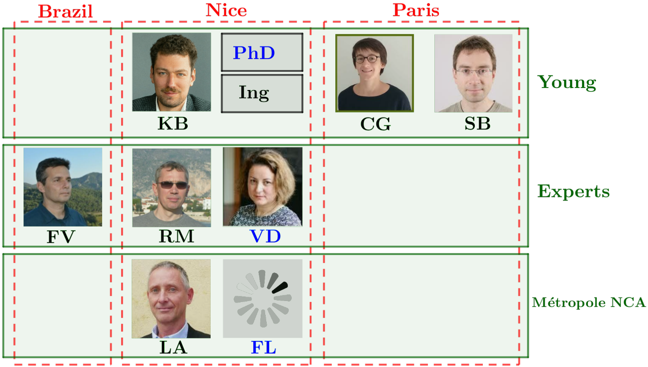
ANR JC project **Top-up**: High-resolution **top**ography **up**scaling for urban flood modeling



CG: Cindy Guichard, Sorbone + Inria

SB: Sébastien Boyaval, LHSV, Ponts PartisTech

FV: Frédéric Valentin, LNCC, Brazil

LA: Ludovic Andres, PAST UCA, Métropole NCA

FL: Florent Largeron, Métropole NCA

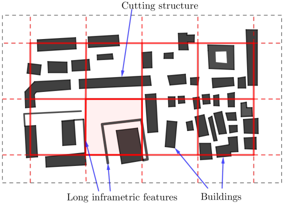
Flow domain: scale $L = 1 - 10km$

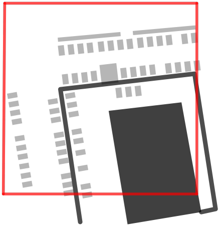
georisques.gouv.fr

Zone inondable par débordement de de cours d'eau

Small structures:

- Buildings, walls, cars
- $l = 0.1 - 100m$

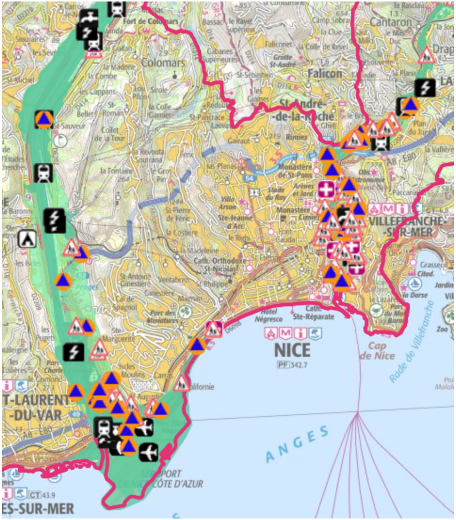




Key difficulty: scale contrast $l \ll L$

Motivation

Flow domain: scale $L = 1 - 10km$

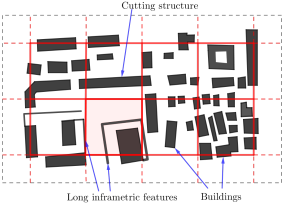


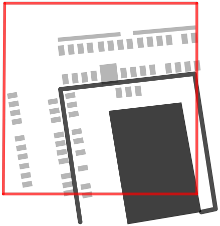
georisques.gouv.fr

Zone inondable + etablissements sensibles

Small structures:

- Buildings, walls, cars
- $l = 0.1 - 100m$





Key difficulty: scale contrast $l \ll L$

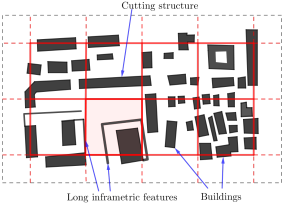
Flow domain: scale $L = 1 - 10km$

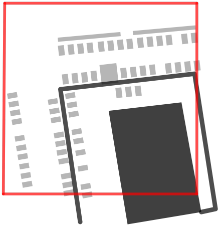
georisques.gouv.fr

Zone inondable + etablissements sensibles

Small structures:

- Buildings, walls, cars
- $l = 0.1 - 100m$





Key difficulty: scale contrast $l \ll L$

Motivation

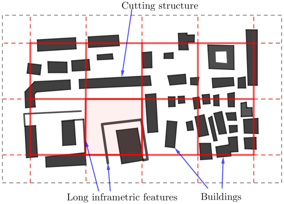
Flow domain: scale $L = 1 - 10km$

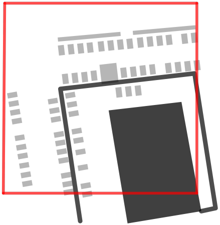
georisques.gouv.fr

Zone inondable + etablissements sensibles

Small structures:

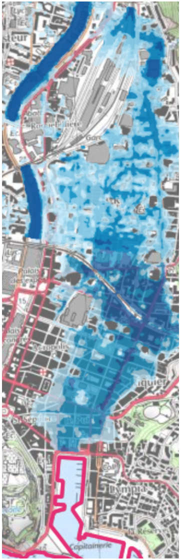
- Buildings, walls, cars
- $l = 0.1 - 100m$





Key difficulty: scale contrast $l \ll L$

Motivation



georisques.gouv.fr

Google Earth

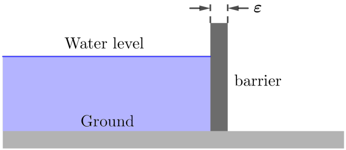
Motivation

Google Earth

Motivation

Google Earth

Google Earth



No solution continuity w.r.t. parameters

Objectives

In conclusion:

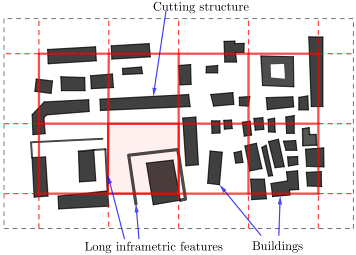
- Small scale structural features has to be accounted for.
- High resolution (infra-metric) topographical data is available.

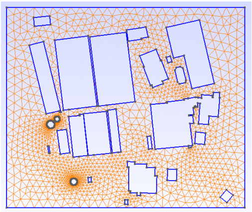
Top-up's question:

- How to integrate small scale structures into the large scale simulations?

Challenge:

- Small cells over large domain = very large system.

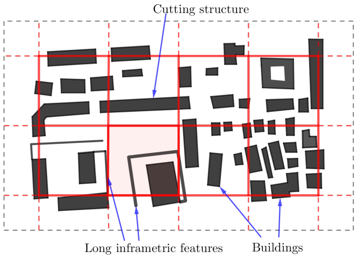


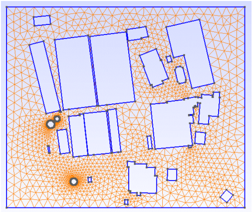


Multi-scale numerical methods

- "upscale" relevant local fine-scale information to the coarse grid
- solve global coarse problem
- goal: parametrize the fine scale solution with few dof
- opportunities: parallelism, offline/online workload distribution

Strategies





Domain Decomposition solve the global fine-scale problem, but preconditioned by

- local fine-scale contributions
- global coarse problem
- goal: weak scalability and coefficient robustness

$$\text{wall-clock time} \leqslant f\left(\frac{H}{h}\right)$$

- opportunities: parallelism

PDE model

- Diffusive Wave equation

DD and Ms for linear problem

- Unified approach to DD and Ms methods

DD and Ms for nonlinear problem

- Few words on NK-DD, DD-NK and POD-DEIM

PDE model

- Diffusive Wave equation

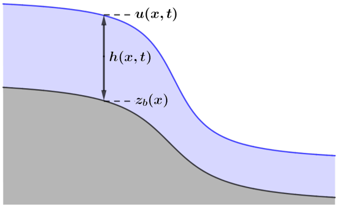
DD and Ms for linear problem

- Unified approach to DD and Ms methods

DD and Ms for nonlinear problem

- Few words on NK-DD, DD-NK and POD-DEIM

What is Diffusive Wave equation?



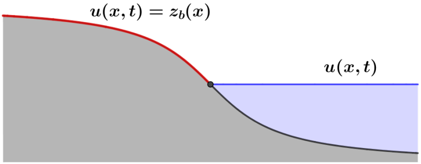
Zero-inertia approximation of Shallow Water equations

$$\partial_t u - \operatorname{div}\left(\kappa \, \frac{(u - z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u\right) = 0$$

- $u(\mathbf{x}, t)$ the unknown free surface elevation;
- $z_b(\mathbf{x})$ the topographical elevation.

Parameters κ, α, γ are related to the empirical bottom friction laws

- $\kappa(\mathbf{x})$ is a "ground roughness" parameter;
- $\alpha > 1$ and $0 < \gamma \leqslant 1$ depend of the flow regimes and the head loss formula.



DWE is a nonhomogeneous doubly nonlinear degenerate parabolic equation

$$\partial_t u - \operatorname{div}\left(\kappa \, \frac{(u - z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u\right) = 0$$

Degenerate diffusion:

- dry regions;
- wetting fronts propagate at finite speed.

$p-$Laplacian like term:

- simplifies in an unlikely laminar regime: $\gamma = 1$;
- diffusion coefficient go to ∞ in lake-at-rest state, can be fixed by laminar regime correction.

DW equation

$$\partial_t u - \mathrm{div}\Big(\kappa\ \frac{(u - z_b)^\alpha}{\|\nabla u\|^{1-\gamma}}\nabla u\Big) = 0$$

Case $z_b = 0$: DWE can be transformed into a $p-$Laplace-porous media equation

$$\partial_t u - \mathrm{div}\left(\tilde{\kappa}\frac{\nabla u^m}{\|\nabla u^m\|^{1-\gamma}}\right) = 0, \qquad m = 1 + \frac{\alpha}{\gamma}.$$

DW equation

$$\partial_t u - \operatorname{div}\Big(\kappa \; \frac{(u-z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u\Big) = 0$$

Case $z_b = 0$: DWE can be transformed into a $p-$Laplace-porous media equation

$$\partial_t u - \operatorname{div}\left(\tilde{\kappa} \frac{\nabla u^m}{\|\nabla u^m\|^{1-\gamma}}\right) = 0, \qquad m = 1 + \frac{\alpha}{\gamma}.$$

Case $\gamma = 1$: Using $h = u - z_b$ we obtain a nonlinear convection-diffusion

$$\partial_t h - \operatorname{div}\Big(\kappa \; h^\alpha \nabla\left(h + z_b\right)\Big) = 0.$$

DW equation

$$\partial_t u - \operatorname{div}\Big(\kappa \frac{(u-z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u\Big) = 0$$

Case $z_b = 0$: DWE can be transformed into a $p-$Laplace-porous media equation

$$\partial_t u - \operatorname{div}\left(\tilde{\kappa} \frac{\nabla u^m}{\|\nabla u^m\|^{1-\gamma}}\right) = 0, \qquad m = 1 + \frac{\alpha}{\gamma}.$$

Case $\gamma = 1$: Using $h = u - z_b$ we obtain a nonlinear convection-diffusion

$$\partial_t h - \operatorname{div}\Big(\kappa\, h^\alpha \nabla (h + z_b)\Big) = 0.$$

- Both simplified equations: existence of weak solution is classical.
- DWE: No general existence result.

Volume conservation

$$\partial_t h + \mathrm{div} h\mathbf{v} = 0$$

Momentum balance with bottom friction

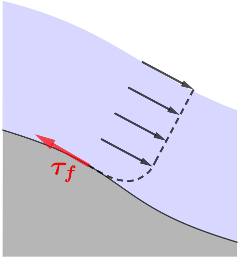
$$\underbrace{\partial_t(h\mathbf{v}) + \mathrm{div}\,(h\mathbf{v}\otimes\mathbf{v})}_{\text{inertia}} + \underbrace{gh\nabla\,(h+z_b)}_{\text{pressure and gravity forces}} = \underbrace{-C_f|\mathbf{v}|\mathbf{v}}_{\text{bottom friction}}$$

To recover DWE

- neglect inertia terms

$$gh\nabla\,(h+z_b) = -C_f|\mathbf{v}|\mathbf{v};$$

- substitute $\mathbf{v}$ into the volume conservation equation.



Volume conservation

$$\partial_t h + \mathrm{div} h\mathbf{v} = 0$$

Momentum balance with bottom friction

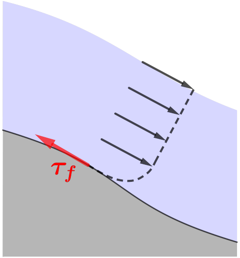
$$\underbrace{\partial_t(h\mathbf{v}) + \mathrm{div}\,(h\mathbf{v}\otimes\mathbf{v})}_{\text{inertia}} + \underbrace{gh\nabla\,(h+z_b)}_{\text{pressure and gravity forces}} = \underbrace{-C_f|\mathbf{v}|\mathbf{v}}_{\text{bottom friction}}$$

To recover DWE

- neglect inertia terms

$$gh\nabla\,(h+z_b) = -C_f|\mathbf{v}|\mathbf{v};$$

- substitute $\mathbf{v}$ into the volume conservation equation.



Friction law

- C_f may depend both on $\mathbf{v}$ and h
- Qualitative difference between laminar and turbulent regime

Friction law: laminar regime

Approximate momentum balance

$$gh\nabla (h + z_b) = -C_f |\mathbf{v}|\mathbf{v};$$

Laminar flow: Analytical formula for C_f *à la* Poiseuille

$$C_f = \frac{C_l}{Re} = C_l \frac{\mu}{\rho|\mathbf{v}|h} \qquad \Rightarrow \qquad C_f |\mathbf{v}|\mathbf{v} = C_l \frac{\mu}{\rho} \frac{\mathbf{v}}{h}$$

Friction law: laminar regime

Approximate momentum balance

$$gh\nabla(h+z_b) = -C_f|\mathbf{v}|\mathbf{v};$$

Laminar flow: Analytical formula for C_f *à la* Poiseuille

$$C_f = \frac{C_l}{Re} = C_l\frac{\mu}{\rho|\mathbf{v}|h} \qquad \Rightarrow \qquad C_f|\mathbf{v}|\mathbf{v} = C_l\frac{\mu}{\rho}\frac{\mathbf{v}}{h}$$

Darcy-like relation

$$h\nabla(h+z_b) = -\frac{1}{K}\frac{\mathbf{v}}{h}, \qquad K = \frac{1}{C_l}\frac{\rho g}{\mu}$$

giving

$$\partial_t h - \operatorname{div}\left(Kh^3\nabla(h+z_b)\right) = 0.$$

Friction law: turbulent regime

Turbulent flow:

- Chézy's law: C_f independent of h and $\mathbf{v}$

Turbulent flow:

- Chézy's law: C_f independent of h and $\mathbf{v}$
- Aldsul's formula:

$$C_f = 0.11 \left(\frac{\varepsilon}{h} + \frac{68}{Re} \right)^{1/4} = \frac{1}{h^{1/4}} \left(a + \frac{b}{|\mathbf{v}|} \right)^{1/4}$$

with ε being the ground roughness.

Turbulent flow:

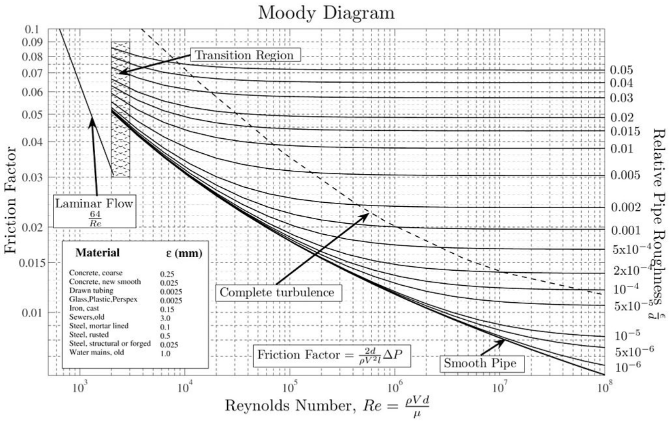
- Chézy's law: C_f independent of h and $\mathbf{v}$
- Aldsul's formula:

$$C_f = 0.11 \left(\frac{\varepsilon}{h} + \frac{68}{Re} \right)^{1/4} = \frac{1}{h^{1/4}} \left(a + \frac{b}{|\mathbf{v}|} \right)^{1/4}$$

  with ε being the ground roughness.
- Colebrook's equation

$$\frac{1}{\sqrt{C_f}} = -2 \log_{10} \left(\frac{\varepsilon/h}{3.7} + \frac{2.51}{Re\sqrt{C_f}} \right)$$

Friction law: Moody chart



DWE using Chézy's law

Approximate momentum balance

$$gh\nabla(h + z_b) = -C_f|\mathbf{v}|\mathbf{v};$$

Assuming C_f independent of h and $\mathbf{v}$

$$\mathbf{v} = -\kappa\, h^{1/2} \frac{\nabla(h + z_b)}{\|\nabla(h + z_b)\|^{1/2}} \qquad \kappa = \sqrt{\frac{g}{C_f}}$$

giving

$$\partial_t h - \mathrm{div}\left(\kappa\, h^{3/2} \frac{\nabla(h + z_b)}{\|\nabla(h + z_b)\|^{1/2}}\right) = 0.$$

Approximate momentum balance

$$gh\nabla(h+z_b) = -C_f|\mathbf{v}|\mathbf{v};$$

Assuming C_f independent of h and $\mathbf{v}$

$$\mathbf{v} = -\kappa\, h^{1/2}\frac{\nabla(h+z_b)}{\|\nabla(h+z_b)\|^{1/2}} \qquad \kappa = \sqrt{\frac{g}{C_f}}$$

giving

$$\partial_t h - \mathrm{div}\left(\kappa\, h^{3/2}\frac{\nabla(h+z_b)}{\|\nabla(h+z_b)\|^{1/2}}\right) = 0.$$

Remarks:

- Exponents may be slightly different.
- Singularity in "diffusion coefficient" is not physical, and should be regularized.

Applicability & justification of DWE

Applicability

- Considered to be accurate for “smooth enough flows” (no tsunami, no dam breaking, no hydraulic jumps...).

Justification

- Formal justification is beyond my knowledge.
- Many applied publications in favor of DW.
- Used in engineering software: SWMM (drainage channels), HEC-RAS (flood modeling).

Why DWE in Top-up?

- DD and Ms friendly (parabolic).

We can go even further and simplify SWE

$$\underbrace{\partial_t(h\mathbf{v}) + \operatorname{div}(h\mathbf{v}\otimes\mathbf{v})}_{\text{inertia}} \quad + \quad \underbrace{gh\nabla(h+z_b)}_{\text{pressure and gravity forces}} \quad = \quad - \quad \underbrace{C_f|\mathbf{v}|\mathbf{v}}_{\text{bottom friction}}$$

as

$$gh\nabla z_b = -C_f|\mathbf{v}|\mathbf{v}$$

which gives a scalar hyperbolic volume conservation equation

$$\partial_t h + \operatorname{div}\left(h^{3/2}\boldsymbol{\xi}_b\right) = 0, \qquad \boldsymbol{\xi}_b = -\kappa\frac{\nabla z_b}{\|\nabla z_b\|^{1/2}}.$$

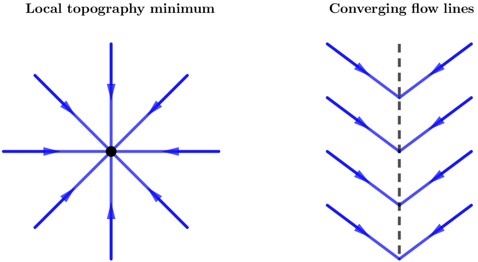
KW equation

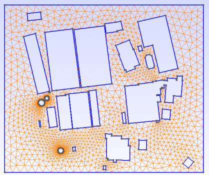
$$\partial_t h + \operatorname{div}\left(h^{3/2}\boldsymbol{\xi}_b\right) = 0, \qquad \boldsymbol{\xi}_b = -\kappa \frac{\nabla z_b}{\|\nabla z_b\|^{1/2}}.$$

can model quasi-uniform channel flows.

Drawbacks:

- Does not move mass if $\nabla z_b = 0$.
- Direct application to 2D is problematic: "depressive topographies", discontinuous converging slopes.





Geometry

- Ω whole flood zone;
- $\Omega_s \subset \overline{\Omega}$ domain occupied by structures.

DWE in a perforated domain

$$\begin{cases} \partial_t u - \operatorname{div}\Big(\kappa \dfrac{(u-z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u\Big) = 0 & = \quad f & \text{in} \quad \Omega\backslash\Omega_s, \\ -\kappa \dfrac{(u-z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u \cdot \mathbf{n} & = \quad 0 & \text{on} \quad \partial\Omega_s, \\ \text{Some BC} & & \text{on} \quad \partial\Omega\backslash\Omega_s \\ \text{Some initial condition} & & \end{cases}$$

PDE model

- Diffusive Wave equation

DD and Ms for linear problem

- Unified approach to DD and Ms methods

DD and Ms for nonlinear problem

- Few words on NK-DD, DD-NK and POD-DEIM

Linear elliptic equation in a perforated domain

$$bu - \mathrm{div}(k\nabla u) = f \qquad \text{in} \quad \Omega \backslash \Omega_s$$

with $b, k > 0$.

- Oversimplified and semi-discretized in time version of DWE

$$\partial_t u - \mathrm{div}\Big(\kappa \, \frac{(u - z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u\Big) = 0$$

with $\gamma = 1$, $\alpha = 0$ and $z_b = 0$.

Linear elliptic equation in a perforated domain

$$bu - \mathrm{div}(k\nabla u) = f \qquad \text{in} \quad \Omega\backslash\Omega_s$$

with $b, k > 0$.

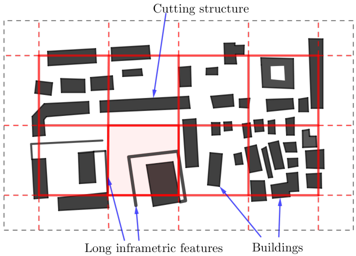
- Oversimplified and semi-discretized in time version of DWE

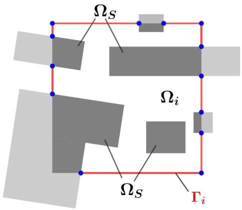
$$\partial_t u - \mathrm{div}\Big(\kappa \, \frac{(u - z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u\Big) = 0$$

  with $\gamma = 1$, $\alpha = 0$ and $z_b = 0$.
- Model equation can be approximated in the full domain Ω

$$b_\varepsilon u - \mathrm{div}(k_\varepsilon \nabla u) = f \qquad \text{in} \quad \Omega$$

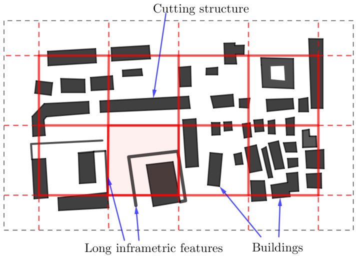
  with $k_\varepsilon|_{\Omega_s}, b_\varepsilon|_{\Omega_s}$ being very small.

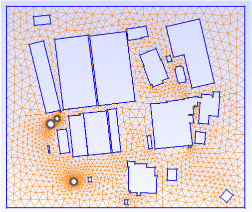




Coarse discretization

- Rectangular grid $\mathcal{T}_H = (\Omega_i)_{i=1,\ldots,N}$ not fitted to Ω_s.
- We denote $\Gamma_i = \partial\Omega_i \backslash \Omega_s$ and $\Gamma = \bigcup_{i=1,\ldots,N} \Gamma_i$.





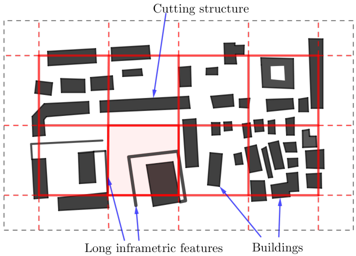
Coarse discretization

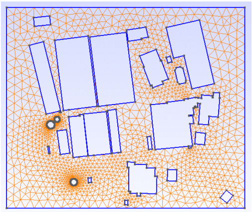
- Rectangular grid $\mathcal{T}_H = (\Omega_i)_{i=1,\ldots,N}$ not fitted to Ω_s.
- We denote $\Gamma_i = \partial\Omega_i \backslash \Omega_s$ and $\Gamma = \bigcup_{i=1,\ldots,N} \Gamma_i$.

Fine discretization

- Fine grid $\mathcal{T}_h$ fitted to Ω_s and $\mathcal{T}_H$.
- We denote by $V_h(\Omega\backslash\Omega_s)$ the associated $\mathbb{P}_1$ finite element space and for $\Upsilon \subset \Omega\backslash\Omega_s$ we set

$$V_h(\Upsilon) = \{v|_\Upsilon, v \in V_h(\Omega\backslash\Omega_s)\}.$$





The vector of unknowns $\mathbf{u}$ can be expressed as $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N, \mathbf{u}_\Gamma)$

- $\mathbf{u}_i$: interior dof of Ω_i;
- $\mathbf{u}_\Gamma$: dof located on Γ.

The discrete system can be written as

$$
\begin{pmatrix}
A_{11} & 0 & \dots & 0 & A_{1\Gamma} \\
0 & A_{22} & & \vdots & A_{2\Gamma} \\
\vdots & 0 & \ddots & 0 & \vdots \\
0 & 0 & \dots & A_{NN} & A_{N\Gamma} \\
A_{\Gamma 1} & A_{\Gamma 2} & \dots & A_{\Gamma N} & A_{\Gamma\Gamma}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \\ \mathbf{u}_\Gamma
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_N \\ \mathbf{f}_\Gamma
\end{pmatrix}.
$$

The discrete system can be written as

$$\begin{pmatrix} A_{11} & 0 & \dots & 0 & A_{1\Gamma} \\ 0 & A_{22} & & \vdots & A_{2\Gamma} \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \dots & A_{NN} & A_{N\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & \dots & A_{\Gamma N} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \\ \mathbf{u}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_N \\ \mathbf{f}_\Gamma \end{pmatrix}.$$

Remarks: The condition number of the matrix scales like $\frac{1}{h^2}$

We can eliminate interior unknowns $\mathbf{u}_i$ in terms of local values of $\mathbf{u}_\Gamma$ as

$$\mathbf{u}_i = A_{ii}^{-1}(\mathbf{f}_i - R_{\Gamma_i}\mathbf{u}_\Gamma)$$

- R_{Γ_i} is the restriction operator from $V_h(\Gamma)$ to $V_h(\Gamma_i)$.

Primal Schur method

This leads to the Schur complement system

$$\mathbb{S}_\Gamma \mathbf{u}_\Gamma = \mathbb{f}_\Gamma$$

that can be expressed in terms of local components S_i and $\mathbf{f}_{\Gamma_i}$ as

$$\left(\sum_{i=1}^{N} R_{\Gamma_i}^T S_i R_{\Gamma_i} \right) \mathbf{u}_\Gamma = \sum_{i=1}^{N} R_{\Gamma_i}^T \mathbf{f}_{\Gamma_i}$$

Primal Schur method

This leads to the Schur complement system

$$\mathbb{S}_\Gamma \mathbf{u}_\Gamma = \mathbb{f}_\Gamma$$

that can be expressed in terms of local components S_i and $\mathbf{f}_{\Gamma_i}$ as

$$\left(\sum_{i=1}^{N} R_{\Gamma_i}^T S_i R_{\Gamma_i}\right) \mathbf{u}_\Gamma = \sum_{i=1}^{N} R_{\Gamma_i}^T \mathbf{f}_{\Gamma_i}$$

Remark: The new system has a better conditioning which scales as $\dfrac{1}{hH}$.

Let DtN_i be an operator which maps

$$H^{1/2}(\Gamma_i) \times L^2(\Omega_i) \ni (u_{\mathrm{bc}}, f) \ \text{ to } \ -k\nabla u \cdot \mathbf{n}_{\Gamma_i} \in H^{-1/2}(\Gamma_i)$$

with

$$\begin{cases} bu - \mathrm{div}(k\nabla u) &= f & \text{in} & \Omega_i \backslash \Omega_s, \\ -k\nabla u \cdot \mathbf{n} &= 0 & \text{on} & \Omega_i \cap \partial\Omega_s \\ u &= u_{\mathrm{bc}} & \text{on} & \Gamma_i. \end{cases}$$

Now

$$\mathrm{DtN}_i^h(\mathbf{u}_{\mathrm{bc}}, \mathbf{f}) = S_i \mathbf{u}_{\mathrm{bc}} - \mathbf{f}_{\Gamma_i}.$$

is a discrete version of DtN_i.

Preconditioned Schur system

We introduce the preconditioned system

$$M^{-1}\mathbb{S}_\Gamma = M^{-1}\mathbb{f}_\Gamma$$

with an additive two-level preconditioner

$$M^{-1} = M_H^{-1} + \sum_i M_i^{-1}$$

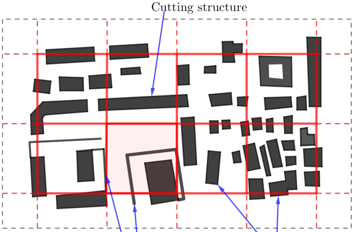
combining

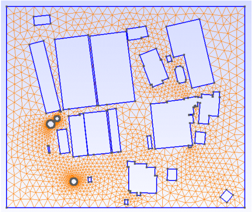
- local components

$$M_i^{-1} = \widetilde{R_{\Gamma_i}}^T S_i^{-1} \widetilde{R_{\Gamma_i}},$$

  where $\widetilde{R_{\Gamma_i}}$ is R_{Γ_i} weighted by a partition of unity;
- the coarse component M_H^{-1} build upon the coarse problem.

Coarse problem





Coarse space

- Let $V_H(\Gamma) \subset V_h(\Gamma)$ be some coarse skeleton space, e.g. edge wise linear (MsFEM).
- Let W_H be the corresponding linear map $W_H : V_H(\Gamma) \to V_h(\Gamma)$.

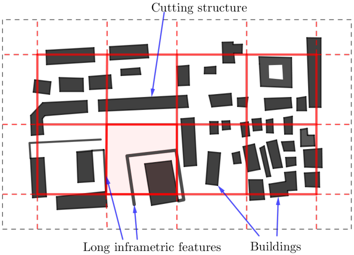
Galerkin method gives the following coarse problem: Find $\mathbf{u}_{H,\Gamma} \in V_H(\Gamma)$ s.t.

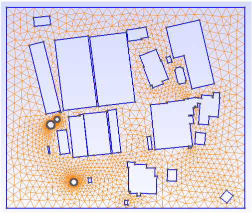
$$\underbrace{W_H^T \mathbb{S}_\Gamma W_H}_{S_H} \mathbf{u}_{H,\Gamma} = W_H^T \mathbb{f}_\Gamma.$$

The coarse component of the preconditioner is defined by

$$M_H^{-1} = W_H S_H^{-1} W_H^T.$$

Coarse problem





We have

$$\text{cond}\left(M^{-1}\mathbb{S}_\Gamma\right) \leqslant C(1+\log(H/h))^2$$

- Scaling with H and h is good;
- C can be very bad, need a coarse space adapted to Ω_s.

Standard Ms methods (MsFEM, MsFVM, ...) work in two steps

- Compute "areal" multi-scale basis functions $\phi_{H,k}$ (for dof k) by solving the fine scale problem

$$\begin{cases} b\phi_{H,k} - \mathrm{div}(k\nabla\phi_{H,k}) &= 0 \quad \text{in} \quad \Omega_i\backslash\Omega_s, \\ -k\nabla\phi_{H,k}\cdot\mathbf{n} &= 0 \quad \text{on} \quad \Omega_i\cap\partial\Omega_s \\ \phi_{H,k} &= \phi_{H,k,i} \quad \text{on} \quad \Gamma_i. \end{cases}$$

- Compute the coarse solution $u_H \in \mathrm{span}\left(\left(\phi_{H,k}\right)_k\right)$ by Galerkin method

$$\int_\Omega bu_H\phi_{H,k} + k\nabla u_H\cdot\phi_{H,k}\mathrm{d}x = \int_\Omega f\phi_{H,k}\mathrm{d}x + \text{BC} \qquad \text{for all} \quad k$$

Standard Ms methods (MsFEM, MsFVM, ...) work in two steps

- Compute "areal" multi-scale basis functions $\phi_{H,k}$ (for dof k) by solving the fine scale problem

$$\begin{cases} b\phi_{H,k} - \mathrm{div}(k\nabla\phi_{H,k}) &= 0 \quad \text{in} \quad \Omega_i \backslash \Omega_s, \\ -k\nabla\phi_{H,k} \cdot \mathbf{n} &= 0 \quad \text{on} \quad \Omega_i \cap \partial\Omega_s \\ \phi_{H,k} &= \phi_{H,k,i} \quad \text{on} \quad \Gamma_i. \end{cases}$$

- Compute the coarse solution $u_H \in \mathrm{span}\left(\left(\phi_{H,k}\right)_k\right)$ by Galerkin method

$$\int_\Omega b u_H \phi_{H,k} + k\nabla u_H \cdot \phi_{H,k} \mathrm{d}x = \int_\Omega f\phi_{H,k}\mathrm{d}x + \text{BC} \qquad \text{for all} \quad k$$

Remarks:

- Key component is a choice of $\phi_{H,k,i}$.
- Multi-scale basis functions can be computed *offline* and in parallel.

For symmetric problems the standard Ms methods are equivalent to:

- Chose a coarse skeleton space $V_H(\Gamma)$ to approximate $V_h(\Gamma)$;
- Solve Schur problem in $V_H(\Gamma)$ using Galerkin projection

$$W_H^T \mathbb{S}_\Gamma W_H \mathbf{u}_{\Gamma,H} = W_H^T \mathbb{f}_\Gamma,$$

Offline/online work load separation

- Costly fine-scale computations can be performed offline
- Important for time dependent problems

Stiffness matrix

$$W_H^T \mathbb{S}_\Gamma W_H = W_H^T \left(\sum_{i=1}^{N} R_{\Gamma_i}^T S_i R_{\Gamma_i} \right) W_H = \sum_{i=1}^{N} \left(R_{\Gamma_i} W_H\right)^T S_i \left(R_{\Gamma_i} W_H\right)$$

can be assembled offline and in parallel.

Ms discretizations: approximate Schur methods

For symmetric problems the standard Ms methods are equivalent to:

- Chose a coarse skeleton space $V_H(\Gamma)$ to approximate $V_h(\Gamma)$;
- Solve Schur problem in $V_H(\Gamma)$ using Galerkin projection

$$W_H^T \mathbb{S}_\Gamma W_H \mathbf{u}_{\Gamma,H} = W_H^T \mathbb{f}_\Gamma,$$

Offline/online work load separation

- Costly fine-scale computations can be performed offline
- Important for time dependent problems

Stiffness matrix

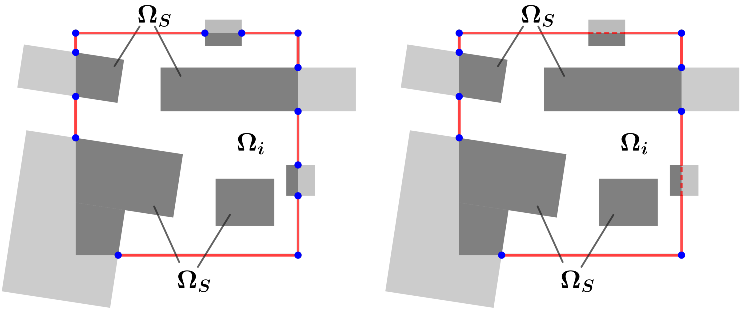
$$W_H^T \mathbb{S}_\Gamma W_H = W_H^T \left(\sum_{i=1}^{N} R_{\Gamma_i}^T S_i R_{\Gamma_i} \right) W_H = \sum_{i=1}^{N} \left(R_{\Gamma_i} W_H \right)^T S_i \left(R_{\Gamma_i} W_H \right)$$

can be assembled offline and in parallel.

RHS: Same goes for the RHS is $\mathbb{f}$ belong to some space of small dimension.

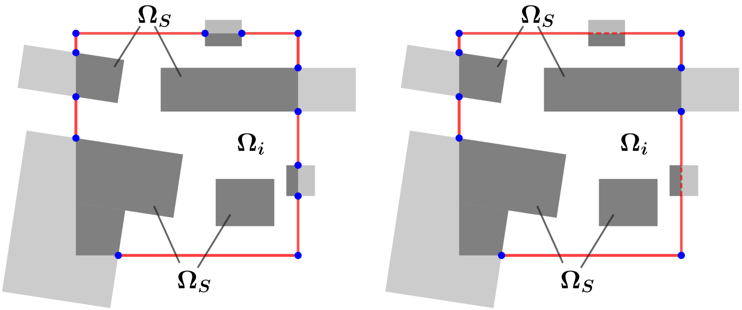
Coarse space

Ad-hoc solution for structure aware coarse space: edge wise polynomials



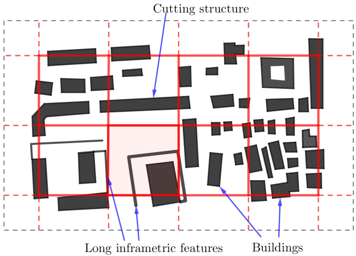
- Risk: to many dofs.
- Solution: remove some edges, *oversampling*.

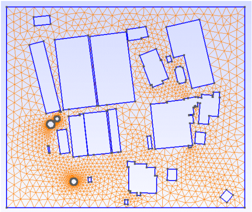
Ad-hoc solution for structure aware coarse space: edge wise polynomials



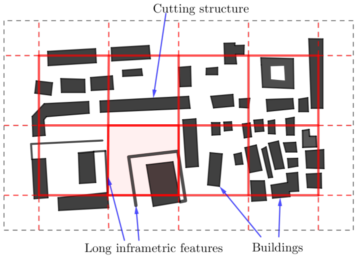
- Risk: to many dofs.
- Solution: remove some edges, *oversampling*.

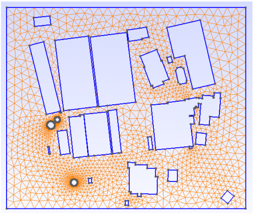
More generic option: methods like GenEO based on the spectrum of the local DtN_i^h operator.





- Coarse grid intersection with structures (Γ_i): *Polygon clipping*
- Global triangular mesh
  - fitted to the structures;
  - fitted to Γ_i.

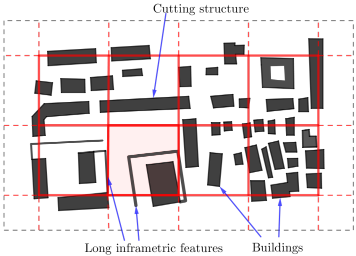


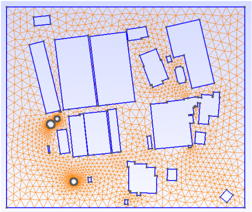


- Coarse grid intersection with structures (Γ_i): *Polygon clipping*
- Global triangular mesh
  - fitted to the structures;
  - fitted to Γ_i.

  May be to hard, so the fallback solution will be
  - Local meshes on $\Omega_i \backslash \Omega_s \Rightarrow$ non-conforming fine mesh $\Rightarrow$ mortar-like strategy.

Meshing and computational geometry (todo list)





- Coarse grid intersection with structures (Γ_i): *Polygon clipping*
- Global triangular mesh
  - fitted to the structures;
  - fitted to Γ_i.

  May be to hard, so the fallback solution will be
  - Local meshes on $\Omega_i \backslash \Omega_s \Rightarrow$ non-conforming fine mesh $\Rightarrow$ mortar-like strategy.
- Dealing with specific GIS data formats: more or less done by Léo Carriba-Demange during last summer.

PDE model

- Diffusive Wave equation

DD and Ms for linear problem

- Unified approach to DD and Ms methods

DD and Ms for nonlinear problem

- Few words on NK-DD, DD-NK and POD-DEIM

What about nonlinear problems?

Semi-discretized DW equation

$$\frac{u - u^{n-1}}{\Delta t} - \mathrm{div}\Big(\kappa \ \frac{(u - z_b)^\alpha}{\|\nabla u\|^{1-\gamma}} \nabla u\Big) = 0$$

We can still write the discrete problem in the form

$$F(u_\Gamma) := \sum_{i=1}^{N} R_{\Gamma_i}^T \, \mathrm{DtN}_i^h \left(R_{\Gamma_i} \mathbf{u}_\Gamma, R_{\Omega_i} \mathbf{f} \right) = 0$$

- Now DtN_i^h is a nonlinear map.

The coarse system reads

$$W_H^T F(W_H \mathbf{u}_{\Gamma_H}) = 0.$$

In principle this system can be solved by Newton's method.

Problems:

a) We can not precompute the action of F on $V_H(\Gamma)$, need to recompute local Dirichlet problems.

b) Even if we knew F, we need to compute the product $\underbrace{W_H^T}_{N\times\mathrm{card}(V_h(\Gamma))} \times \underbrace{F}_{\mathrm{card}(V_h(\Gamma))\times 1}$

Possible solutions: Positive thinking:

a) can be performed in parallel;

b) can be performed mostly in parallel.

The coarse system reads

$$W_H^T F(W_H \mathbf{u}_{\Gamma_H}) = 0.$$

In principle this system can be solved by Newton's method.

Problems:

a) We can not precompute the action of F on $V_H(\Gamma)$, need to recompute local Dirichlet problems.

b) Even if we knew F, we need to compute the product $\underbrace{W_H^T}_{N \times \text{card}(V_h(\Gamma))} \times \underbrace{F}_{\text{card}(V_h(\Gamma)) \times 1}$

Possible solutions: Positive thinking:

a) can be performed in parallel;

b) can be performed mostly in parallel.

ROM-DEIM:

a) (local) model reduction, involving e.g. POD;

b) Discrete Empirical Interpolation Method.

Nonlinear DD method

Newton-Krylov-DD (NK-DD):

- Linearize first, then apply DD to precondition the problem.
- More standard approach.
- Idea: GenEO to enrich the *ad-hoc* coarse space.

DD-Newton-Krylov (DD-NK):

- First DD, then linearize.
- More recent approach.
- Deals with steep nonlinearities locally .

Thank you for your attention!