MANIFOLD CALCULUS IS DUAL TO FACTORIZATION HOMOLOGY

HIRO LEE TANAKA

ABSTRACT. These are notes from a talk given at the 2012 Talbot Workshop. It discusses relationships between the study of factorization homology and manifold calculus. It ends by classifying all symmetric monoidal functors which are equivalent to their Taylor series approximation.

Contents

1. Introduction

Yesterday and today, we were introduced to the theory of manifold calculus. If the functors we consider are context-free, manifold calculus studies contravariant functors

$$F : \mathsf{Mfld}_n^{\mathsf{op}} \to \mathcal{C}$$

where $\mathcal{C}$ is some appropriate category. (Recall that Mfld_n is the category of n-manifolds with morphisms embeddings.) What I want to do today is study *covariant* functors

$$A : \mathsf{Mfld}_n \to \mathcal{C}.$$

I will do this through the lens of *factorization homology*.

Already from this 'op', it seems like these two subjects should be dual to each other. And indeed, if there's one thing you remember from my talk, it's that the theory of manifold calculus and the theory of factorization homology are dual to each other. Let me first define things, and then I'll get into this 'dual' statement more concretely.

Conventions. I come from a school where every category is an ∞-category (so in particular, it is possible to think of a category as enriched over spaces). I also come from a school where any category term X is actually an ∞-X. (So when I say functor, I mean functor of ∞-categories, and when I say colimit, I mean a colimit in the sense of ∞-categories.) The point being, I will explicitly state that something is a 'strict' colimit when it is a colimit in the sense of ordinary category theory. Otherwise, every categorical notion I state (symmetric monoidal, tensored over spaces, et cetera) is in the context of ∞-categories.

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2. Factorization Homology

As advertised, we study functors $A : \mathsf{Mfld}_n \to \mathcal{C}$.

Definition 2.1. We define the following categories enriched over spaces:

- Let Mfld_n be the category of smooth n-manifolds. The hom space $\mathsf{Mfld}_n(M, N)$ is given by the space of embeddings from M to N. We will also denote this space by $\mathsf{Emb}(M, N)$.
- Let Disk_n be the full subcategory of manifolds which are disjoint unions of finitely many $\mathbb{R}^n$.
- Let $\mathsf{Disk}_n^{\leq k}$ be the full subcategory of Disk_n where objects are at most k disjoint copies of $\mathbb{R}^n$.

Remark 2.2. The empty manifold is an object of all the categories above.

Definition 2.3 (For this week)**.** Let $T_k A$ be the left Kan extension of A from $\mathsf{Disk}_n^{\leq k}$ to Mfld_n. By the universal property of left Kan extensions we have maps

$$\begin{array}{ccccc} T_0 A & \longrightarrow & T_1 A & \longrightarrow & \cdots \\ \downarrow & \swarrow & & & \\ A & & & & \end{array}$$

We call this the *Taylor tower* of A.

We say a functor A is *polynomial of degree* k if the natural map $T_k A \to A$ is an equivalence. We let $T_\infty A$ denote the colimit of the sequence $T_k A$.

Remark 2.4. Note that I make no mention of being excisive with respect to cubes, or with being a cosheaf with respect to any kind of topology. The main reason for this is that we have the theorem of Brito and Weiss, that Kan extensions give rise to polynomial functors in whatever sense you mean. So I will forget about an extrinsic definition of polynomial functors altogether.

Remark 2.5. The arrows are in the opposite direction from manifold calculus, as are the numerics. So it's like a co-Postnikov tower for the functor A, or, if you like, a cell decomposition.

Also, note that since colimits commute with each other, $T_\infty A$ is actually the left Kan extension of A from Disk_n to Mfld_n.

Definition 2.6. Let $M \in \mathsf{Mfld}_n$. The *factorization homology* of M with coefficients in A is

$$\int_M A := T_\infty A(M) \in \mathcal{C}.$$

We say that A is *convergent* if the natural transformation $T_\infty A \to A$ is an equivalence.

Remark 2.7. I call this *convergent* and not analytic, since the latter term usually has a condition restricting the rate of convergence.

Remark 2.8 (Framed manifolds)**.** One can change Mfld to be the category of manifolds with certain kinds of structure, where the embeddings must respect that structure up to homotopy. For instance, if we demand that every manifold is equipped with a trivialization of the tangent bundle (this is a fairly heavy restriction on the admissible manifolds), the resulting category $\mathsf{Mfld}^{\mathsf{fr}}$ has morphisms which are embeddings, together with a homotopy between the domain manifold's framing with the pullback framing.

Example 2.9. There is the natural inclusion of categories

$$\iota : \mathsf{Mfld}_n^{\mathsf{fr}} \to \mathsf{Spaces}$$

given by taking the underlying topological space of a manifold. This is a convergent functor.

Proof. Note that this is actually equivalent to a corepresentable functor. Namely,

$$\mathsf{Emb}^{\mathsf{fr}}(\mathbb{R}^n, -) \simeq \iota.$$

This is because the space of embeddings of $\mathbb{R}^n$ into M is homotopy equivalent to the space of embeddings of a point into M – this is given by the restriction to $0 \in \mathbb{R}^n$. On the other hand, the left Kan extension is computed object-wise by the coend

$$\begin{aligned} \mathsf{Emb}^{\mathsf{fr}}(-, M) \otimes_{\mathsf{Disk}_n} \iota &\simeq \mathsf{Emb}^{\mathsf{fr}}(-, M) \otimes_{\mathsf{Disk}_n} \mathsf{Emb}^{\mathsf{fr}}(\mathbb{R}^n, -) \\ &\simeq \mathsf{Emb}^{\mathsf{fr}}(\mathbb{R}^n, M). \end{aligned}$$

The last line follows because we are taking a coend with respect to a corepresentable functor. □

The same method of proof, by realizing that $\amalg_k \mathbb{R}^n$ is an object in $\mathsf{Disk}^{\leq k}$, leads to the following:

Proposition 2.10. *The functor* $\mathsf{Emb}((\mathbb{R}^n)^{\amalg k}, -)$ *is a polynomial functor of degree k. If we work with framed manifolds, the functor assigns to any M the homotopy type of $M^k \backslash \Delta$; the k-fold product without the fat diagonal.*

Remark 2.11 (Non-framed case)**.** For non-framed manifolds, the space of embeddings of $\mathbb{R}^n$ retracts not to the configuration space of one point on M, but the configuration space of one point together with a choice of an element in $O(n)$. More specifically is, it is the choice of a point in the $O(n)$-bundle over M associated to TM. In general if we work in the category of manifolds with reduction of structure group to G, then the representable functor $\mathsf{Emb}((\mathbb{R}^n)^{\amalg k}, -)$ assigns to any manifold a point in $P^k \backslash \Delta$. Here P is the G-bundle associated to TM and Δ is the fact diagonal. The punchline is that the space of embeddings deformation retracts onto the space of derivatives of an embedding.

So this gives us two more examples of convergent functors. The functor $\mathsf{Mfld} \to \mathsf{Spaces}$ given by $M \mapsto P \to M$, where P is the associated $O(n)$ bundle to TM, is convergent using the same method above. The forget functor $M \mapsto M$ is also convergent, as it is obtained by taking the quotient of P by the $O(n)$ action, and colimits commute.

Example 2.12. Let $n \geq 2$. Let U be $\mathbb{R}^n - \{0\}$. Then $A = \mathsf{Emb}(U, -)$ is not convergent.

2.1. **Symmetric Monoidal Version and Excision.** Note that the category of manifolds has a symmetric monoidal structure, given by disjoint union. So it makes sense to talk about symmetric monoidal functors out of this category.

Let $\mathcal{C}$ be a category with a fixed monoidal structure. (This could be Spaces with $\amalg$, or with $\times$. It could be $\mathsf{Spectra}$ with $\wedge$ or $\vee$ – let's not get into the construction of a symmetric monoidal structure for spectra – and it could even be chain complexes with $\oplus$ or $\otimes$.) From hereon we only consider symmetric monoidal functors

$$A : (\mathsf{Mfld}_n, \amalg) \to (\mathcal{C}, \otimes).$$

The first observation I'd like to make is that, if we restrict A to the category Disk_n, one recovers the structure of an algebra over the operad of framed little n-disks. If you don't know what that is, you should take this as a definition:

Definition 2.13. A Disk_n-algebra in the category $(\mathcal{C}, \otimes)$ is a symmetric monoidal functor $A : (\mathsf{Disk}_n, \amalg) \to (\mathcal{C}, \otimes)$.

Example 2.14. If $n = 1$, we have the structure of an associative algebra A together with an involution $\tau : A \to A$. What do I mean? It's a map s.t. $\tau^2 = id_A$ and $\tau(a)\tau(b) = \tau(ab)$.

If $n = 2$, we get an E_2-algebra A with an $O(2)$ action. (Draw a picture.)

If we think of symmetric monoidal functors out of $\mathsf{Mfld}_n^{\mathsf{fr}}$, then we would get algebras over the usual n-disks operad.

Remark 2.15. This is the first time this week that we've ever even thought of considering functors with this kind of restriction. But there are two good reasons for this – first, Disk_n algebras have been popping up a lot lately. They are objects of interest to derived algebraic geometers, who should study schemes over Disk_n-algebras, they show up in topological field theories, and they interpolate between the stable and unstable world—they're in some sense the bridge between spectra, which are like infinite loop spaces, and spaces, which need not be loop spaces at all.

One can still define factorization homology as a left Kan extension of A along the inclusion $\mathsf{Disk}_n \to \mathsf{Mfld}_n$. It shouldn't be a surprise that factorization homology has some special properties, once we impose the extra condition that A be symmetric monoidal. The following is proven by Francis in [Francis12], and it is also proven in [AyalaFrancisT12].

Theorem 2.16 (Excision, [Francis12], [AyalaFrancisT12])**.** *Suppose M has a cover by two open manifolds $N_0, N_1 \subset M$, and that the intersection $N_0 \cap N_1$ can be written as a product manifold $V \times \mathbb{R}$. Fix this identification. Then*

$$\int_M A \simeq \int_{N_0} A \bigotimes_{\int_{V \times \mathbb{R}} A} \int_{N_1} A.$$

The idea is that $V \times \mathbb{R}$ is a Disk_1 algebra in the category of manifolds, since we have the functor $\mathsf{Disk}_1 \to \mathsf{Mfld}$ given by $\mathbb{R} \mapsto V \times \mathbb{R}$, and $(f : \mathbb{R} \to \mathbb{R}) \mapsto id_V \times f$. Moreover N_0 has the structure of a right module, and N_1 has the structure of a left module. Since A is a symmetric monoidal functor, these module structures remain in $\mathcal{C}$, and we can take the bar construction in $\mathcal{C}$.

And this gives us a criterion for detecting convergent functors:

Definition 2.17. Say that a symmetric monoidal functor H satisfies *excision* if the above theorem is true when one replaces $\int_\bullet A$ with H.

Corollary 2.18. *Let $A : \mathsf{Mfld}_n \to \mathcal{C}$ be a symmetric monoidal functor. If A does not satisfy excision, A is not convergent.*

But in fact, one can go further. We have a complete classification of convergent symmetric monoidal functors. First, note the obvious fact:

Proposition 2.19. *There is an equivalence of categories*

$$\mathsf{Disk}_n\text{-}\mathsf{Mod}^L \simeq \mathsf{Fun}^\omega(\mathsf{Mfld}_n, \mathcal{C}).$$

between the category of left modules over the Disk_n-operad, and the category of convergent functors. There is also a symmetric monoidal version

$$\mathsf{Disk}_n - \mathsf{Alg} \simeq \mathsf{Fun}^{\omega,\otimes}(\mathsf{Mfld}_n, \mathcal{C}).$$

between the category of Disk_n-algebras and convergent symmetric monoidal functors.

Proof. A Kan extension is completely determined by the values of a functor on the category from which one is extending. □

We also have the following theorem of Francis from [Francis12]:

Theorem 2.20 ([Francis12])**.** *Let $\mathcal{H}(\mathcal{C}) \subset \mathsf{Fun}^\otimes(\mathsf{Mfld}, \mathcal{C})$ be the full subcategory of symmetric monoidal functors H such that H satisfies excision. There is an equivalence of categories*

$$\mathcal{H}(\mathcal{C}) \simeq \mathsf{Disk}_n - \mathsf{Alg}(\mathcal{C}).$$

The equivalence is given by factorization homology in one direction, and by restriction to Disk_n in the other.

Corollary 2.21. *A symmetric monoidal functor is convergent if and only if it satisfies excision.*

Remark 2.22. The category $\mathcal{H}$ is called the category of *homology theories for n-manifolds.* This is because it satisfies (a version of) the usual Eilenberg-Steenrod axioms familiar from classical topology. First, while a homology theory is a functor out of Spaces, we now consider functors out of Mfld. Second, while a usual homology theory satisfies excision (i.e., Mayer-Vietoris), a functor in $\mathcal{H}$ sends some strict pushouts of manifolds to a *tensor product.*

Example 2.23. Let $A : \mathsf{Mfld}_1 \to \mathcal{C}$. Then A assigns the structure of an E_1-algebra to $\mathbb{R}$. By the above theorem, we see that the factorization homology of A over the circle is given by the bar construction

$$A(\mathbb{R}) \bigotimes_{A(\mathbb{R}) \otimes A(\mathbb{R})^{op}} A(\mathbb{R}).$$

If A takes values in chain complexes, this is the Hochschild homology of the algebra $A(\mathbb{R})$. In spectra, this is also talled THH, or the *topological Hochschild homology.*

Note that S^1 has a group of diffeomorphisms given by $\mathbb{Z}_2 \ltimes S^1$. Well, the functoriality of $\int_\bullet A$ means this action is transferred to the object $\int_{S^1} A$ in $\mathcal{C}$. The circle action by S^1 is (according to Lurie, but without written proof) precisely the action of cyclic homology. So the quotient by S^1's action will be topological cyclic homology.

Example 2.24. Let $X \in \mathsf{Spaces}$ and let $\mathsf{Free}(X)$ be the free $\mathsf{Disk}_n^{\mathsf{fr}}$ algebra generated by X. Then

$$\int_M \mathsf{Free}(X) \simeq \coprod_{k \geq 0} \mathsf{Conf}^k(M; X)$$

where the latter is the space of configurations of k points in M, each point labeled by an element of X. These points are unordered.

To see this, note that there's a functor $\Sigma \to \mathsf{Disk}_n^{\mathsf{fr}}$, where Σ is the category of (possibly empty) finite sets with bijections. Then the left Kan extension of the functor $\Sigma \to X$ given by $S \mapsto X^S$ along the inclusion $\Sigma \to \mathsf{Disk}_n^{\mathsf{fr}}$ yields the free $\mathsf{Disk}_n^{\mathsf{fr}}$-algebra on X. Composing with $\mathsf{Disk}_n^{\mathsf{fr}} \to \mathsf{Mfld}_n^{\mathsf{fr}}$, we get the Kan extension

$$\mathsf{Mfld}((\mathbb{R}^n)^{\amalg(-)}, M) \bigotimes_{\Sigma} X^{(-)}.$$

But this is the same thing as embeddings of disks into M, labeled by points of X. And this is homotopy equivalent to the space of points on M labeled by X.

More generally, if we study the category of manifolds with a reduction of the tangent bundle to some G, together with G-framed embeddings, then we would find the configuration space of points on P labeled by X, where P is the associated G-budle to TM. Note this space has an action of G by virtue of the fact that G is a principle G-bundle.

Now let's study the Taylor tower. In the framed case, we have

$$T_1 \mathsf{Free}(X) \simeq \mathsf{Emb}^{\mathsf{fr}}(\mathbb{R}^n, M) \times \coprod_k \mathsf{Conf}^k(\mathbb{R}^n; X).$$

In the non-framed case, we have

$$\mathsf{Emb}^G(\mathbb{R}^n, M) \times_G \coprod_k \mathsf{Conf}^k(P_{\mathbb{R}^n}; X).$$

As before, now the configuration space is that of points on $P_{\mathbb{R}^n}$, the principle G-bundle on $\mathbb{R}^n$.

Example 2.25. This example was told to me by Geoffroy Horel. Let A be an E_n algebra. Its degree one Taylor polynomial is given by the free E_∞ algebra generated by A. Namely, to a disjoint union of k disks, one assigns $A^{\amalg k}$.

Then $\int_{S^1} A^{\amalg} \simeq A \vee \Sigma A$.

On the other hand the universal property of the Taylor tower gives a map to the actual factorization homology. So we get a map

$$A \vee \Sigma A \to THH(A).$$

This is a well-known map. It is know to be an equivalence for a certain E_1 spectrum.

Remark 2.26 (Factorization homology usually)**.** This is usually the setting in which one considers factorization homology – the symmetric monoidal context. This is why I said that the definition above is a definition *for this week*.

Manifold Calculus	Factorization Homology
$F : \mathsf{Mfld}_n^{\mathsf{op}} \to \mathcal{C}$	$A : \mathsf{Mfld}_n \to \mathcal{C}$
Right Kan extension	Left Kan extension
Right Disk_n-modules	Left Disk_n-modules
$T_\infty F(M) \simeq \mathrm{Ext}_{\mathsf{Disk}_n\text{-}\mathsf{Mod}_R}(\mathsf{Emb}_n, F)$	$\int_M A \simeq \mathrm{Tor}_{\mathsf{Disk}_n}(\mathsf{Emb}_n, \int)$

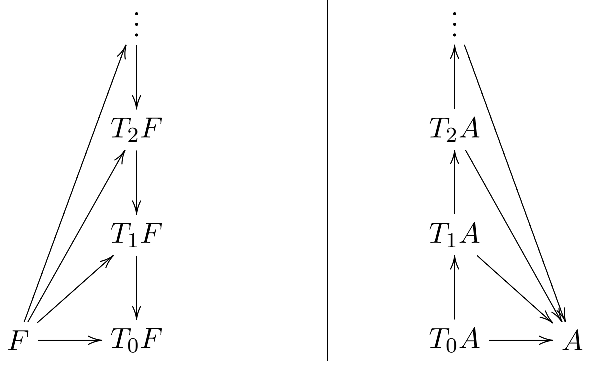

TABLE 1. Dual notions

3. Manifold Calculus Revisited

Now I'd like to state a reformulation of manifold calculus – from how we considered it earlier this week, to the way that Pedro and Michael Weiss consider it in their paper [BW12].

Recall that I complained when Dan said he'd consider good functors

$$F : \mathsf{Open}(M)^{op} \to \mathcal{C}.$$

My complaint was that while F would assign equivalent objects to isotopic open sets, F wouldn't be sensitive to *how* they were isotopic.

Well, here's a way to fix that situation. We could instead consider functors

$$F : \mathsf{Mfld}_{/M}^{op} \to \mathcal{C}$$

from the overcategory. (This is an overcategory in the sense of ∞-categories. So for instance, a morphism from $X \to Y$ would be a homotopy commutative triangle with edges $X \to Y, Y \to M, X \to M$.)

But if we've gone this far, we may as well consider functors out of all of Mfld:

$$F : \mathsf{Mfld}^{\mathsf{op}} \to \mathcal{C}.$$

And that's what we'll do here.

Definition 3.1 (Boavida De Brito-Weiss [BW12])**.** Given $F : \mathsf{Mfld}^{op} \to \mathcal{C}$, let T_kF be the right Kan extension of $F|_{\mathsf{Disk}_n^{\leq k}}$ along the inclusion $\mathsf{Disk}_n^{\leq k} \to \mathsf{Mfld}_n$. We call this the *$k$th Polynomial approximation to F.*

That this is equivalent to the notion of polynomial approximation is Theorem 5.3 and Proposition 4.2 of Pedro and Michael Weiss's pre-print [BW12].

By the universal property of right Kan extensions, we have the usual Taylor tower

$$\begin{array}{ccccc} T_0F & \longleftarrow & T_1F & \longleftarrow & \cdots \\ \uparrow & \nearrow & & \nearrow & \\ F. & & & & \end{array}$$

I will say that A is *convergent* if it is equal to $T_\infty A := \mathsf{holim}\, T_i A$. Note that by definition we see

$$T_\infty(F)(M) \simeq \hom_{\mathsf{Disk}_n\text{-}\mathsf{Mod}^R}(\mathsf{Emb}(-, M), F).$$

3.1. **Symmetric Monoidal Version.** Now we can study the symmetric monoidal such functors. Note that the restriction of such a functor F to Disk_n gives rise to a *$disk_n$-coalgebra*, not algebra. I don't know about a lot of examples of such coalgebras.

Regardless, we can consider this as a functor $\mathsf{Mfld} \to \mathcal{C}^{op}$ being extended by left Kan extensions. Then everything we said about excision holds, but with every diagram flipped in direction. In other words, we have

Corollary 3.2. *The functor $T_\infty F$ satisfies coexcision if F is symmetric monoidal. That is: Suppose M has a cover by two open manifolds $N_0, N_1 \subset M$, and that the intersection $N_0 \cap N_1$ can be written as a product manifold $V \times \mathbb{R}$. Fix this identification. Then*

$$T_\infty F(M) \simeq \mathrm{Cobar}(T_\infty F(N_0), T_\infty F(V \times \mathbb{R}), T_\infty F(N_1).$$

Dual to the factorization homology picture, $V \times \mathbb{R}$ now has the structure of an E_1 coalgebra. N_0 and N_1 are left and right comodules over $V \times \mathbb{R}$. Hence it makes sense to take the cobar construction, or cotensor product, of these two modules.

Again by dualizing, we classify convergent functors in manifold calculus:

Corollary 3.3. *There is an equivalence of categories between convergent symmetric monoidal functors $F : \mathsf{Mfld}^{op} \to \mathcal{C}$ and homology theories $H : \mathsf{Mfld}^{\mathsf{op}} \to \mathcal{C}$.*

Where here, by homology theory, we mean a functor satisfying the 'coexcision axiom,' where the bar construction from Theorem 2.16 is replaced by a cobar construction.

References

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