Robust Numerical Solution of Richards' Equation Under Dry Conditions

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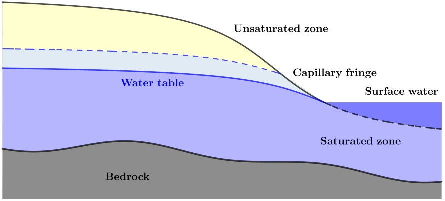
CMWR 2022

Gdańsk, June 20, 2022

Introduction

Unsaturated/saturated groundwater flow

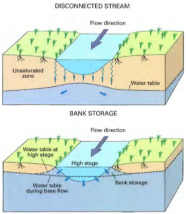
$$\phi \partial_t s - \operatorname{div}\left(\frac{\mathbb{K}k(s)}{\mu}(\nabla p - \rho \mathbf{g})\right) = 0, \qquad s = S(p)$$



Michel et al. '19

Applications:

- Water resource estimation
- Irrigation
- Contaminant transport
- Interaction with surface water
- ...



Reilly '01

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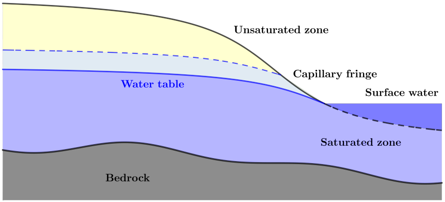

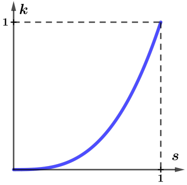

Relative permeability k as a function of saturation

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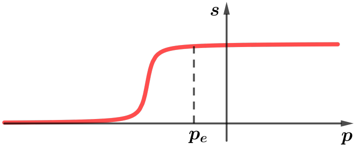

Retention curve $s = S(p)$

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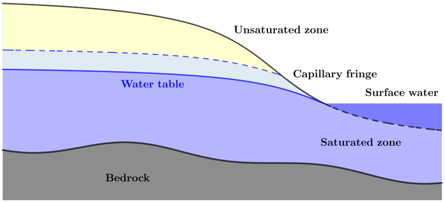
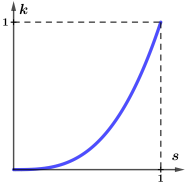
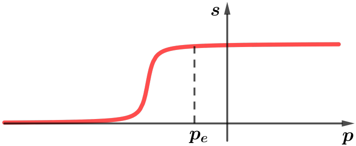




Relative permeability k as a function of saturation

Substantially nonlinear PDE

- Multiple "regimes"
- Numerically challenging: dry-wet transition; heterogeneous soils; **data-dependent performance**



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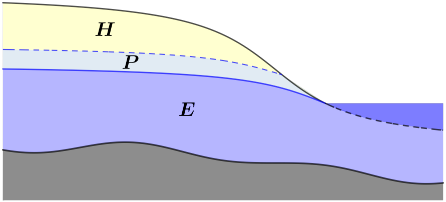

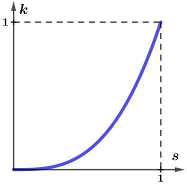

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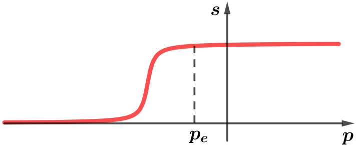

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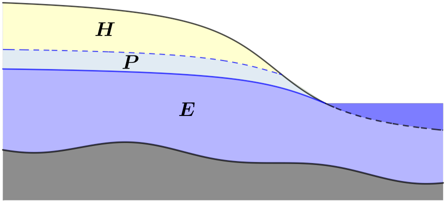

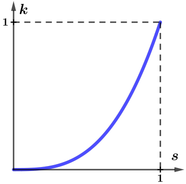

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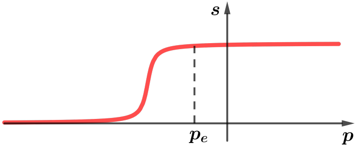

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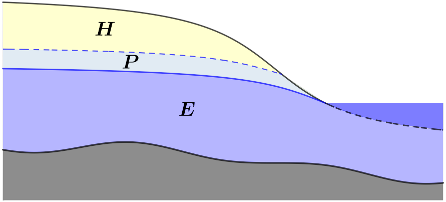

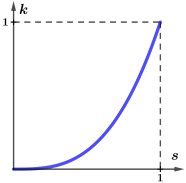

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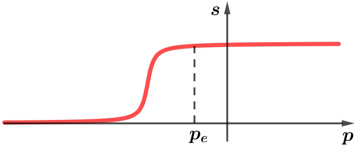

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Nonlinear convergence

PDE	
↓	mesh, discretization
Nonlinear algebraic system	
↓	linearization scheme
Linear system	
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Factors affecting the nonlinear convergence

- PDE formulation
- Mesh and discretization
- Linearization scheme: Newton, Picard, L-schemes, Quasi-Newton

  Globalization methods: backtracking, trust region, ...
- Nonlinear Preconditioning:

  Point-wise: variable substitution

  Domain Decomposition: Schwarz, Schur, Substructuring

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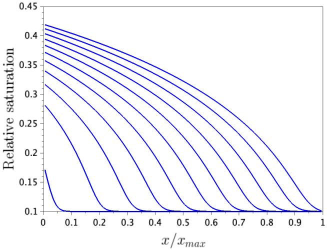
Test case: 1D water injection into a dry soil

Saturation profiles



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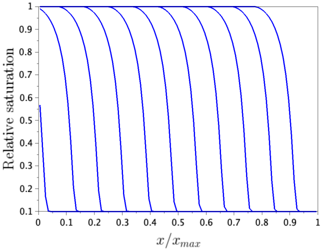

Pressurized flow

- Soil properties: Loam with Van Genuchten *kr*-*Pc* relation
- Input: Dry initial conditions $s_0 \in [10^{-6}, \mathbf{10^{-1}}]$
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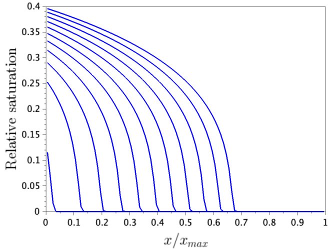

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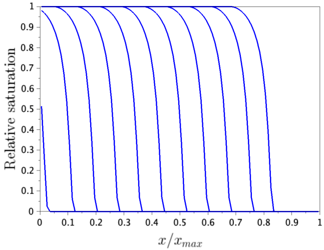

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Primary variable selection and (conservative) formulations

Original problem

$$\partial_t s - \operatorname{div}\left(k(s)\nabla p\right) = 0, \qquad s = S(p)$$

Major options: eliminate, reformulate.

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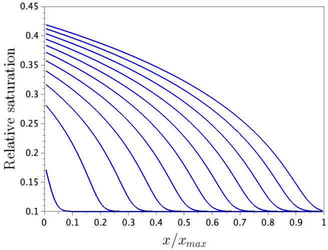

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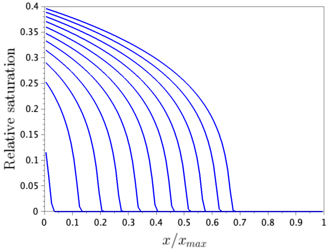

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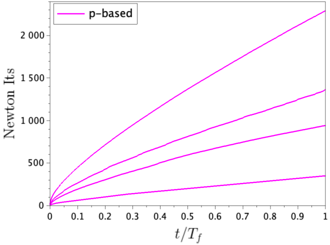

$s_0 \in \{0.1, 0.02, 0.01, 0.001\}$

- Space disc.: $Nx = 100$
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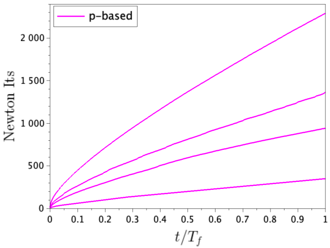

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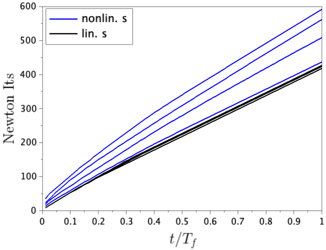

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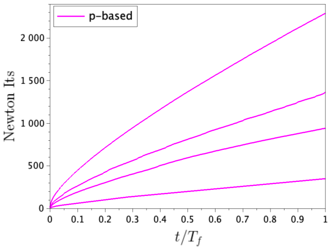

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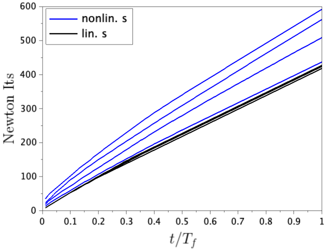

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- $p-$formulation: convergence is local, not robust
- linear $s-$formulation: fully robust, seemingly unconditional convergence
- nonlin. $s-$formulation: in between

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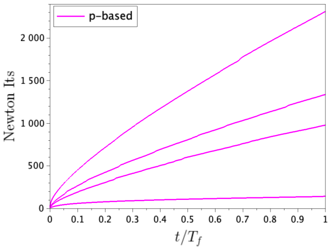

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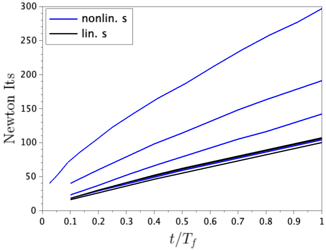

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Primary variable selection and (conservative) formulations

Nonlinear p-formulation

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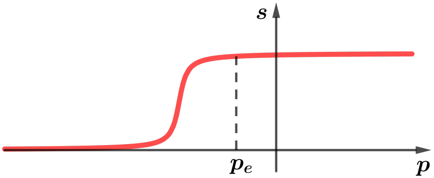
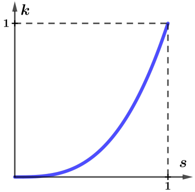
Bibliography: *Haverkamp et al. '77, Vauclin '79, Van Genuchten '82, Hills et al. '89, Celia et al. '90, Kirkland, Hills, & Wierenga '92*

What is so terribly wrong with the nonlinear p−formulation?





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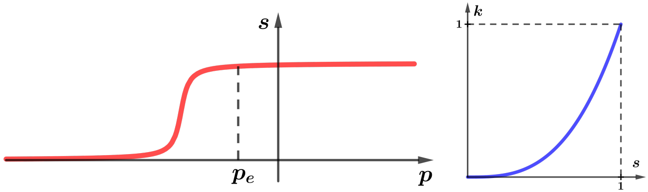

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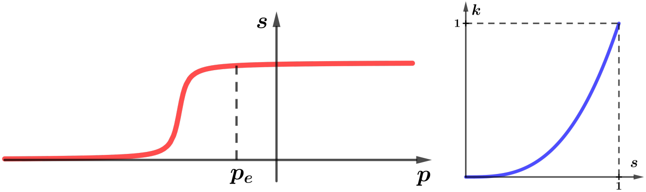

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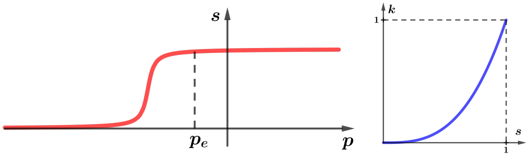
For small s gives $0 \approx 0$.

Nonlinear vs linear s−formulation



Upwind discretization of $\mathbf{k(s)P'(s)\nabla s}$

$$F_{KL} \propto k(s_{KL})P'(s_{KL})(s_K - s_L)$$

Compensation: small $\times$ large $=$ bounded

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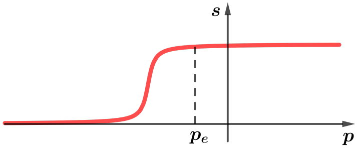

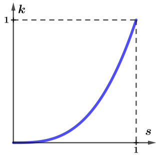

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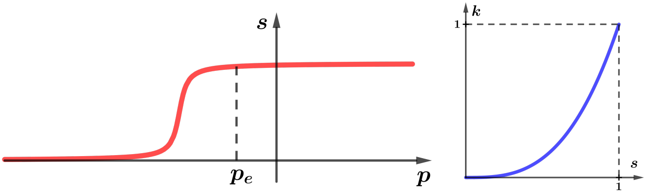
Upwind discretization of $\mathbf{k(s)\nabla P(s)}$

$$F_{KL} \propto k(s_{KL})(p_K - p_L) = k(s_{KL})P'(s^{\star}_{KL})(s_K - s_L)$$

(!) No compensation

Nonlinear vs linear $s-$formulation



Other issues of the nonlinear s-formulation

$$\partial_t s - \mathrm{div}\,(k(s)\nabla P(s)) = 0$$

- Floating point overflows $\Rightarrow$ need for regularization
- Does not combines well with nonlinear preconditioning

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Objective: get best of the available formulations

- First step: nonlin. p + nonlin. s
- Second step: nonlin. p + lin. s

Variable switching

Original problem

$$\partial_t s - \operatorname{div}(k(s)\nabla p) = 0, \qquad s = S(p)$$

Variable substitution

$$\tau \mapsto \overline{s}(\tau), \overline{p}(\tau) \qquad \text{s.t.} \qquad \overline{s}(\tau) = S(\overline{p}(\tau)) \text{ for all } \tau$$

Variable switching

$$\tau \propto \begin{cases} \overline{s}(\tau) & \tau \leq \tau_{sw} \\ \overline{p}(\tau) & \tau > \tau_{sw} \end{cases}$$

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$$\tau \propto \begin{cases} \overline{s}(\tau) & \tau \leq \tau_{sw} \\ \overline{p}(\tau) & \tau > \tau_{sw} \end{cases}$$

Reformulated problem (nonlinear)

$$\partial_t \overline{s}(\tau) - \operatorname{div}\left(k(\overline{s}(\tau))\nabla \overline{p}(\tau)\right) = 0$$

Variable switching

Original problem

$$\partial_t s - \operatorname{div}(k(s)\nabla p) = 0, \qquad s = S(p)$$

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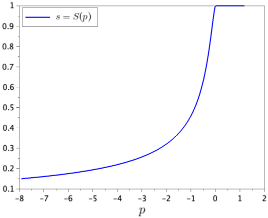

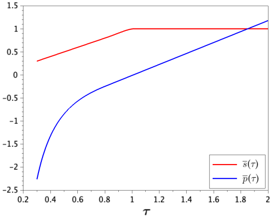

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	Variably sat.	Heterogeneities	Performance
Nonlin. p	Okay	Okay	Poor
Nonlin. s	Unsat	Okay	Okayish
Nonlin. τ	**Okay**	**Okay**	**Okayish**

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Bibliography:

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Reformulated problem (partly linearized)

$$\partial_t \overline{s}(\tau) - \operatorname{div}\left(k(\overline{s}(\tau))\overline{p}'(\tau)\nabla\tau\right) = 0$$

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Nonlin. p	Okay	Okay	Poor
Lin. s	Unsat	Not trivial	Good
Lin. τ	**Okay**	**???**	**Good**

Bibliography:

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Nonlin. p	Okay	Okay	Poor
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Bibliography:

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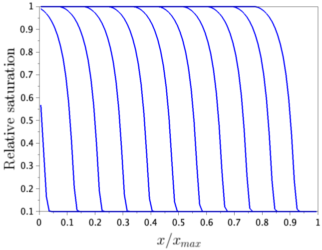
Test case: 1D water injection into a dry soil

Saturation profiles



Pressurized flow

- Soil properties: Loam with Van Genuchten *kr*-*Pc* relation
- Input: Dry initial conditions $s_0 \in [10^{-6}, \mathbf{10^{-1}}]$
- Output: Newton iterations over time

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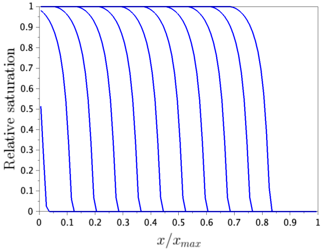

Pressurized flow

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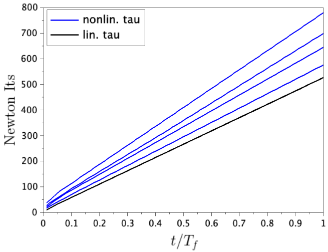
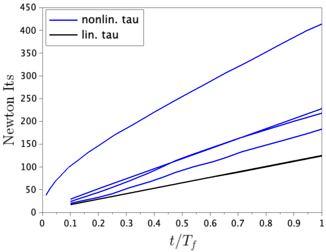
- Output: Newton iterations over time

Performance: nonlinear vs. linear variable switching formulation



$\Delta t_{max} = T_f/\mathbf{100}$



$\Delta t_{max} = T_f/\mathbf{10}$

- Initial condition: $s_0 \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-6}\}$
- Space disc.: $Nx = 100$

Outline

Introduction

Primary variables and formulations

Preconditioned Newton method

Conclusion

Preconditioned Newton method

Illustration over a toy problem and a couple of toy methods

Find $\boldsymbol{u} \in \mathbb{R}^N$ s.t.

$$f(\boldsymbol{u}) + A\boldsymbol{u} = \boldsymbol{b}, \qquad \text{with} \qquad f(\boldsymbol{u})_i = f_i(u_i)$$

Splitting $A = P - Q$:

$$f(\boldsymbol{u}) + P\boldsymbol{u} = Q\boldsymbol{u} + \boldsymbol{b}$$

Preconditioned Newton method

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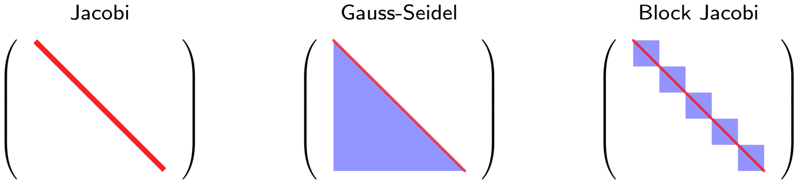

Preconditioned Newton method

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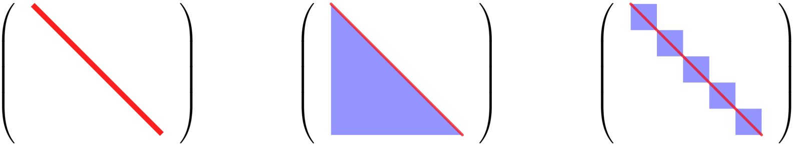
Jacobi — Gauss-Seidel — Block Jacobi



Preconditioned Newton method

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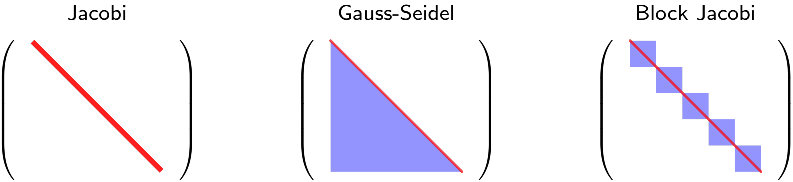

Stationary iterations

$$\boldsymbol{u}_{k+1} = M^{-1}\left(N(\boldsymbol{u}_k)\right)$$

Newton's method applied to

$$\boldsymbol{u} - M^{-1}\left(N(\boldsymbol{u})\right) = 0$$

Preconditioned Newton method

Illustration over a toy problem and a couple of toy methods

Find $\boldsymbol{u} \in \mathbb{R}^N$ s.t.

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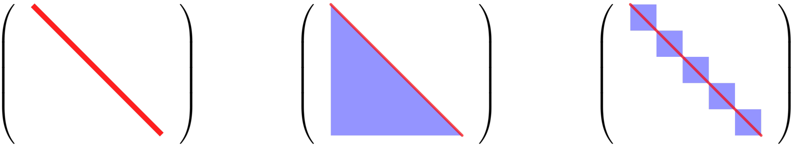
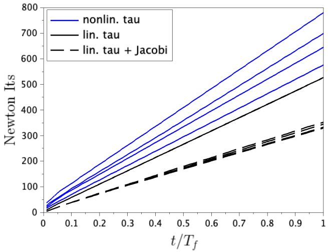
Jacobi   Gauss-Seidel   Block Jacobi



Extensions

- General systems $F(\boldsymbol{u}) = 0$
- General DD methods (e.g. ASPIN, RASPEN)

Performance: Jacobi-Newton method



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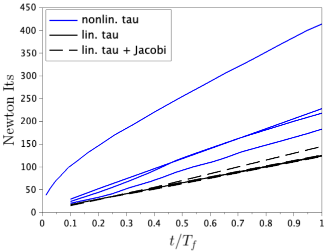

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- Initial condition: $s_0 \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-6}\}$
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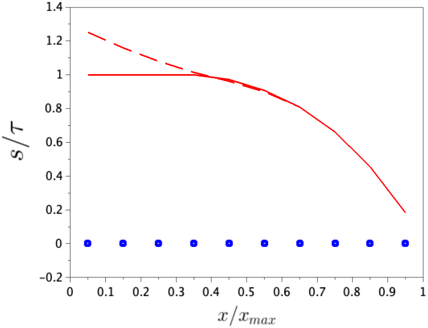
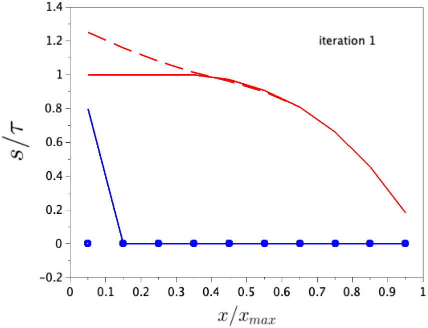
Jacobi-Newton iterates



#unk = 10

Jacobi-Newton iterates



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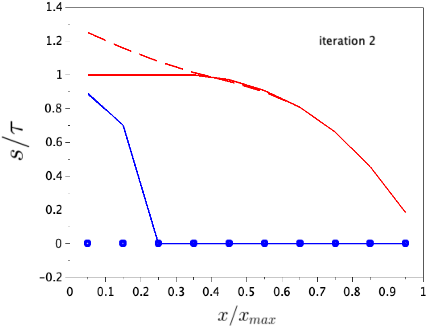
Jacobi-Newton iterates



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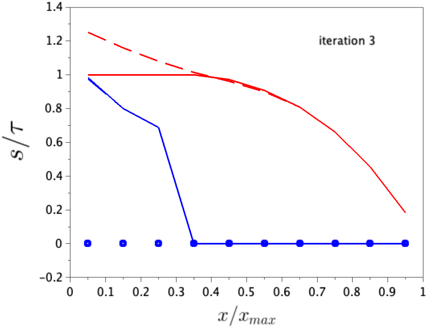

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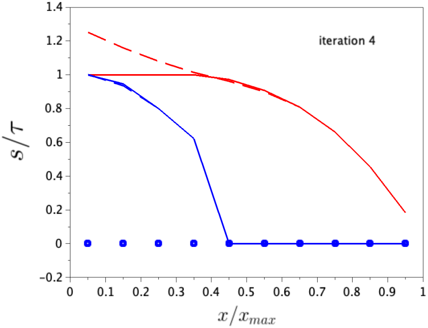

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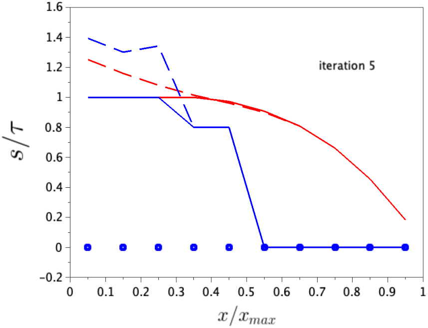

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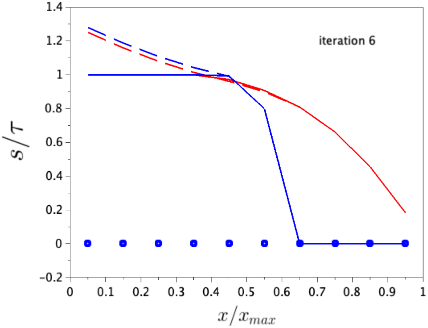

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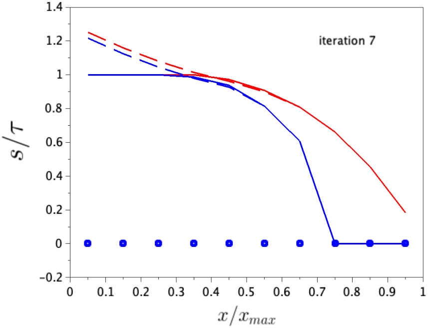

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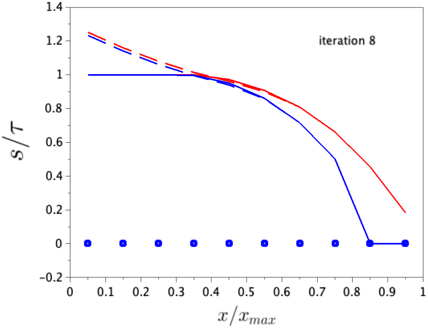

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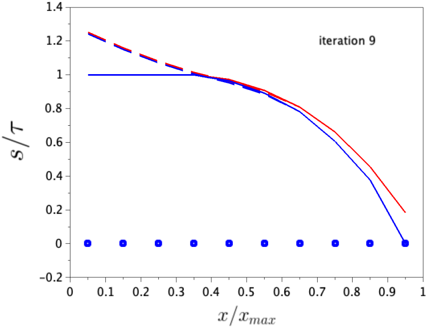
Jacobi-Newton iterates



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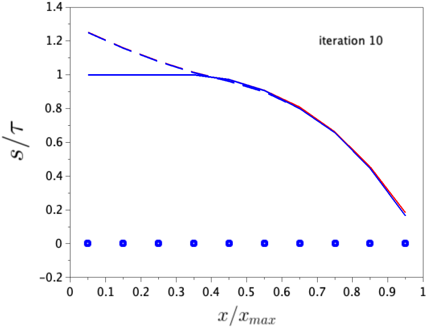
Jacobi-Newton iterates



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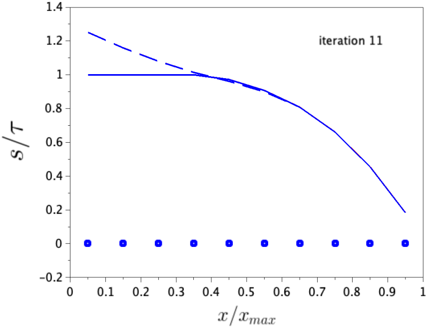
Jacobi-Newton iterates



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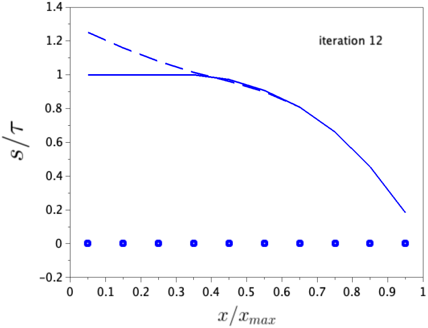
Jacobi-Newton iterates



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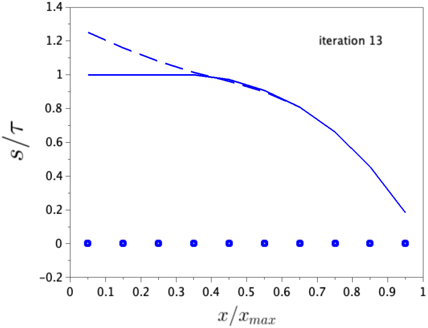

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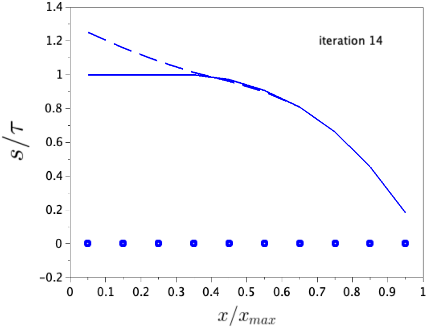
Jacobi-Newton iterates



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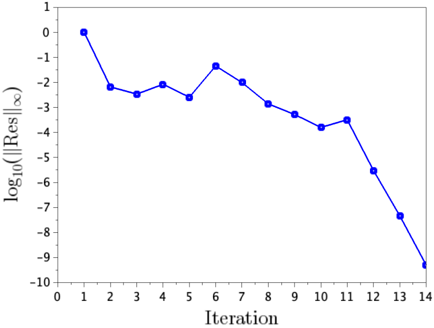

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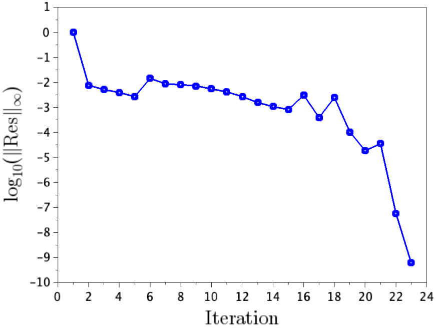

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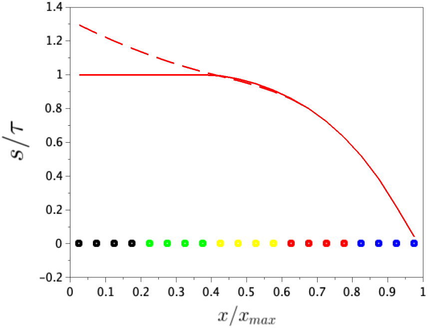
Block Jacobi-Newton iterates



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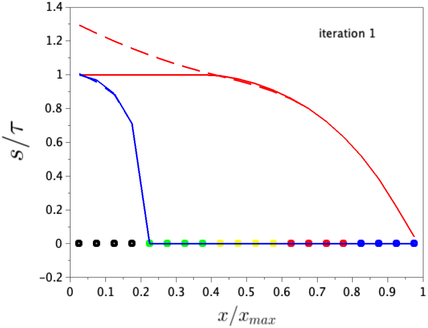
Block Jacobi-Newton's iterates



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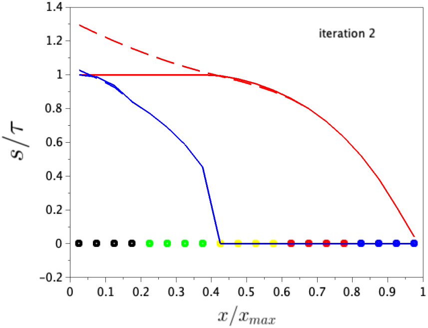

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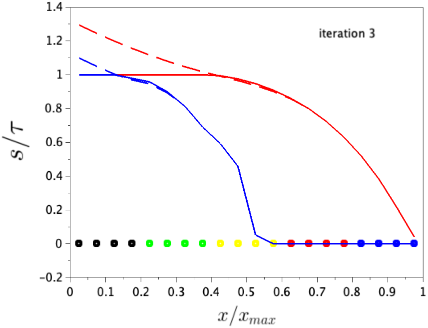

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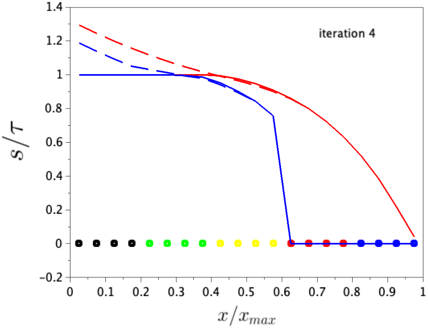

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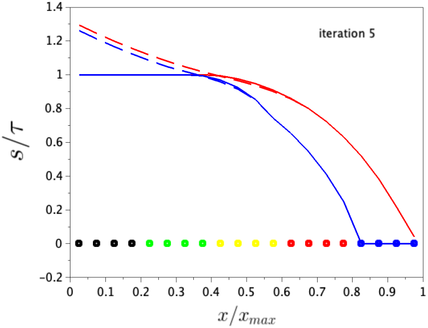

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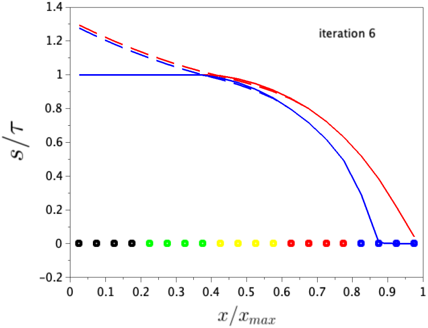

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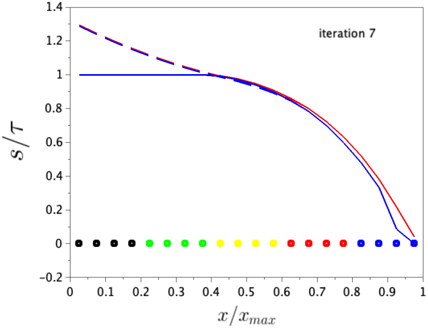

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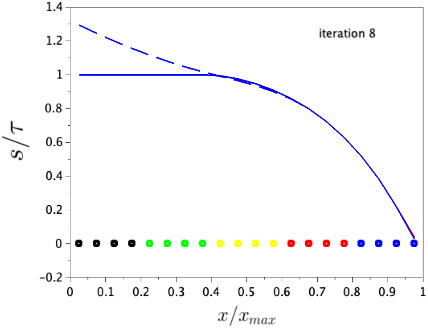

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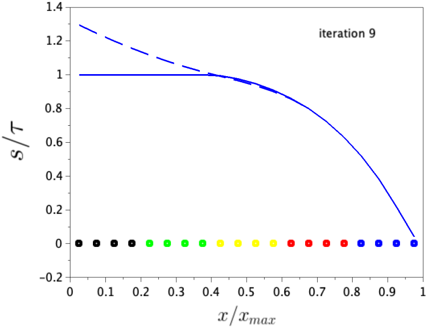

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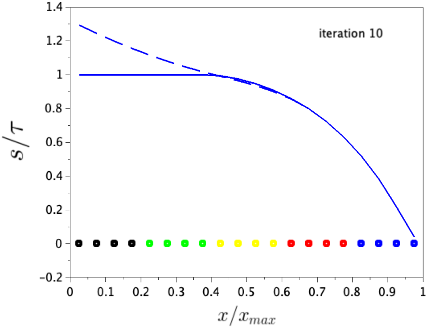

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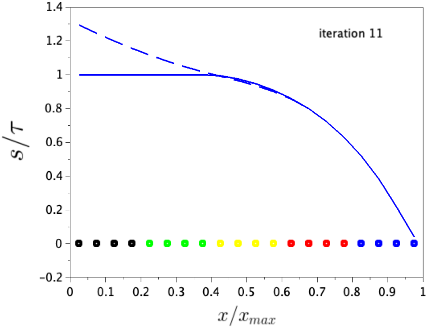

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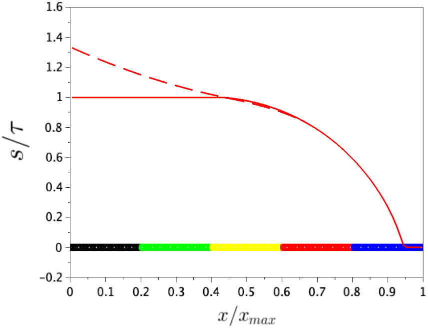

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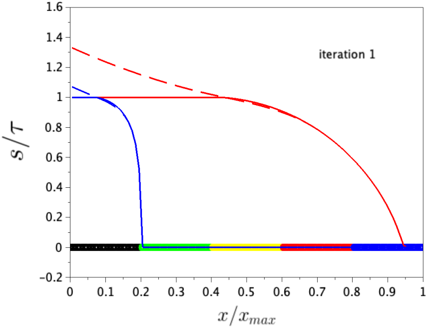

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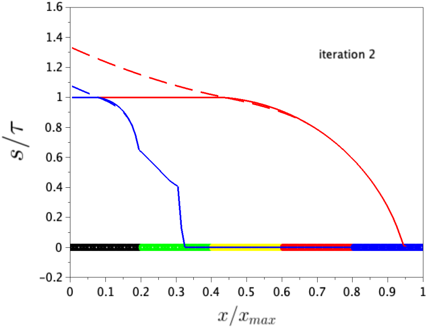

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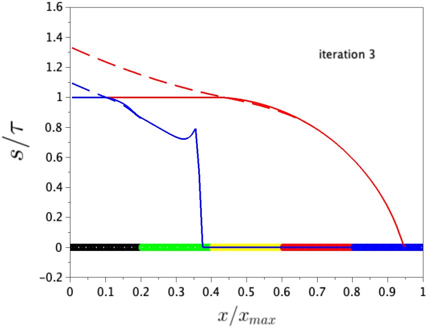

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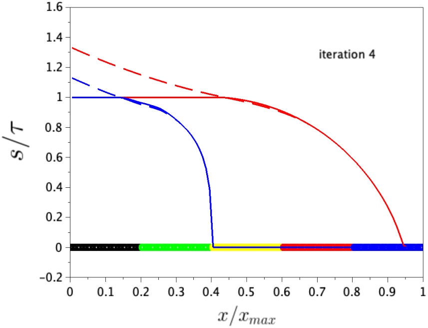
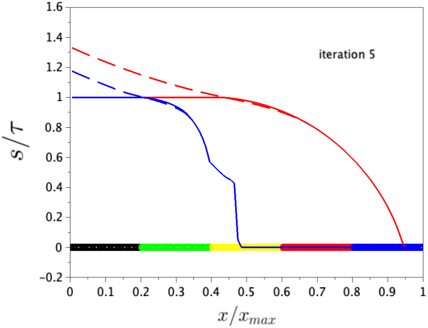


#unk = 100

Block Jacobi-Newton's iterates



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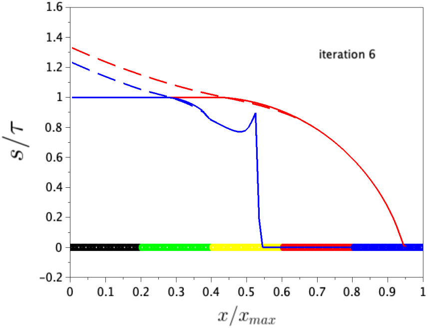
Block Jacobi-Newton's iterates



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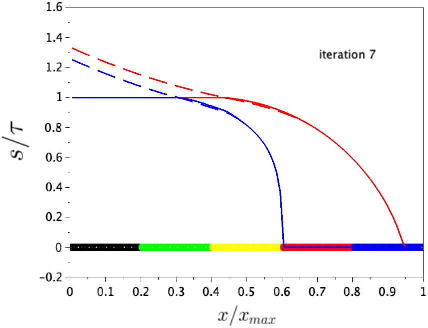
Block Jacobi-Newton's iterates



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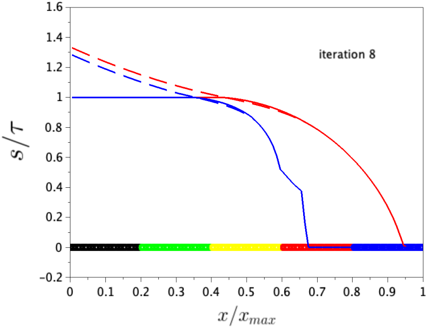
Block Jacobi-Newton's iterates



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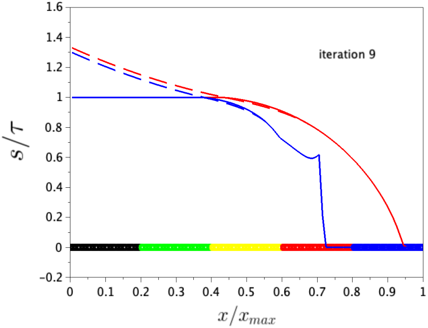
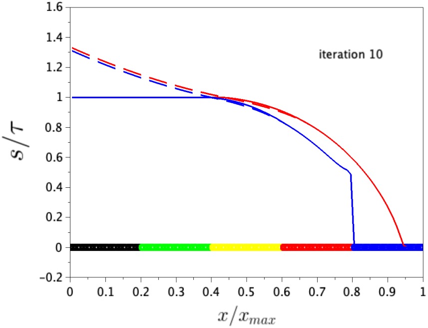
Block Jacobi-Newton's iterates



$\#\text{unk} = 100$

Block Jacobi-Newton's iterates



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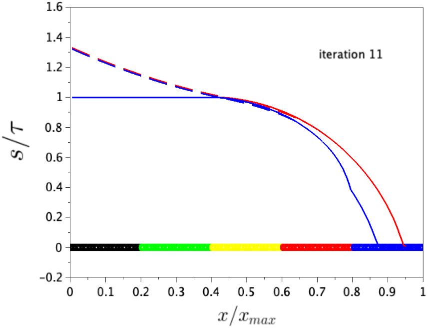

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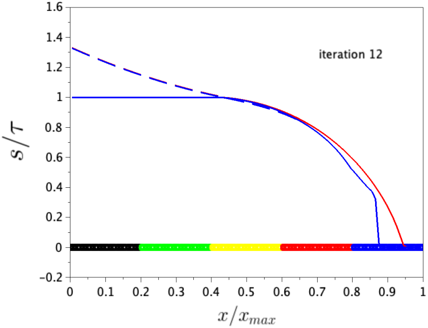

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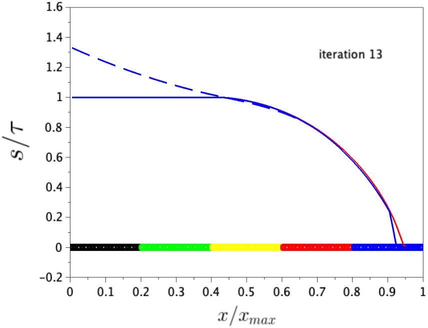

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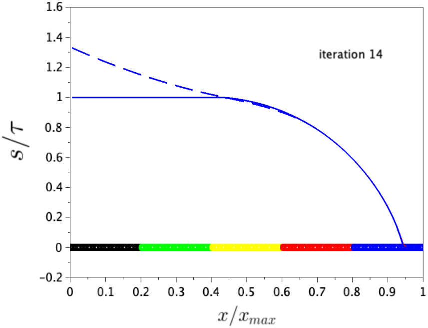
Block Jacobi-Newton's iterates



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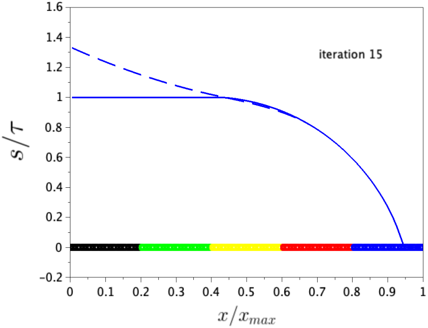
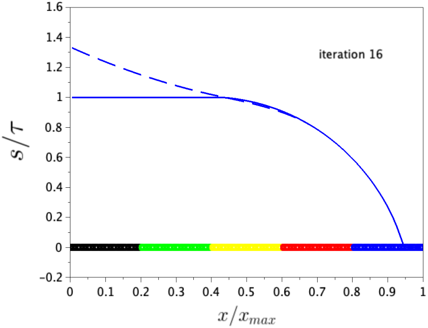
Block Jacobi-Newton's iterates



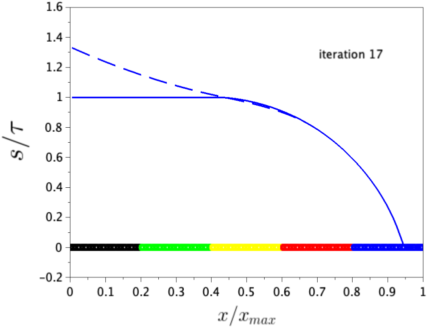
#unk = 100

Block Jacobi-Newton's iterates



#unk = 100

Block Jacobi-Newton's iterates



#unk = 100

Outline

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Ways of improving robustness

- Heuristic variable substitution and reformulation
- Nonlinear preconditioning
- Large time steps

Analysis

- Global monotone convergence using
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Perspectives & ongoing work

- Heterogeneous problems
- Nonlinear two-level RAS methods, front tracking
- Connections to space-time DD?

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