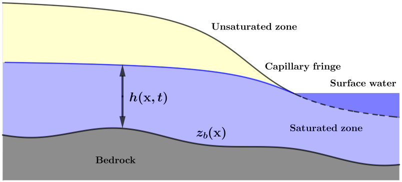
Dupuit model



Large scale aquifer model Dupuit-(Forchheimer, Boussinesq)[1]:

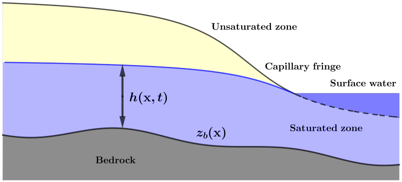
$$\eta \partial_t h - \mathrm{div}_{\mathrm{x}} \left(h \nabla (h + z_b) \right) = r$$

- Depth averaged
- Shallow flow assumptions

[1] Dupuit 1863, Forchheimer 1901, Boussinesq 1903, 1904

[2] Blendinger 1999

Dupuit model



Large scale aquifer model Dupuit-(Forchheimer, Boussinesq)[1]:

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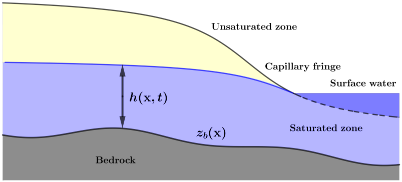
Empirical parameters

- Specific storage η
- Recharge r

[1] Dupuit 1863, Forchheimer 1901, Boussinesq 1903, 1904

[2] Blendinger 1999

Dupuit model



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Limitations

- Unsaturated zone is not modeled
- No flow trough the capillary fringe[2]

[1] Dupuit 1863, Forchheimer 1901, Boussinesq 1903, 1904

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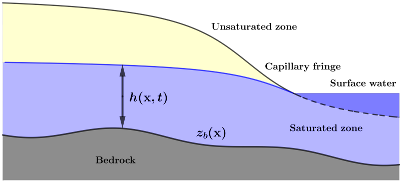
Dupuit model



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Questions

- Connection to Richards' ?
- Meaning of η and r?

[1] Dupuit 1863, Forchheimer 1901, Boussinesq 1903, 1904

[2] Blendinger 1999

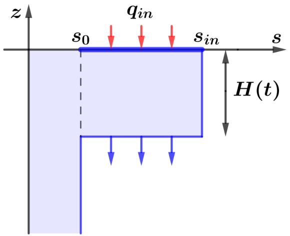
Green-Ampt

1D Richards' equation

$$\partial_t s + \partial_z k(s)\boldsymbol{g} = 0$$

Assumptions:

- Semi-infinite domain
- Constant initial saturation s_0
- No capillarity*



Infiltration saturation: $k(s_{in})g = q_{in}$

Infiltration velocity

$$(s_{in} - s_0)\frac{\mathrm{d}H}{\mathrm{d}t} = (k(s_{in}) - k(s_0))g$$

Green-Ampt

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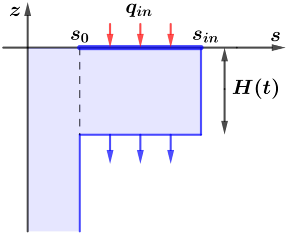

What if $q_{in} > k(1)g$?

- Surface runoff
- Pressurized font

$$(1 - s_0)\frac{\mathrm{d}H}{\mathrm{d}t} = (k(1) - k(s_0))g + k(1)\frac{H_p - 0}{H}$$

H_p - pounding water depth.

Green-Ampt

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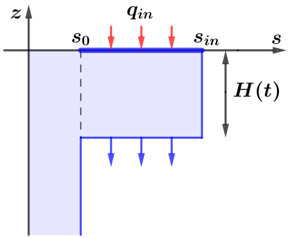

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H_p - pounding water depth.

Remarks

- Simple 0D model that can be coupled with surface flow
- No memory
- What if reach the groundwater table?

Dupuit 2D + Richards' 1D model

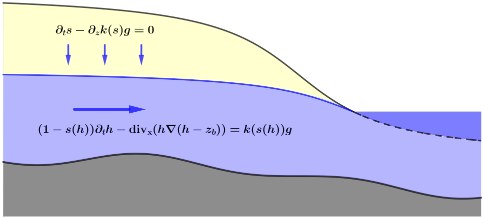
Assumptions:

- No capillarity
- No pressurized flow above groundwater-table*
- Shallow flow

Decomposition

- Above water table: set of 1D Richards' equations
- Below water table: Dupuit

Coupled model[1,...,2]



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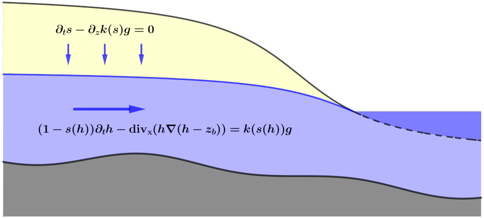

Observations:

- Richards' on moving domain
- Can be extend to pressurized infiltration fronts

Dupuit 2D + Richards' 1D model

Assumptions:

- No capillarity
- No pressurized flow above groundwater-table⋆
- Shallow flow

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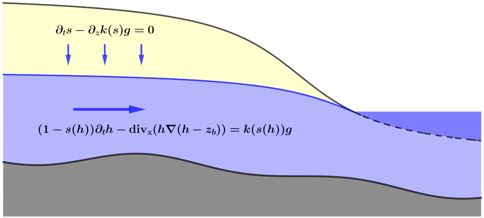


Open questions

Well-posedness

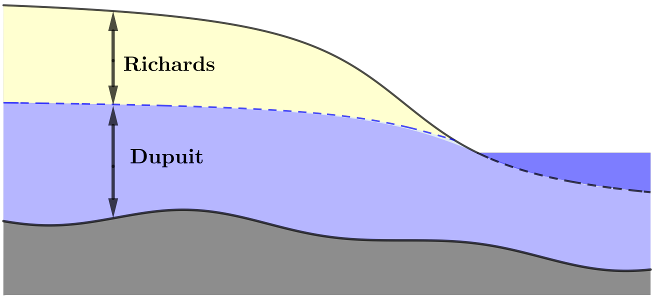
- No time regularity for $h(x, t)$.
- Front collisions?

Efficient numerical scheme

- Moving domain
- Front collision

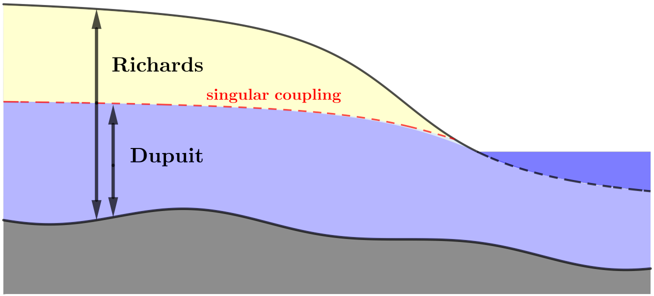
Fixed domain Dupuit 2D + Richards' 1D model

Moving domain model



Fixed domain Dupuit 2D + Richards' 1D model

Fixed domain model



Fixed domain Dupuit 2D + Richards' 1D model

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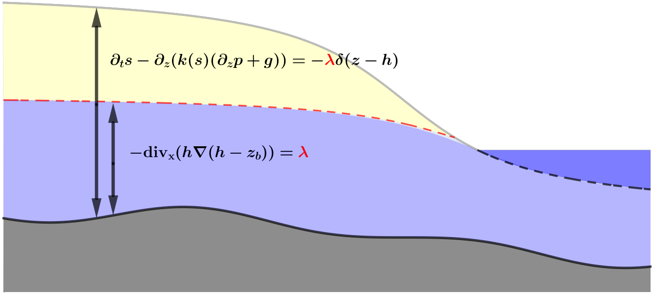


Pros:

- Fixed mesh for Richards'
- Pressurized fronts

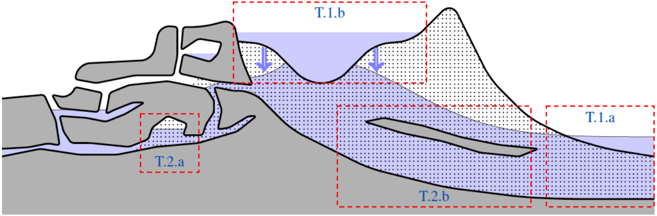
Cons:

- Singular coupling term

GeoFun ANR project

Collaborations with M. Parisot & M. Carreau as well as

- N. Aguillon, E. Audusse, R. Masson



Conclusions on Richards' equation

Reach nonlinear model and exiting research subject

Many open questions

Practically relevant challenges

Porosity

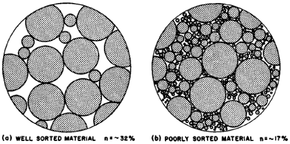
Table 2.5.1

Typical Porosity Values of Natural Sedimentary Materials[a]

Sedimentary Material	Porosity Value (percent)	Sedimentary Material	Porosity Value (percent)
Peat soil	60–80	Fine-to-medium mixed sand	30–35
Soils	50–60	Gravel	30–40
Clay	45–55	Gravel and sand	30–35
Silt	40–50	Sandstone	10–20
Medium-to-coarse mixed sand	35–40	Shale	1–10
Uniform sand	30–40	Limestone	1–10

[a] After Todd 1959.

Typical porosity values[1]



Effect of sorting on porosity[1]

Go back

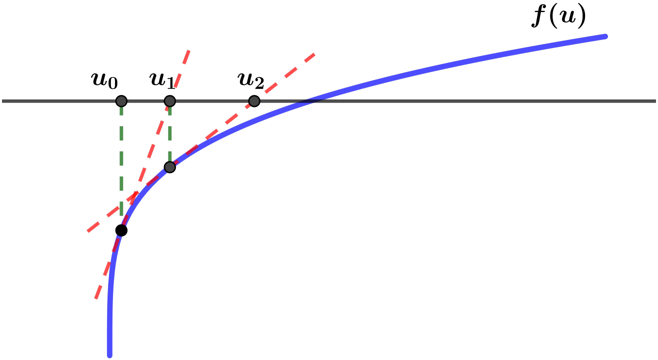
[1] Jacob Bear, Dynamics of Fluids in Porous Media, 1972

Newton's method for scalar concave problem

Newton's method for

$$f(u) = 0, \qquad u \in \mathbb{R}$$

- f concave and increasing



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