

## ON THE GROUPS $J(X)$ —I

J. F. ADAMS

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### §1. INTRODUCTION

ATIYAH [6] has defined certain groups, which he has called  $J(X)$ . For our purposes, we shall define the groups  $J(X)$  as follows. Let  $X$  be a good space, for example, a finite-dimensional  $CW$ -complex. Let  $K_R(X)$  be the Grothendieck–Atiyah–Hirzebruch group [7, 8, 1] defined in terms of real vector bundles over  $X$ . Let  $T(X)$  be the subgroup of  $K_R(X)$  generated by elements of the form  $\{\xi\} - \{\eta\}$ , where  $\xi$  and  $\eta$  are orthogonal bundles whose associated sphere-bundles are fibre homotopy equivalent. (We think of  $T(X)$  as the subgroup of fibre-homotopy-trivial virtual bundles.) We define

$$J(X) = K_R(X)/T(X).$$

If  $X$  is connected we have

$$K_R(X) = Z + \tilde{K}_R(X),$$

where  $\tilde{K}_R(X)$  denotes the subgroup of virtual bundles whose virtual dimension is zero. We have  $T(X) \subset \tilde{K}_R(X)$ , so we may define

$$\tilde{J}(X) = \tilde{K}_R(X)/T(X).$$

We then have

$$J(X) = Z + \tilde{J}(X).$$

It is not hard to see that the group which we call  $\tilde{J}(X)$  is isomorphic to that which Atiyah originally introduced and called  $J(X)$  [6]. It was natural for Atiyah to concentrate on  $\tilde{J}(X)$ , since the summand  $Z$  is not interesting, and since Atiyah's theorem that  $\tilde{J}(X)$  is finite [6, Proposition (1.5)] would not be true for  $J(X)$ .

Atiyah has also shown that the groups  $J(X)$  have useful applications. If we take  $X$  to be a projective space (either real, complex or quaternionic) then the resulting group  $J(X)$  holds the answer to classical questions about the existence of cross-sections of appropriate Stiefel fiberings [6, Theorem (6.5)]. If we take  $X$  to be a sphere, then the resulting group  $\tilde{J}(X)$  is (up to isomorphism) the image of the classical  $J$ -homomorphism in an appropriate dimension [6, Proposition (1.4)]. It would therefore be amply worth-while to give means for computing the groups  $J(X)$ . The present series of four papers represents a start in this direction.

We shall attempt to compute the group  $J(X)$  by introducing two further groups  $J'(X)$ ,  $J''(X)$ . For the moment we need only emphasise three points about these groups.

(i) These groups are defined as quotients of  $K_R(X)$ ; that is, we shall give definitions of the form

$$J'(X) = K_R(X)/V(X)$$

$$J''(X) = K_R(X)/W(X).$$

(ii) The groups  $J'(X)$ ,  $J''(X)$  are computable.

(iii) The group  $J'(X)$  is intended to serve as a lower bound for  $J(X)$ , and the group  $J''(X)$  is intended to serve as an upper bound for  $J(X)$ , in a sense which we will now explain.

We shall say that " $J'(X)$  is a lower bound for  $J(X)$ " if  $T(X) \subset V(X)$ , so that the quotient map  $K_R(X) \rightarrow J'(X)$  factors through an epimorphism  $J(X) \rightarrow J'(X)$ . We shall prove that this is so for all  $X$ .

We shall say that " $J''(X)$  is an upper bound for  $J(X)$ " if  $W(X) \subset T(X)$ , so that the quotient map  $K_R(X) \rightarrow J(X)$  factors through an epimorphism  $J''(X) \rightarrow J(X)$ . It is plausible to conjecture that this is so for all  $X$ ; but so far we can prove this only in favourable cases, for example,  $X = RP^n$  (real projective space),  $X = CP^n$  (complex projective space), and  $X = S^m$  with  $m \not\equiv 0 \pmod{8}$ .

In such favourable cases we can proceed to compute the groups  $J'(X)$ ,  $J''(X)$ ; and if we find that the quotient map  $J''(X) \rightarrow J'(X)$  is an isomorphism, then the group  $J(X)$  is completely determined, being isomorphic to both  $J'(X)$  and  $J''(X)$ .

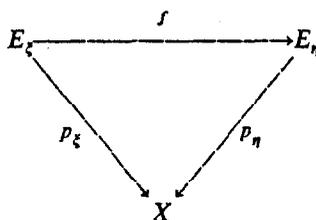
We will now try to explain that the groups  $J'(X)$ ,  $J''(X)$  merely formalise two reasonable methods of attacking our problem. Let us start with the first. We shall sometimes wish to show that two bundles  $\xi$ ,  $\eta$  represent different elements of  $J(X)$ . This is the sort of problem which one usually attacks by introducing suitable invariants. For example, the theory of characteristic classes sometimes allows one to prove that two bundles  $\xi$ ,  $\eta$  are not fibre homotopy equivalent. This method has been pressed further by Atiyah (private communications; cf. [6] p. 291, lines 14, 15; p. 309, lines 6, 7) and Bott [9, 10]. Instead of characteristic classes with values in the ordinary cohomology groups  $H^*(X; G)$ , they use characteristic classes with values in the extraordinary cohomology groups  $K_A(X)$ . By using the best techniques available in this direction, one defines the group  $J'(X)$ ; if two bundles  $\xi$ ,  $\eta$  have different images in  $J'(X)$ , then they represent different elements of  $J(X)$ .

The group  $J'(X)$ , then, is essentially due to Atiyah and Bott. In particular, the notation  $J'(X)$  is taken from unpublished work of Atiyah; it originally stood for a somewhat cruder lower-bound group. We adopt the notation  $J''(X)$  by analogy with  $J'(X)$ .

Let us now turn to the second method of attack. We shall sometimes wish to show that two bundles  $\xi$ ,  $\eta$  represent the same element of  $J(X)$ , although they represent different elements of  $K_R(X)$ . This is the sort of problem which one usually attacks by giving geometrical constructions. In this direction we offer Theorem (1.1) below.

Let  $\xi$ ,  $\eta$  be sphere-bundles over a finite  $CW$ -complex  $X$ , with total spaces  $E_\xi$ ,  $E_\eta$  and projections  $p_\xi$ ,  $p_\eta$ . By a 'fibrewise map  $f: E_\xi \rightarrow E_\eta$ ', we shall mean a map  $f$  such that

the following diagram is commutative:



Let  $k$  be a positive integer.

**THEOREM (1.1).** *If there is a fibrewise map  $f: E_\xi \rightarrow E_\eta$  of degree  $\pm k$  on each fibre, then there exists a non-negative integer  $e$  such that the Whitney multiples  $k^e \xi, k^e \eta$  are fibre homotopy equivalent.*

If we put  $k = 1$  this is a theorem of Dold [11]. Therefore one may regard this theorem as a ‘mod  $k$ ’ analogue of Dold’s theorem.

By using Theorem (1.1), one can prove certain cases of the following conjecture (in which the operation  $\Psi^k$  is as in [1]).

**CONJECTURE (1.2).** *If  $k$  is an integer,  $X$  is a finite CW-complex and  $y \in K_R(X)$ , then there exists a non-negative integer  $e = e(k, y)$  such that  $k^e(\Psi^k - 1)y$  maps to zero in  $J(X)$ .*

More precisely, we shall prove the following cases of Conjecture (1.2).

**THEOREM (1.3).** *Assume that  $X$  is a finite CW-complex and that  $y$  is a linear combination of  $O(1)$  and  $O(2)$  bundles. Then there exists  $e = e(k, y)$  such that  $k^e(\Psi^k - 1)y$  maps to zero in  $J(X)$ .*

**THEOREM (1.4).** *Assume that  $X$  is a sphere  $S^{2n}$  and that  $y$  lies in the image of*

$$r : K_C(S^{2n}) \rightarrow K_R(S^{2n}).$$

*Then there exists  $e = e(k, y)$  such that  $k^e(\Psi^k - 1)y$  maps to zero in  $J(X)$ .*

In Part II we shall see that Theorem (1.4) leads to the result on the  $J$ -homomorphism which was announced in [2, Theorem (3); 3, Theorem (3)].

The definition of the group  $J''(X)$  will be arranged so that if Conjecture (1.2) is true for all  $k$  and all  $y$  in  $K_R(X)$ , then  $J''(X)$  is an upper bound for  $J(X)$ .

The arrangement of the present series of papers is as follows. The main object of Part I is to prove Theorem (1.1), the ‘mod  $k$  Dold theorem’. We shall also prove Theorems (1.3) and (1.4). Parts II and III are devoted to a systematic account of the groups  $J'(X)$  and  $J''(X)$ . In Part IV we shall apply the methods of  $K$ -theory to study the homotopy groups of spheres. Here we are concerned not only with the image of the  $J$ -homomorphism; we apply the methods of  $K$ -theory to give invariants defined for every homotopy class in the appropriate homotopy group.

A separate paper with G. Walker [5] will study the case  $X = CP^n$ . This paper depends essentially on Parts I and II of this series, but is independent of Parts III and IV.

The results of Parts I, II, III and [5] have been summarised elsewhere [2, 3, 4]. The results of Part IV have so far appeared only in lectures and mimeographed form, but there is some overlap with work of E. Dyer [13].

The present paper is arranged as follows. Theorem (1.1) is proved in §3; it depends on lemmas proved in §2. Theorems (1.3) and (1.4) are proved in §4.

## §2. FUNCTION SPACES

Dold's theorem is proved [11] by using the topological monoid  $H(n)$  of homotopy equivalences from  $S^{n-1}$  to  $S^{n-1}$ ; the key idea is to take  $H(n)$  seriously as a 'structural group'. This idea was developed further by Dold and Lashof [12].

We shall prove our 'mod  $k$ ' analogue of Dold's theorem by using the space  $G(n)$  of all maps from  $S^{n-1}$  to  $S^{n-1}$  (of whatever degree.) We write  $G(n, k)$  for the component of  $G(n)$  which consists of maps of degree  $k$ . These spaces are to be given the compact-open topology.

Various maps can be defined on the spaces  $G(n)$ , and in §3 we shall need to quote lemmas about the effect of these maps on the homotopy groups of  $G(n)$ . It is the object of this section to supply these results, which are stated as Lemmas (2.1) and (2.4).

We recall that if  $r \leq n - 3$ , the homotopy group  $\pi_r(G(n, k))$  can be identified with the stable homotopy group  $\pi_r^S$  of the  $r$ -stem. We will give details of the identification below; it is hoped that these details will remove any doubt about the sign-conventions employed; we follow 'homology' conventions. The conventions appear more natural if one writes maps of spaces on the right of their arguments, and we therefore do so throughout this section.

If we are given a sphere map  $\alpha : S^r \rightarrow G(n, k)$  or a homotopy class  $\beta \in \pi_r(G(n, k))$ , we shall write  $[\alpha]$ , or  $[\beta]$ , for the corresponding element in  $\pi_r^S$  (assuming always that  $r \leq n - 3$ ).

Our identification of  $\pi_r(G(n, k))$  with  $\pi_r^S$  passes through various intermediate groups, and we make the same convention for these groups; if  $\gamma$  is an element of one of these intermediate groups, we shall write  $[\gamma]$  for the corresponding element in  $\pi_r^S$ .

We now discuss the various maps defined on  $G(n)$ .

If  $g \in G(n)$ , we shall define  $\bar{g} : G(n) \rightarrow G(n)$  by composition with  $g$ , so that

$$(x)(f\bar{g}) = ((x)f)g \quad (x \in S^{n-1}, f \in G(n).)$$

If  $g \in G(n, t)$ , then  $\bar{g}$  maps  $G(n, s)$  into  $G(n, st)$ . Our first lemma describes the induced homomorphism  $\bar{g}_*$  of homotopy groups.

LEMMA (2.1). *If  $\alpha \in \pi_r(G(n, s))$ ,  $g \in G(n, t)$  and  $r \leq n - 3$ , then*

$$[\bar{g}_*\alpha] = t[\alpha].$$

The join product

$$j : G(n) \times G(m) \rightarrow G(n + m)$$

is defined by  $j(f, g) = f * g$ , where  $S^{n+m-1}$  is regarded as the join  $S^{n-1} * S^{m-1}$  of  $S^{n-1}$  and

$S^{m-1}$ . The product  $j$  maps  $G(n, s) \times G(m, t)$  into  $G(n + m, st)$ . Our second lemma describes the induced homomorphism  $j_*$ . For this purpose we identify  $\pi_r(G(n, s) \times G(m, t))$  with the direct sum  $\pi_r(G(n, s)) + \pi_r(G(m, t))$ , as usual.

LEMMA (2.2). *If  $\alpha \in \pi_r(G(n, s))$ ,  $\beta \in \pi_r(G(m, t))$  and  $r \leq \min(n - 3, m - 3)$ , then*

$$[j_*(\alpha + \beta)] = t[\alpha] + s[\beta].$$

The iterated join product

$$j^{(v)} : G(n_1) \times G(n_2) \times \dots \times G(n_v) \rightarrow G\left(\sum_{i=1}^v n_i\right)$$

is defined by

$$j^{(v)}(f_1, f_2, \dots, f_v) = f_1 * f_2 * \dots * f_v.$$

Our third lemma describes the induced homomorphism  $j^{(v)}_*$ . For this purpose we identify  $\pi_r(G(n_1, s_1) \times G(n_2, s_2) \times \dots \times G(n_v, s_v))$  with the direct sum

$$\sum_{i=1}^v \pi_r(G(n_i, s_i)),$$

as usual.

LEMMA (2.3). *If  $\alpha_i \in \pi_r(G(n_i, s_i))$  for  $1 \leq i \leq v$  and  $r \leq \text{Min}(n_i - 3)$ , then*

$$\left[ j^{(v)}_* \left( \sum_{i=1}^v \alpha_i \right) \right] = \sum_{i=1}^v \frac{s_1 s_2 \dots s_v}{s_i} [\alpha_i].$$

This follows immediately from Lemma (2.2), by induction over  $v$ .

LEMMA (2.4). *Let  $\alpha : S^r \rightarrow G(n, s)$  be a sphere map, and define  $\beta : S^r \rightarrow G(nv, s^v)$  by*

$$\beta(x) = \alpha(x) * \alpha(x) * \dots * \alpha(x) \quad (v \text{ factors}).$$

Then

$$[\beta] = vs^{v-1}[\alpha].$$

This follows immediately from Lemma (2.3). In fact, let

$$\gamma : S^r \rightarrow \prod_{i=1}^v G(n, s)$$

be the map all of whose components are  $\alpha$ ; then  $\beta$  is just the composite

$$S^r \xrightarrow{\gamma} \prod_{i=1}^v G(n, s) \xrightarrow{j^{(v)}} G(nv, s^v).$$

The reader is now warned that the rest of this section consists of routine homotopy theory designed to establish Lemmas (2.1) and (2.2); if these lemmas are found credible, the rest of this section may be omitted.

We now proceed to give the identification of  $\pi_r(G(n, k))$  with  $\pi_r^S$ . Following [15], we first define  $F(n)$  to be the subspace of  $G(n)$  which consists of maps leaving the base-point fixed. Similarly, we define  $F(n, k) = F(n) \cap G(n, k)$ . We have an obvious fibering

$$F(n, k) \longrightarrow G(n, k) \longrightarrow S^{n-1},$$

so

$$i_* : \pi_r(F(n, k)) \longrightarrow \pi_r(G(n, k))$$

is an isomorphism for  $r \leq n - 3$ .

The space  $F(n)$  is an  $H$ -space, under the following multiplication. We choose a fixed map  $\phi : S^{n-1} \rightarrow S^{n-1} \vee S^{n-1}$  of type  $(1, 1)$  (preserving the base-points); we use this to define the product  $\phi(f \vee g)$  of any two maps  $f, g$  in  $F(n)$ . Since the map  $\phi$  is determined up to a homotopy (if  $n \geq 3$ ), the product map  $\phi^* : F(n) \times F(n) \rightarrow F(n)$  is determined up to a homotopy. The product  $\phi^*$  is homotopy-associative and homotopy-commutative (assuming  $n \geq 3$ ), and has a homotopy-unit.

The product  $\phi^*$  maps  $F(n, s) \times F(n, t)$  into  $F(n, s + t)$ . Thus the arcwise-components of  $F(n)$  form a group under the product  $\phi^*$  (namely the group  $Z$ ). It follows that the arcwise-components  $F(n, s)$  are  $r$ -simple for each  $r$  (so that the choice of base-points for their homotopy groups is immaterial); moreover, the homotopy groups of the various arcwise-components may be identified, using left or right translations. Since the product is homotopy-commutative (assuming  $n \geq 3$ ) it is immaterial whether we use left translations or right translations.

We may identify the space  $F(n, 0)$  with  $\Omega^{n-1}(S^{n-1})$ , and so identify  $\pi_r(F(n, 0))$  with  $\pi_{n-1+r}(S^{n-1})$ . We give this identification explicitly. Let  $S^p \times S^q$  denote the reduced product  $S^p \times S^q / S^p \vee S^q$ . Suppose given a map

$$h : S^r, e \longrightarrow F(n, 0), \omega$$

where  $e$  is the base-point in  $S^r$  and  $\omega$  is the constant map at the base-point. Then we define the corresponding map

$$h' : S^{n-1} \times S^r \longrightarrow S^{n-1}$$

by the following formula:

$$(x, y)h' = (x)((y)h) \quad (x \in S^{n-1}, y \in S^r).$$

If  $r \leq n - 3$ , we may identify  $\pi_{n-1+r}(S^{n-1})$  with  $\pi_r^S$ . For this purpose it only remains to indicate our sign-convention for suspension. We define the suspension of  $g : S^p \rightarrow S^q$  to be  $1 \times g : S^1 \times S^p \rightarrow S^1 \times S^q$ .

We now return to the proof of Lemmas (2.1), (2.2). We begin by replacing the spaces  $G(n, s)$  by spaces  $F(n, s)$ . In fact, if we alter  $g$  inside  $G(n, t)$ , then we alter  $\bar{g}$  by a homotopy, and do not alter  $\bar{g}_*$ ; we may therefore suppose  $g \in F(n, t)$ , so that  $\bar{g}$  maps  $F(n)$  into  $F(n)$ . Similarly, the join product  $j$  maps  $F(n, s) \times F(m, t)$  into  $F(n + m, st)$ , provided that we take the base-point in  $S^{n-1} * S^{m-1}$  somewhere on the segment joining the base-points in  $S^{n-1}$  and  $S^{m-1}$ . Lemmas (2.1) and (2.2) will therefore follow from the following results.

LEMMA (2.5). *If  $\alpha \in \pi_r(F(n, s))$ ,  $g \in F(n, t)$  and  $r \leq n - 3$ , then*

$$[\bar{g}_* \alpha] = t[\alpha].$$

LEMMA (2.6). *If  $\alpha \in \pi_r(F(n, s))$ ,  $\beta \in \pi_r(F(m, t))$  and  $r \leq \text{Min}(n - 3, m - 3)$ , then*

$$[j_*(\alpha + \beta)] = t[\alpha] + s[\beta].$$

We begin with Lemma (2.5). This will evidently follow from the following result.

LEMMA (2.7). (i) If  $g \in F(n, t)$  we have a commutative diagram of the following form, in which  $i_s, i_{st}$  are the identifications made earlier in this section:

$$\begin{array}{ccc} \pi_r(F(n, 0)) & \xrightarrow{i_s} & \pi_r(F(n, s)) \\ \bar{g}_* \downarrow & & \downarrow \bar{g}_* \\ \pi_r(F(n, 0)) & \xrightarrow{i_{st}} & \pi_r(F(n, st)). \end{array}$$

(ii) Lemma (2.5) is true if  $s = 0$ .

*Proof.* We begin with part (i). The homomorphism  $\bar{g}_*i_s$  is induced by a map of spaces which sends  $f \in F(n, 0)$  into the following composite:

$$S^{n-1} \xrightarrow{\phi} S^{n-1} \vee S^{n-1} \xrightarrow{f \vee h} S^{n-1} \xrightarrow{g} S^{n-1}$$

(Here  $h$  is a fixed map of degree  $s$ .) The homomorphism  $i_{st}\bar{g}_*$  is induced by a map of spaces which sends  $f \in F(n, 0)$  into the following composite:

$$S^{n-1} \xrightarrow{\phi} S^{n-1} \vee S^{n-1} \xrightarrow{f \vee k} S^{n-1}.$$

(Here  $k$  is a fixed map of degree  $st$ .) If we take  $k = hg$ , the two maps of  $F(n, 0)$  become equal. This proves part (i).

We turn to part (ii). Let

$$y \longrightarrow (y)h : S^r, e \longrightarrow F(n, 0), \omega$$

be a representative map for  $\alpha$ . Then a representative map  $r_1$  for  $[\alpha]$  is

$$(x, y) \longrightarrow (x)((y)h) : S^{n-1} \times S^r \longrightarrow S^{n-1}.$$

Also a representative map for  $\bar{g}_*\alpha$  is

$$y \longrightarrow ((y)h)\bar{g} : S^r, e \longrightarrow F(n, 0), \omega.$$

Therefore a representative map  $r_2$  for  $[\bar{g}_*\alpha]$  is

$$(x, y) \longrightarrow (x)[((y)h)\bar{g}] : S^{n-1} \times S^r \longrightarrow S^{n-1}.$$

Evidently  $r_2 = r_1g$ . But since  $g$  is a map of degree  $t$ , in the stable homotopy group  $\pi_r^S$  we have  $[r_2] = t[r_1]$ ; that is,  $[\bar{g}_*\alpha] = t[\alpha]$ . This proves part (ii).

We now turn to Lemma (2.6). We recall that in defining  $j$  we have regarded  $S^{n+m-1}$  as  $S^{n-1} * S^{m-1}$ . However, for our purposes it makes no difference if we now replace  $S^{n-1} * S^{m-1}$  by the quotient

$$\frac{S^{n-1} * S^{m-1}}{(S^{n-1} * e') \cup (e * S^{m-1})}$$

Here  $e, e'$  denote the base-points in  $S^{n-1}, S^{m-1}$ . Since  $S^{n-1} * e'$  and  $e * S^{m-1}$  are cells with only the segment  $e * e'$  in common, the quotient map

$$S^{n-1} * S^{m-1} \longrightarrow \frac{S^{n-1} * S^{m-1}}{(S^{n-1} * e') \cup (e * S^{m-1})}$$

is a homotopy equivalence. We may interpret the quotient space as  $S^{n-1} \times S^1 \times S^{m-1}$ .

If  $f, g$  are maps of  $S^{n-1}, S^{m-1}$  which preserve base-points, then  $f * g$  passes to the quotient, and may be interpreted as  $f \times 1 \times g$ . In what follows, then, the 'join' symbol  $*$  will be interpreted as referring to these quotient spaces and maps.

We now remark that since  $j_*$  is a homomorphism, Lemma (2.6) will be proved if we can calculate  $j_*(\alpha + 0)$  and  $j_*(0 + \beta)$ . This calculation is equivalent to calculating the homomorphisms induced by two 'translation' maps. In fact, let  $f, g$  be fixed maps in  $F(n, s), F(m, t)$ ; then we can define maps

$$f^L : F(m, t) \longrightarrow F(n + m, st)$$

$$g^R : F(n, s) \longrightarrow F(n + m, st)$$

by  $(h)f^L = f * h, (h)g^R = h * g$ . We have

$$j_*(\alpha + 0) = g_*^R(\alpha), j_*(0 + \beta) = f_*^L(\beta).$$

It will thus be sufficient to prove the following results.

LEMMA (2.8). (i) *If  $f \in F(n, s)$  we have a commutative diagram of the following form, in which  $i, i_*$  are the identifications made earlier in this section:*

$$\begin{array}{ccc} \pi_r(F(m, 0)) & \xrightarrow{i} & \pi_r(F(m, t)) \\ f_*^L \downarrow & & \downarrow f_*^L \\ \pi_r(F(n + m, 0)) & \xrightarrow{i_*} & \pi_r(F(n + m, st)). \end{array}$$

(ii) *Similarly for  $g_*^R$ .*

(iii) *If  $g \in F(m, t), \alpha \in \pi_r(F(n, 0))$  (so that  $s = 0$ ), then*

$$[g_*^R \alpha] = i[\alpha].$$

(iv) *Similarly for  $f_*^L \beta$ .*

*Proof.* We begin with part (i). The homomorphism  $f_*^L i$  is induced by a map of spaces which sends  $h \in F(m, 0)$  into

$$f * (\phi(h \vee k)),$$

where  $k$  is a fixed map of degree  $t$ . Owing to the fact that we are using the 'quotient' join, we can identify  $S^{n-1} * (S^{m-1} \vee S^{m-1})$  with  $(S^{n-1} * S^{m-1}) \vee (S^{n-1} * S^{m-1})$ , and so write  $f * (\phi(h \vee k))$  in the form

$$(1 * \phi)((f * h) \vee (f * k)).$$

The homomorphism  $i_* f_*^L$  is induced by a map of spaces which sends  $h \in F(m, 0)$  into

$$\phi'((f * h) \vee k'),$$

where  $\phi'$  is a map of type  $(1, 1)$  and  $k'$  is a fixed map of degree  $st$ . If we take  $\phi' = 1 * \phi, k' = f * k$  the two maps of  $F(m, 0)$  become equal. This proves part (i); part (ii) is closely similar.

We now turn to part (iii). Let

$$h : S^r, e \longrightarrow F(n, 0), \omega$$

be a representative map for  $\alpha$ . Then a representative map  $r_1$  for  $[\alpha]$  is given by

$$(x, y) \longrightarrow (x)((y)h) : S^{n-1} \times S^r \longrightarrow S^{n-1}.$$

A representative map for  $g_*^R \alpha$  is given by assigning to each point  $y \in S^r$  the map

$$(x, u, v) \longrightarrow ((x)(y)h), u, (v)g) : S^{n-1} \times S^1 \times S^{m-1} \longrightarrow S^{n-1} \times S^1 \times S^{m-1}.$$

Therefore a representative map  $r_2$  for  $[g_*^R \alpha]$  is given by

$$(x, u, v, y) \longrightarrow ((x)((y)h), u, (v)g) : S^{n-1} \times S^1 \times S^{m-1} \times S^r \longrightarrow S^{n-1} \times S^1 \times S^{m-1}.$$

This map may be factored in the form

$$\begin{aligned} (x, u, v, y) &\xrightarrow{\rho} (u, v, x, y) \xrightarrow{1 \times 1 \times r_1} (u, v, (x)((y)h)) \\ &\xrightarrow{\sigma} ((x)((y)h), u, v) \xrightarrow{1 \times 1 \times g} ((x)((y)h), u, (v)g). \end{aligned}$$

According to our definition of suspension, the stable class  $[1 \times 1 \times r_1]$  is equal to  $[r_1]$ . The permutation maps  $\rho$  and  $\sigma$  have the same degree  $(-1)^{(n-1)m}$ , and the map  $1 \times 1 \times g$  has degree  $t$ . Therefore in the stable homotopy group  $\pi_r^S$  we have  $[r_2] = t[r_1]$ , that is,  $[g_*^R \alpha] = t[\alpha]$ . This proves part (iii); the proof of part (iv) is closely similar, except that we do not need any permutation maps in the last step.

This completes the proof of Lemma (2.8), and establishes all the results of this section.

### §3. PROOF OF 'DOLD'S THEOREM MOD $k$ '

In this section we shall prove Theorem (1.1).

We begin by fixing some notation. Let  $\xi, \xi'$  be sphere bundles over  $X$ ; I shall allow myself to speak of 'a fibrewise map  $f: \xi \rightarrow \xi'$ '; this is an abuse of language, or not, according to one's precise definition of a sphere-bundle.

Let  $f: \xi \rightarrow \xi'$  be a fibrewise map of sphere bundles over  $X$ , and let  $g: \eta \rightarrow \eta'$  be another such. Then we can clearly construct their Whitney sum

$$f \oplus g : \xi \oplus \eta \rightarrow \xi' \oplus \eta'$$

by taking joins on each fibre; it is again a fibrewise map of sphere bundles over  $X$ . By iterating this procedure we can construct Whitney multiples

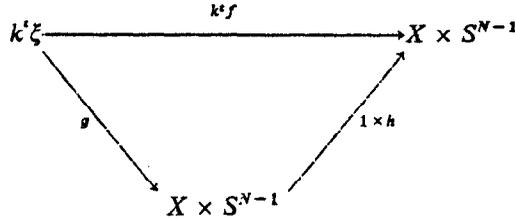
$$mf : m\xi \rightarrow m\xi',$$

where  $m$  is any non-negative integer.

We shall write  $X \times S^{n-1}$  to indicate a product bundle over  $X$ .

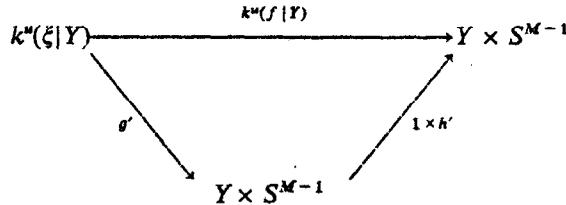
Our first lemma contains the main part of the proof. We shall suppose given (i) an integer  $k > 0$ , (ii) an  $(n-1)$ -sphere bundle  $\xi$  over a finite  $CW$ -complex  $X$  such that  $\dim(X) \leq n-3$ , and (iii) a fibrewise map  $f: \xi \rightarrow X \times S^{n-1}$  of degree  $\pm k$  on each fibre. It clearly follows that we can orient  $\xi$  so that the fibrewise map  $f$  has degree  $k$  on each fibre, and we will suppose this done.

LEMMA (3.1). *There exists (i) an integer  $t \geq 0$ , (ii) a fibrewise map  $g : k^t \xi \rightarrow X \times S^{N-1}$  (where  $N = nk^t$ ) of degree 1 on each fibre, and (iii) a map  $h : S^{N-1} \rightarrow S^{N-1}$ , such that the following diagram of fibrewise maps is fibre homotopy commutative:*

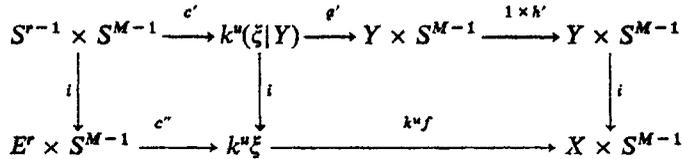


*Remark.* It is clear that the degree of  $h$  must be  $k^{(k^t)}$ .

*Proof.* *CW-complexes* may be constructed by an inductive process, in which one attaches cells to what has already been constructed. The present proof (like many proofs about *CW-complexes*) consists of a corresponding induction. If  $X$  consists solely of 0-cells, then the result is clearly true, with  $t = 0$ . Let us suppose that  $X$  is formed by attaching a cell  $E^r$  to the subcomplex  $Y$ , with characteristic map  $c : E^r, S^{r-1} \rightarrow X, Y$ ; and let us suppose, as our inductive hypothesis, that the result is true for  $Y$ ; that is, we can find a fibre homotopy commutative diagram of the following form:



(Here  $g'$  is supposed to be a map of degree 1 on each fibre.) Consider the induced bundle  $c^*(k^u \xi)$  over  $E^r$ ; it can be represented as a product bundle  $E^r \times S^{M-1}$ ; we now have the following fibre homotopy commutative diagram of fibrewise maps:



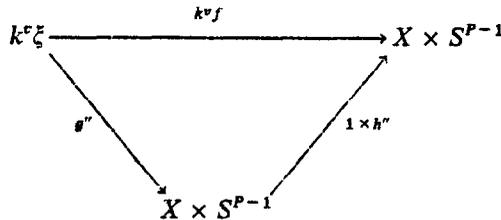
(Here  $c', c''$  lie over  $c$ .) The map  $g'c'$  is equivalent to a map  $\theta : S^{r-1} \rightarrow G(M, 1)$ , where  $G(M, 1)$  denotes the space of all maps from  $S^{M-1}$  to  $S^{M-1}$  of degree 1 (as in §2). If  $r = 1$  then  $\theta$  can be extended over  $E^r$ , since  $G(M, 1)$  is arcwise-connected; we proceed to examine the case  $r > 1$ . Let  $K$  be the degree of  $h'$ , which is  $k^{(k^u)}$ , as remarked above; and let us define  $\bar{h}' : G(m, 1) \rightarrow G(m, K)$  by composition with  $h'$ , as in §2; then the diagram shows that  $\bar{h}'\theta : S^{r-1} \rightarrow G(m, K)$  can be extended over  $E^r$ . Now by Lemma (2.1) we have  $[\bar{h}'\theta] = K[\theta]$ , so  $K[\theta] = 0$ .

If we take the Whitney sum of the diagram with itself  $m$  times, we evidently replace  $\theta(x)$  by  $\theta(x) * \theta(x) * \dots * \theta(x)$  ( $m$  factors). According to Lemma (2.4) this replaces  $[\theta]$  by

$m[\theta]$ . If we take  $m = K$ , we replace  $[\theta]$  by 0, and therefore we can find a fibrewise extension of  $K(g'c')$  over  $E^r \times S^{P-1}$  (where  $P = KM$ ). This of course defines a fibrewise map

$$g'' : Kk^u\xi \longrightarrow X \times S^{P-1}$$

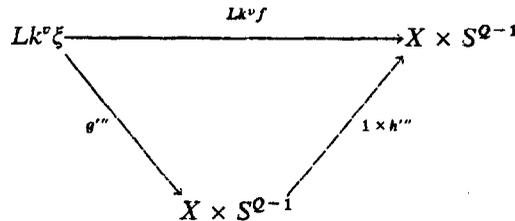
extending  $Kg'$ . (If  $r = 1$  we arrive at the same conclusion with  $K$  replaced by 1.) At this stage we have the following diagram of fibrewise maps:



Here  $k^v = Kk^u$ ;  $h''$  is the join of  $K$  copies of  $h'$ ; and the diagram is known to be fibre homotopy commutative on  $k^v\xi|Y$ .

The obstruction to extending a fibrewise homotopy over  $I \times E^r$  is a map  $\phi$  from the boundary of  $I \times E^r$  to  $G(P, L)$ , where  $L$  is the present degree on the fibres (that is,  $k^{(k^v)}$ .) The map  $\phi$  represents an element of  $\pi_r(G(P, L))$ . We can of course alter  $g''$  by using any element  $\alpha$  of  $\pi_r(G(P, 1))$ , and this alters  $[\phi]$  by  $[\tilde{h}''\alpha] = L[\alpha]$  (Lemma (2.1)).

Let us now investigate the effect of taking the Whitney sum of this diagram with itself  $m$  times. We evidently replace  $\phi(x)$  by  $\phi(x) * \phi(x) * \dots * \phi(x)$  ( $m$  factors). According to Lemma (2.4), this replaces  $[\phi]$  by  $mL^{m-1}[\phi]$ . Since  $L$  is replaced by  $L^m$ , we can alter the obstruction  $mL^{m-1}[\phi]$  by  $L^m[\alpha]$ . We now take  $m = L$ ; the obstruction becomes  $L^L[\phi]$  modulo  $L^L[\alpha]$ , that is, zero. We conclude that we can construct the following fibre homotopy commutative diagram of fibrewise maps:



Here  $Q = LP$ ,  $g'''$  has degree 1 on each fibre and  $h'''$  is the join of  $L$  copies of  $h''$ . Since  $Lk^v$  is a power of  $k$ , this completes the induction and proves Lemma (3.1).

**COROLLARY (3.2).** *Suppose given (i) an integer  $k > 0$ , (ii) an  $(n - 1)$ -sphere bundle  $\xi$  over a finite CW-complex  $X$ , and (iii) a fibrewise map  $f: \xi \rightarrow X \times S^{n-1}$  of degree  $\pm k$  on each fibre. Then there exists an integer  $t$  such that the bundle  $k^t\xi$  is fibre homotopy equivalent to a trivial bundle.*

*Proof.* The result is true for  $k = 1$ , by Dold's theorem [11]. We may thus suppose  $k > 1$ . For a suitable choice of  $s$  the bundle  $k^s\xi$  and the fibrewise map  $k^s f: k^s\xi \rightarrow X \times S^{N-1}$  (where  $N = nk^s$ ) satisfy the dimensional restriction of Lemma (3.1). (The degree of  $k^s f$  on each fibre is  $k^{(k^s)}$ .) The conclusion of Lemma (3.1) provides a fibrewise map  $g$  of degree 1 on

each fibre, and by Dold's theorem  $g$  must be a fibre homotopy equivalence. That is, there exists an integer  $t$  such that  $k^{(tk^*)}\xi$  is fibre homotopy equivalent to a trivial bundle. This completes the proof.

We will now deduce Theorem (1.1) from Corollary (3.2).

*Proof of Theorem (1.1).* The result is true for  $k = 1$ , by Dold's theorem [11]. We may thus suppose  $k > 1$ . Suppose given a fibrewise map  $f: \xi \rightarrow \eta$  of degree  $\pm k$  on each fibre. There exists a sphere bundle  $\zeta$  such that  $\eta \oplus \zeta = \tau$ , where  $\tau$  is a trivial bundle. The map

$$f \oplus 1: \xi \oplus \zeta \longrightarrow \eta \oplus \zeta = \tau$$

has degree  $\pm k$  on each fibre. By Corollary (3.2), there exists an integer  $t$  such that  $k^t\xi \oplus k^t\zeta$  is fibre homotopy equivalent to  $k^t\tau$ . Adding  $k^t\eta$ , we see that  $k^t\xi \oplus k^t\tau$  is fibre homotopy equivalent to  $k^t\eta \oplus k^t\tau$ . That is,  $k^t\xi$  and  $k^t\eta$  are stably fibre homotopy equivalent. Now since  $k > 1$ , we can make the dimension of  $k^t\xi$  as large as we please by increasing  $t$ ; in particular we can make it so large that 'stable fibre homotopy equivalence' implies 'fibre homotopy equivalence' (cf. [6], pp. 293, 294). This completes the proof.

§4. APPLICATION OF 'DOLD'S THEOREM MOD  $k$ '

In this section we shall prove Theorems (1.3) and (1.4).

LEMMA 4.1. *Assume that  $X$  is a finite CW-complex and that  $y \in K_R(X)$  is a linear combination of  $O(1)$  bundles. Then there exists  $e$  (depending only on  $\dim(X)$ ) such that*

$$k^e(\Psi^k - 1)y = 0 \quad \text{in } K_R(X).$$

*Proof.* Since  $k^e(\Psi^k - 1)y$  is linear in  $y$ , it is sufficient to consider the case in which  $y$  is an  $O(1)$  bundle. In this case it is sufficient to consider the case in which  $y$  is the canonical real line bundle over  $RP^n$ , because any other  $O(1)$  bundle can be induced from this by a map  $f: X \rightarrow RP^n$ , where  $n = \dim(X)$ . We now divide cases according to the parity of  $k$ . If  $k$  is odd,  $\Psi^k(y) = y$  by [1, (5.1) (iii) or (7.4)(i)], and therefore  $(\Psi^k - 1)y = 0$ . If  $k$  is even,  $(\Psi^k - 1)y = 1 - y$ , and by [1, (7.4)] there exists  $e$  depending only on  $n$  such that

$$2^e(\Psi^k - 1)y = 0.$$

Since  $k$  is even,

$$k^e(\Psi^k - 1)y = 0.$$

This completes the proof.

*Proof of Theorem (1.3).* The result is trivial for  $k = 0$ ; also by [1] we have  $\Psi^{-k} = \Psi^k$ , so we may assume  $k > 0$ .

Since  $k^e(\Psi^k - 1)y$  is linear in  $y$ , it is sufficient to prove the result when  $y$  is an  $O(1)$  bundle and when  $y$  is an  $O(2)$  bundle. Lemma (4.1) deals with the case in which  $y$  is an  $O(1)$  bundle, so we may suppose that  $y$  is an  $O(2)$  bundle.

We will now recall something of the representation-theory of  $O(2)$ .  $O(2)$  is a group of matrices acting on column vectors  $(x_1, x_2)'$ . Let us write

$$X_1 + iX_2 = (x_1 + ix_2)';$$

thus  $X_1$  and  $X_2$  are polynomials of degree  $r$  in  $x_1$  and  $x_2$ . Each matrix in  $O(2)$  induces an orthogonal transformation of  $(X_1, X_2)'$ ; we have thus defined a representation  $\mu_r : O(2) \rightarrow O(2)$ . We also have  $\lambda_2$ , the determinant representation of  $O(2)$ , and  $\lambda_0$ , the trivial representation of degree 1.

By checking the characters we easily find:

$$\Psi^k = \begin{cases} \mu_k & (k \text{ odd}) \\ \mu_k - \lambda_2 + \lambda_0 & (k \text{ even}). \end{cases}$$

Now we have

$$(\lambda_2 - \lambda_0)y = (\lambda_2 y) - 1,$$

where  $\lambda_2 y$  is an  $O(1)$ -bundle. By the argument of Lemma (4.1), if  $k$  is even there exists  $e$  such that

$$k^e(\lambda_2 - \lambda_0)y = 0 \quad \text{in } K_R(X).$$

It remains then to prove that there exists  $e$  such that

$$k^e(\mu_k - 1)y$$

maps to zero in  $J(X)$ .

Consider the map  $\phi : S^1 \rightarrow S^1$  defined by

$$\phi(x_1 + ix_2) = (x_1 + ix_2)^k.$$

By construction,  $\phi$  is equivariant with respect to the homomorphism  $\mu_k : O(2) \rightarrow O(2)$  of groups operating on  $S^1$ . Therefore it defines a map of bundles, say

$$f : y \longrightarrow \mu_k y.$$

The map  $f$  has degree  $\pm k$  on each fibre. Therefore Theorem (1.1) applies; there is an integer  $e$  such that the multiples  $k^e y, k^e \mu_k y$  are fibre homotopy equivalent. Thus  $k^e(\mu_k y - y)$  maps to zero in  $J(X)$ . This completes the proof.

*Proof of Theorem (1.4).* We must recall some facts about the representability of our functors; the following details are taken from [6, pp. 293, 294]. Let  $O(m)$  be the orthogonal group, and let  $H(m)$  be the monoid of homotopy equivalences from  $S^{m-1}$  to  $S^{m-1}$ ; then we have an inclusion map  $i(m) : O(m) \rightarrow H(m)$ . By passing to classifying spaces (in the sense of [12]) we obtain

$$Bi(m) : BO(m) \rightarrow BH(m).$$

Consider the induced function

$$(Bi(m))_* : \pi(X, BO(m)) \longrightarrow \pi(X, BH(m)),$$

where  $\pi(X, Y)$  means the set of homotopy classes of maps from  $X$  to  $Y$ , and  $m$  is taken sufficiently large, depending on  $\dim(X)$ . Then there is a natural  $(1-1)$  correspondence between  $\tilde{J}(X)$  and  $\text{Im}(Bi(m))_*$ .

Now let  $W$  be the Cartesian product of  $n$  copies of  $S^2$ ; say  $W = S^2 \times S^2 \times \dots \times S^2$ . We shall argue by considering the relation between  $S^{2n}$  and  $W$ . Let  $V$  be the set of points in  $W$

which have at least one co-ordinate at the base-point in  $S^2$ ; then we have  $W = V \cup E^{2n}$ . The attaching map  $\alpha : S^{2n-1} \rightarrow V$  may be used to start a sequence of cofiberings, which up to homotopy type is the following:

$$S^{2n-1} \xrightarrow{\alpha} V \xrightarrow{i} W \xrightarrow{q} S^{2n} \xrightarrow{S\alpha} SV.$$

(Here  $SV$  and  $S\alpha$  are the suspensions of  $V$  and  $\alpha$ .) The cofibering

$$W \xrightarrow{q} S^{2n} \xrightarrow{S\alpha} SV$$

induces the following sequence of sets:

$$\pi(W, BH(m)) \xleftarrow{q^*} \pi(S^{2n}, BH(m)) \xleftarrow{(S\alpha)^*} \pi(SV, BH(m)).$$

This sequence is exact, in the sense that if  $q^*x = 0$ , then  $x = (S\alpha)^*y$  for some  $y$ . But it is well known [14, Theorem (4.1)] that  $S\alpha : S^{2n} \rightarrow SV$  is homotopic to the constant map; therefore  $q^*x = 0$  implies  $x = 0$ . This shows that the map

$$q^* : \bar{J}(S^{2n}) \longrightarrow \bar{J}(W)$$

is monomorphic. By adding  $Z$ , we see that

$$q^* : J(S^{2n}) \longrightarrow J(W)$$

is monomorphic.

Now suppose given a class  $y \in K_R(S^{2n})$  lying in the image of

$$r : K_C(S^{2n}) \longrightarrow K_R(S^{2n}),$$

so that  $y = rz$ , where  $z \in K_C(S^{2n})$ . Then in  $K_R(W)$  we have  $q^*y = rq^*z$ . Now every element in  $K_C(W)$  is a linear combination of complex line bundles; in particular,  $q^*z$  is such a linear combination. Therefore  $q^*y = rq^*z$  is a linear combination of  $SO(2)$  bundles. Theorem (1.3) thus applies to  $q^*y$ , and there exists  $e = e(k, y)$  such that the element

$$k^e(\Psi^k - 1)q^*y = q^*k^e(\Psi^k - 1)y$$

maps to zero in  $J(W)$ . Since we have shown that

$$q^* : J(S^{2n}) \longrightarrow J(W)$$

is monomorphic, it follows that  $k^e(\Psi^k - 1)y$  is zero in  $J(S^{2n})$ . This completes the proof.

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*Department of Mathematics,  
University of Manchester.*