The nerve theorem and Grothendieck’s hypothesis on homotopy types

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1. Monadic squares

2. Nerves and theories

3. Higher categories and wreath products

4. Grothendieck’s hypothesis and $\Theta_n$-spaces
A *monadic square* is a commutative diagram of functors

\[
\begin{array}{ccc}
\mathcal{E}_1' & \xrightarrow{G'} & \mathcal{E}_2' \\
U_1 & = & U_2 \\
\mathcal{E}_1 & \xrightarrow{G} & \mathcal{E}_2
\end{array}
\]

such that

(i) \( U_1, U_2 \) are monadic functors with left adjoints \( F_1, F_2 \);

(ii) the induced 2-cell \( \phi = \epsilon_2 G' F_1 \circ F_2 G \eta_1 \) (the “mate”)

\[
\begin{array}{ccc}
\mathcal{E}_1' & \xrightarrow{G'} & \mathcal{E}_2' \\
F_1 & \phi & F_2 \\
\mathcal{E}_1 & \xrightarrow{G} & \mathcal{E}_2
\end{array}
\]

is invertible.
A monadic square is a commutative diagram of functors

\[
\begin{array}{ccc}
\mathcal{E}'_1 & \xrightarrow{G'} & \mathcal{E}'_2 \\
\downarrow U_1 & = & \downarrow U_2 \\
\mathcal{E}_1 & \xrightarrow{G} & \mathcal{E}_2
\end{array}
\]

such that

(i) \( U_1, U_2 \) are monadic functors with left adjoints \( F_1, F_2 \);

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\downarrow U_1 & = & \downarrow U_2 \\
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such that

(i) $U_1, U_2$ are monadic functors with left adjoints $F_1, F_2$;
(ii) the induced 2-cell $\phi = \epsilon_2 G' F_1 \circ F_2 G \eta_1$ (the “mate”)

is invertible.
Let \((T_1, \mu_1, \eta_1), (T_2, \mu_2, \eta_2)\) be monads on \(E_1, E_2\) respectively.

A (strong) monad morphism \((G, \psi) : (E_1, T_1) \to (E_2, T_2)\) is a functor \(G : E_1 \to E_2\) together with an (invertible) 2-cell \(\psi : T_2 G \Rightarrow G T_1\) such that \(G \eta_1 = \psi \circ \eta_2 G\) and \(\psi \circ \mu_2 G = G \mu_1 \circ G \psi T_1 \circ T_2 \psi G\).

A strong monad morphism \((G, \psi)\) induces a monadic square

\[
\begin{array}{ccc}
\text{Alg}_{T_1} & \xrightarrow{G'} & \text{Alg}_{T_2} \\
U_1 \downarrow & = & \downarrow U_2 \\
E_1 & \xrightarrow{G} & E_2
\end{array}
\]

with \(G'(X, \xi : T_1 X \to X) = (GX, G\xi \circ \psi : T_2 GX \to GX)\).

Conversely, a monadic square induces a strong monad morphism from which it derives up to canonical equivalence.
Let \((T_1, \mu_1, \eta_1), (T_2, \mu_2, \eta_2)\) be monads on \(\mathcal{E}_1, \mathcal{E}_2\) respectively.

A (strong) monad morphism \((G, \psi) : (\mathcal{E}_1, T_1) \to (\mathcal{E}_2, T_2)\) is a functor \(G : \mathcal{E}_1 \rightarrow \mathcal{E}_2\) together with an (invertible) 2-cell \(\psi : T_2 G \Rightarrow GT_1\) such that \(G \eta_1 = \psi \circ \eta_2 G\) and \(\psi \circ \mu_2 G = G \mu_1 \circ G \psi T_1 \circ T_2 \psi G\).

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Conversely, a monadic square induces a strong monad morphism from which it derives up to canonical equivalence.
The nerve theorem and Grothendieck’s hypothesis on homotopy types
Monadic squares

**Proposition**

In any monadic square like above, if $G$ is faithful (resp. fully faithful resp. an equivalence) then so is $G'$.

**Proposition**

For a fully faithful functor $G$, the essential image factorisation of $G$ decomposes the monadic square into two monadic squares

$$
\begin{array}{ccc}
\mathcal{E}_1' & \xrightarrow{\sim} & \text{Im}(G) \times_{\mathcal{E}_2} \mathcal{E}_2' \\
U_1 \downarrow & & \downarrow \\
\mathcal{E}_1 & \xrightarrow{\sim} & \text{Im}(G) \leftarrow \mathcal{E}_2
\end{array}
$$

In particular, the essential image of $G'$ is given by restriction of the monadic functor $U_2$ to the essential image of $G$. 
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\begin{array}{ccc}
\mathcal{E}_1' & \sim & \text{Im}(G) \times \mathcal{E}_2' \\
\downarrow U_1 & & \downarrow \text{ff} \\
\mathcal{E}_1 & \sim & \text{Im}(G) \\
\end{array}
\]

In particular, the essential image of $G'$ is given by restriction of the monadic functor $U_2$ to the essential image of $G$. 
A category with arities \((\mathcal{E}, \Theta_0)\) is a category \(\mathcal{E}\) equipped with a small dense subcategory \(i_0 : \Theta_0 \hookrightarrow \mathcal{E}\), i.e. the induced nerve functor \(\nu_0 : \mathcal{E} \to \widehat{\Theta}_0 : X \mapsto \mathcal{E}(i_0(-), X)\) is fully faithful.

For each object \(X\) of \(\mathcal{E}\) the functor \(\xi_X : i_0/X \to \Theta_0 \hookrightarrow \mathcal{E}\) induces a colimit cocone \(\text{colim}_{i_0/X} \xi_X \xrightarrow{\approx} X\).

A monad with arities on \((\mathcal{E}, \Theta_0)\) is a monad \(T\) such that the composite functor \(\nu_0 \circ T\) preserves the \(\Theta_0\)-colimit cones.

The theory \(\Theta_T\) induced by a monad with arities \(T\) is obtained by factoring \(\Theta_0 \xrightarrow{i_0} \mathcal{E} \xrightarrow{F_T} \text{Alg}_T\) into a bijective-on-objects functor \(j : \Theta_0 \to \Theta_T\) followed by a fully faithful functor \(\Theta_T \to \text{Alg}_T\).

\(\Theta_T\) is called homogeneous if \(\Theta_T\) admits a generic/free factorisation system \(\Theta_T = (\Theta_{T,\text{gen}}, \Theta_0)\).
A **category with arities** \((\mathcal{E}, \Theta_0)\) is a category \(\mathcal{E}\) equipped with a small dense subcategory \(i_0 : \Theta_0 \hookrightarrow \mathcal{E}\), i.e. the induced **nerve functor** \(\nu_0 : \mathcal{E} \to \hat{\Theta}_0 : X \mapsto \mathcal{E}(i_0(-), X)\) is fully faithful.

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Example (algebraic theories and symmetric operads)

Consider sets with arities $\mathcal{I}_0$ the subcategory of finite sets.

- A monad $\mathcal{T}$ has arities $\mathcal{I}_0$ iff $\mathcal{T}$ preserves filtered colimits;
- $\Theta_{\mathcal{T}}$ is (the dual of) Lawvere’s algebraic theory for $\mathcal{T}$-algebras;
- $\Theta_{\mathcal{T}}$ is homogeneous iff $\mathcal{T}$ is induced by a symmetric operad.

Theorem (B. ’02, Leinster ’04, Weber ’07, Mellies ’10)

For a monad with arities $\mathcal{T}$ on $(\mathcal{E}, \Theta_0)$, the theory $\Theta_{\mathcal{T}}$ is dense in $\text{Alg}_{\mathcal{T}}$. The essential image of $\nu_{\mathcal{T}} : \text{Alg}_{\mathcal{T}} \to \hat{\Theta}_{\mathcal{T}}$ is spanned by those $X : \Theta_{\mathcal{T}}^{\text{op}} \to \text{Sets}$ whose restriction $j^*X$ belongs to $\text{Im}(\nu_0)$.

Remark

If $\mathcal{E} = \hat{\mathcal{C}}$ and $\Theta_0$ contains the representables, the essential image of $\nu_0 : \hat{\mathcal{C}} \to \hat{\Theta}_0$ is spanned by sheaves on $\Theta_0$. The essential image of $\nu_{\mathcal{T}} : \text{Alg}_{\mathcal{T}} \to \hat{\Theta}_{\mathcal{T}}$ is then given by a restricted sheaf condition.
Example (algebraic theories and symmetric operads)

Consider sets with arities $\mathcal{I}_0$ the subcategory of finite sets.
- A monad $T$ has arities $\mathcal{I}_0$ iff $T$ preserves filtered colimits;
- $\Theta_T$ is (the dual of) Lawvere’s algebraic theory for $T$-algebras;
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Theorem (B. ’02, Leinster ’04, Weber ’07, Mellies ’10)

For a monad with arities $T$ on $(\mathcal{E}, \Theta_0)$, the theory $\Theta_T$ is dense in $\text{Alg}_T$. The essential image of $\nu_T : \text{Alg}_T \to \hat{\Theta}_T$ is spanned by those $X : \Theta_T^{\text{op}} \to \text{Sets}$ whose restriction $j^* X$ belongs to $\text{Im}(\nu_0)$.

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**Example (algebraic theories and symmetric operads)**

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- A monad $T$ has arities $\mathcal{T}_0$ iff $T$ preserves filtered colimits;
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**Remark**

If $\mathcal{E} = \hat{\mathcal{C}}$ and $\Theta_0$ contains the representables, the essential image of $\nu_0 : \hat{\mathcal{C}} \to \hat{\Theta}_0$ is spanned by sheaves on $\Theta_0$. The essential image of $\nu_T : \text{Alg}_T \to \hat{\Theta}_T$ is then given by a *restricted* sheaf condition.
Example (algebraic theories and symmetric operads)

Consider sets with arities $T_0$ the subcategory of finite sets.
- A monad $T$ has arities $T_0$ iff $T$ preserves filtered colimits;
- $\Theta_T$ is (the dual of) Lawvere’s algebraic theory for $T$-algebras;
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Theorem (B. ’02, Leinster ’04, Weber ’07, Mellies ’10)

For a monad with arities $T$ on $(\mathcal{E}, \Theta_0)$, the theory $\Theta_T$ is dense in $\text{Alg}_T$. The essential image of $\nu_T : \text{Alg}_T \to \hat{\Theta}_T$ is spanned by those $X : \Theta^{\text{op}}_T \to \text{Sets}$ whose restriction $j^*X$ belongs to $\text{Im}(\nu_0)$.

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Consider sets with arities $\mathcal{I}_0$ the subcategory of finite sets.

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Proof of the nerve theorem.

Since $T$ is a monad with arities on $(\mathcal{E}, \Theta_0)$ the square

$$
\begin{array}{ccc}
\text{Alg}_T & \overset{\nu_T}{\longrightarrow} & \hat{\Theta}_T \\
U_T & \downarrow & \downarrow j^* \\
\mathcal{E} & \overset{\nu_0}{\longrightarrow} & \hat{\Theta}_0
\end{array}
$$

is pseudo-monadic and $\nu_0$ is fully faithful.

A theory on $(\mathcal{E}, \Theta_0)$ is a bijective-on-objects faithful functor $j : \Theta_0 \to \Theta_T$ such that $j^*j_!$ preserves the essential image of $\nu_0$.

Theorem (B. ’02, Mellies ’10)

There is a canonical one-to-one correspondence between monads with arities on $(\mathcal{E}, \Theta_0)$ and theories on $(\mathcal{E}, \Theta_0)$. 
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U_T & \downarrow \text{diag} & \downarrow j^* \\
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Theorem (B. ’02, Mellies ’10)

There is a canonical one-to-one correspondence between monads with arities on $(\mathcal{E}, \Theta_0)$ and theories on $(\mathcal{E}, \Theta_0)$. 
Each *finite level tree* $S$ defines a globular set $S_*$ with $\text{ht}(S) = \dim(S_*)$ (Batanin’s star-construction ’98).

The category of arities $\Theta_0$ is the full subcategory of $\hat{G}$ spanned by the $S_*$ where $S$ runs through the set of finite level trees.

The Grothendieck topology on $\Theta_0$ induced by the nerve $\nu_0 : \hat{G} \to \hat{\Theta}_0$ has the characteristic property that a presheaf $X$ on $\Theta_0$ is a sheaf if and only if $X$ transforms the canonical colimit cones

$$\text{colim}_{\sigma \in \text{el}(S_*)} \sigma \xrightarrow{\cong} S_*$$

into limit cones.

A theory $\Theta_A$ on $(\hat{G}, \Theta_0)$ is called *globular*. The presheaves $X$ such that $j^*X$ is a sheaf are called $\Theta_A$-*models*. According to the nerve theorem they correspond to $A$-algebras for a monad $A$ on $\hat{G}$. 
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Each *finite level tree* $S$ defines a globular set $S_*$ with $\text{ht}(S) = \dim(S_*)$ (Batanin’s star-construction ’98).

The category of arities $\Theta_0$ is the full subcategory of $\hat{\mathcal{G}}$ spanned by the $S_*$ where $S$ runs through the set of finite level trees.

The Grothendieck topology on $\Theta_0$ induced by the nerve $\nu_0 : \hat{\mathcal{G}} \to \hat{\Theta}_0$ has the caracteristic property that a presheaf $X$ on $\Theta_0$ is a sheaf if and only if $X$ transforms the canonical colimit cones

$$\text{colim}_{\sigma \in \text{el}(S_*)} \sigma \xrightarrow{\sim} S_*$$

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A globular theory $\Theta_A$ is called \textit{homogeneous} if there is a factorisation system $\Theta_A = (\Theta_{A, \text{gen}}, \Theta_0)$ such that each \textit{generic} operator $\phi : S \to T$ satisfies $\text{ht}(S) \geq \text{ht}(T)$.

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- There is a canonical one-to-one correspondence between homogeneous globular theories and globular operads;
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Example (\( n=1 \), Segal condition)

The terminal graphical theory is the simplex category \( \Delta \).

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\begin{array}{ccc}
\text{Cat} & \sim & \text{Mod}_\Delta \\
\Downarrow U & & \Downarrow j^* \\
\hat{\mathcal{G}}_1 & \sim & \text{Sh}(\Delta_0) \\
& & \Downarrow \\
& & \hat{\Delta}_0
\end{array}
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\( \Delta_0 = \{ \text{distance-preserving operators} \} \),
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**Corollary**

- There is a canonical one-to-one correspondence between homogeneous $n$-globular theories and globular $n$-operads;
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The terminal $n$-globular theory $\Theta_n$ is dense in $n\text{Cat}$ for each $n \geq 1$.

\[ \Theta_n \to n\text{Cat} \]
\[ \Theta_{n+1} \to (n+1)\text{Cat} \]

The \textit{wreath product} $\Delta \wr A$ is the category

- with objects $([m], a_1, \ldots, a_m) \in \Delta \times A^m$, $m \geq 0$;
- with morphisms

\[ (\phi; \phi_1, \ldots, \phi_m) : ([m], a_1, \ldots, a_m) \to ([n], b_1, \ldots, b_n) \]

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This identification is compatible with the theory structures.

Remark (Batanin's category of quasi-bijections '10)

- If \( A \) is augmented over Segal's category \( \Gamma \) then so is \( \Delta \wr A \).
- There is thus a canonical functor \( \gamma_n : \Theta_n \rightarrow \Gamma \) for each \( n \geq 1 \).
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Each topological space $X$ is (weakly) homotopy equivalent to the inverse limit of its Postnikov tower

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In principle this allows to reconstruct the homotopy type of $X$ through cohomological invariants, called *Postnikov invariants* of $X$.

The fundamental groupoid $\Pi_1(X)$ captures the homotopy type of the Postnikov section $X_{\leq 1}$, but it is known that for $n \geq 3$ there cannot exist a strict fundamental $n$-groupoid capturing the homotopy type of $X_{\leq n}$ for all $X$.

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Strict $n$-categories are $\Theta_n$-sets fulfilling a restricted sheaf condition.

“Weak” $n$-categories are $\Theta_n$-spaces which are fibrant for a Quillen model structure on $\Theta_n$-spaces, introduced by Rezk ’10.

These fibrant $\Theta_n$-spaces (the Rezk $n$-categories) are essentially those $\Theta_n$-spaces $X$ for which $j^*X$ is a homotopy sheaf on $\Theta_{n,0}$. Rezk proves Grothendieck’s hypothesis for his $n$-groupoids.

There are discrete versions Rezk’s $n$-categories:

- Segal $n$-categories, i.e. fibrant objects for a suitable model structure on $\Theta_n$-spaces which are discrete on $\Theta_{n-1}$;
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These discrete-sized model structures have been shown to exist only for $n = 1$, cf. Joyal-Tierney ’07 !!!
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P.-A. Melliès – *Segal condition meets computational effects*, see his homepage.


