Descent

The notion of *descent*, piecing together a global picture out of local pieces and glueing data, permeates Grothendieck's work. The history of this idea dates to the middle ages with mapmakers drawing an ever more precise picture of the world, as modern terminology of "atlases" and "charts" reminds us. It is crucial to the notion of *cohomology*, where we first meet higher glueing data.

Descent comes into Grothendieck's philosophy and work in a myriad of forms, starting with his papers "Technique de descente et théorèmes d'existence en géométrie algébrique". The technical requirements of this theory incited him to introduce the notion of *fibered category*. He extended the domain of application of this point of view in a revolutionary way by introducing the notion of "Grothendieck topology", integrating all of Galois theory and giving us etale cohomology. A further transformation occured with the notion of *topos*. Another incarnation of the idea was *cohomological descent* used in Deligne's papers on Hodge theory, where simplicial objects enter in a way which differs significantly from their original occurences in algebraic topology.

Between the 1960's and the 1980's, the notion of descent for objects of a category, slowly gave way to a notion of "higher descent" for objects in generalized categorical situations. Examples include Breen's calculation of etale *Ext* groups using the cohomology of simplicial Eilenberg-MacLane presheaves, and the theory of twisted complexes of Toledo and Tong. Stasheff and Wirth investigated higher cocycles in the 1960's although Wirth's thesis was only recently made public. In algebraic topology, Quillen's theory of model categories was gradually applied to simplicial diagrams. Illusie introduced the notion of weak equivalence of simplicial presheaves on a Grothendieck site.

Grothendieck came out of isolation with the manuscript *La poursuite des champs*, which at its start refers to a letter from Joyal to Grothendieck developping the model category structure on simplicial sheaves. This led to Jardine's model category structure for simplicial presheaves enhancing Illusie's weak equivalences. We enter into the modern period in which Jardine's model structure and its variants have been used and developped with applications in a wide range of mathematics including Thomason's work in K-theory and then Voevodsky's theory of $\mathbb{A}^1$-homotopy and motives. Algebraic stacks, the first step in the "higher descent" direction, are now used without restraint in all of algebraic geometry.

In Grothendieck's vision as set out in "La poursuite des champs", higher descent is just the same as usual descent, but for n-stacks of n-categories over a site. The theory of 2-categories was developped early on by Benabou, having occured also in the book of Gabriel and Zisman. The theory of strict n-categories was thoroughly investigated by Street and the Australian school, and Brown and Loday introduced other related algebraic objects which could model homotopy types. Grothendieck set out the goal of finding an adequate theory of *weak n-categories* where composition would be associative only up to a coherent system of higher equivalences. Similar ideas were being developped by Dwyer and Kan in algebraic topology, and Cordier and Porter in category theory. Several definitions of weak n-categories have been proposed, by Baez-Dolan, Tamsamani, Batanin and others. It is now well understood that the homotopy coherence problems inherent in higher categories, are basically the same as those which were studied by topologists for delooping machines. Segal's simplicial approach and May's operadic approaches play important roles in all of the current definitions. Maltsiniotis points out that Batanin's definition is the closest to Grothendieck's original idea. The topologists, notably Rezk and Bergner, have developped model structures on simplicial categories and simplicial spaces, and Joyal gives a model structure on the restricted Kan complexes originally defined by Boardman and Vogt in the 1960's. Cisinski and Maltsiniotis have built on a somewhat different direction of "La poursuite des champs" which aims to characterize the algebraic models for homotopy theory.

My own work in this area is inspired by the phrase in "La poursuite des champs" where Grothendieck forsees n-stacks as the natural coefficients for higher nonabelian cohomology. With Hirschowitz, we have developped the notion of n-stack based on Tamsamani's definition of n-category, and proven that the association $U \mapsto \{n\text{-stacks on } U\}$ is an $n+1$-stack.

The theory of "derived algebraic geometry" originated by Kontsevich, Kapranov and Ciocan-Fontanine is now cast by Toen, Vezzosi and Lurie in a foundational framework which relies on higher categories and higher stacks for glueing. In the future derived geometry should be a key ingredient in Hodge theory for higher nonabelian cohomology, to be compared with Katzarkov, Pantev and Toen's Hodge theory on the schematic homotopy type. The latter is a higher categorical version of Grothendieck's reinterpretation of Galois theory, forseen in "La poursuite des champs", or really its Tannakian counterpart. Grothendieck also mentionned, somewhat cryptically, a potential application to stratified spaces. The respective theses of Treumann and Dupont go in this direction by using exit-path n-categories to classify constructible complexes of sheaves.

Up-to-the-minute developments include Hopkins and Lurie's proof of a part of the Baez-Dolan system of conjectures relating higher categories to topological quantum field theory. And derived algebraic geometry permits us to imagine a local notion of descent as was explained to me by David Ben-Zvi: using the derived non-transverse intersections, Schlessinger-Stasheff-Deligne-Goldman-Millson theory should be viewed as descent for the inclusion of a point into a local formal space, with the neighborhood intersection being the derived loop space of Ben-Zvi and Nadler.

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