Feynman categories, derived modular envelopes and moduli spaces

Clemens Berger\textsuperscript{1}

University of Nice

Topology Feest Utrecht
August 28, 2018

\textsuperscript{1}joint with Ralph Kaufmann (Purdue University)
1. Moduli space of bordered Riemann surfaces
2. Feynman categories
3. Symmetric, cyclic and modular operads
4. Non-symmetric, planar-cyclic and surface-modular operads
5. $W$-construction and derived modular envelopes
6. Perspectives and open problems
Feynman categories, derived modular envelopes and moduli spaces

Moduli space of bordered Riemann surfaces

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of hyperbolic metrics on a surface $S_{g,n}$ of genus $g$ with $n$ punctures where $\chi(S_{g,n}) < 0$ and $n > 0$.
- $\mathcal{M}_G$ moduli space of admissible metrics on ribbon graph $G$.

Theorem (Mumford, Strebel, Penner, Kontsevich, ...)

$\mathcal{M}_{g,n} \simeq \bigcup G \mathcal{M}_G$ where the metric ribbon graphs $G$ are of type $(g, n)$ and at least trivalent.

Proposition (Igusa)

$\bigcup G \mathcal{M}_G \simeq |\text{nerve}(\text{rb}_{g,n})|$ where the ribbon category $\text{rb}_{g,n}$ is generated by orientation preserving edge contractions between ribbon graphs of type $(g, n)$. 
Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a surface $S_{g,n}$ of genus $g$ with $n$ punctures where $\chi(S_{g,n}) < 0$ and $n > 0$.
- $\mathcal{M}_G$ moduli space of *admissible metrics* on ribbon graph $G$.

Theorem (Mumford, Strebel, Penner, Kontsevich, ...)

$\mathcal{M}_{g,n} \simeq \bigcup_G \mathcal{M}_G$ where the metric ribbon graphs $G$ are of type $(g, n)$ and at least trivalent.

Proposition (Igusa)

$\bigcup_G \mathcal{M}_G \simeq |\text{nerve}(\text{rb}_{g,n})|$ where the *ribbon category* $\text{rb}_{g,n}$ is generated by orientation preserving edge contractions between ribbon graphs of type $(g, n)$. 
Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a surface $S_{g,n}$ of genus $g$ with $n$ punctures where $\chi(S_{g,n}) < 0$ and $n > 0$.
- $\mathcal{M}_G$ moduli space of *admissible metrics* on ribbon graph $G$.

Theorem (Mumford, Strebel, Penner, Kontsevich, ...)

$\mathcal{M}_{g,n} \simeq \bigcup_{G} \mathcal{M}_G$ where the metric ribbon graphs $G$ are of type $(g, n)$ and at least trivalent.

Proposition (Igusa)

$\bigcup_{G} \mathcal{M}_G \simeq |\text{nerve}(rb_{g,n})|$ where the *ribbon category* $rb_{g,n}$ is generated by orientation preserving edge contractions between ribbon graphs of type $(g, n)$. 
Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a surface $S_{g,n}$ of *genus* $g$ with $n$ *punctures* where $\chi(S_{g,n}) < 0$ and $n > 0$.
- $\mathcal{M}_G$ moduli space of *admissible metrics* on ribbon graph $G$.

Theorem (Mumford, Strebel, Penner, Kontsevich, ...)

$\mathcal{M}_{g,n} \simeq \bigcup_G \mathcal{M}_G$ where the metric ribbon graphs $G$ are of type $(g, n)$ and at least trivalent.

Proposition (Igusa)

$\bigcup_G \mathcal{M}_G \simeq |\text{nerve}(\text{rb}_{g,n})|$ where the *ribbon category* $\text{rb}_{g,n}$ is generated by orientation preserving edge contractions between ribbon graphs of type $(g, n)$. 
Feynman categories, derived modular envelopes and moduli spaces

Moduli space of bordered Riemann surfaces

**Definition (moduli space for oriented surfaces/ribbon graphs)**

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a surface $S_{g,n}$ of genus $g$ with $n$ punctures where $\chi(S_{g,n}) < 0$ and $n > 0$.
- $\mathcal{M}_G$ moduli space of *admissible metrics* on ribbon graph $G$.

**Theorem (Mumford, Strebel, Penner, Kontsevich, ...)**

$\mathcal{M}_{g,n} \simeq \bigcup G \mathcal{M}_G$ where the metric ribbon graphs $G$ are of type $(g, n)$ and at least trivalent.

**Proposition (Igusa)**

$\bigcup G \mathcal{M}_G \simeq \text{nerve}(\text{rb}_{g,n})|$ where the *ribbon category* $\text{rb}_{g,n}$ is generated by orientation preserving edge contractions between ribbon graphs of type $(g, n)$. 
Definition (moduli space for oriented surfaces/ribbon graphs)

- \( \mathcal{M}_{g,n} \) moduli space of *hyperbolic metrics* on a surface \( S_{g,n} \) of genus \( g \) with \( n \) punctures where \( \chi(S_{g,n}) < 0 \) and \( n > 0 \).
- \( \mathcal{M}_G \) moduli space of *admissible metrics* on ribbon graph \( G \).

Theorem (Mumford, Strebel, Penner, Kontsevich, ...)

\( \mathcal{M}_{g,n} \cong \bigcup_G \mathcal{M}_G \) where the metric ribbon graphs \( G \) are of type \((g, n)\) and at least trivalent.

Proposition (Igusa)

\( \bigcup_G \mathcal{M}_G \cong |\text{nerve}(\text{rb}_{g,n})| \) where the *ribbon category* \( \text{rb}_{g,n} \) is generated by orientation preserving edge contractions between ribbon graphs of type \((g, n)\).
Definition (bordered case)

\( \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \) moduli space of hyperbolic metrics on a surface \( S_{g,s}^{p_1,\ldots,p_\nu} \) of genus \( g \) with \( s \) punctures and \( \nu \) cyclic boundary components containing \( p_i > 0 \) marked points respectively.

Theorem (Penner, Igusa, B-K)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \cong \bigcup G \mathcal{M}_G \cong \left| \mathcal{Rb}_{g,s}^{p_1,\ldots,p_\nu} \right| \] where the flagged ribbon graphs \( G \) are of type \((g, s; p_1, \ldots, p_\nu)\) and at least trivalent.

Proof sketch (via doubling construction).

(bordered R. surface with \( \chi < 0 \)) \( \sim \) (involutive hyperbolic surface)
(flagged ribbon graph with \( \chi < 0 \)) \( \sim \) (involutive ribbon graph)
involution = orientation-reversing with separating fixpoint set
Definition (bordered case)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \] moduli space of hyperbolic metrics on a surface \( S_{g,s}^{p_1,\ldots,p_\nu} \) of genus \( g \) with \( s \) punctures and \( \nu \) cyclic boundary components containing \( p_i > 0 \) marked points respectively.

Theorem (Penner, Igusa, B-K)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \cong \bigcup_G \mathcal{M}_G \cong |rb_{g,s}^{p_1,\ldots,p_\nu}| \] where the flagged ribbon graphs \( G \) are of type \((g, s; p_1, \ldots, p_\nu)\) and at least trivalent.

Proof sketch (via doubling construction).

(bordered R. surface with \( \chi < 0 \)) \( \sim \) (involutive hyperbolic surface) (flagged ribbon graph with \( \chi < 0 \)) \( \sim \) (involutive ribbon graph) involution = orientation-reversing with separating fixpoint set
Definition (bordered case)

\[ M_{g,s}^{p_1,\ldots,p_\nu} \] moduli space of hyperbolic metrics on a surface \( S_{g,s}^{p_1,\ldots,p_\nu} \) of genus \( g \) with \( s \) punctures and \( \nu \) cyclic boundary components containing \( p_i > 0 \) marked points respectively.

Theorem (Penner, Igusa, B-K)

\[ M_{g,s}^{p_1,\ldots,p_\nu} \cong \bigcup_G M_G \cong |\text{rb}_{g,s}^{p_1,\ldots,p_\nu}| \] where the flagged ribbon graphs \( G \) are of type \((g, s; p_1, \ldots, p_\nu)\) and at least trivalent.

Proof sketch (via doubling construction).

(bordered R. surface with \( \chi < 0 \)) \( \leadsto \) (involutive hyperbolic surface)

(flagged ribbon graph with \( \chi < 0 \)) \( \leadsto \) (involutive ribbon graph)

involution = orientation-reversing with separating fixpoint set
Definition (bordered case)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \] moduli space of hyperbolic metrics on a surface \( S_{g,s}^{p_1,\ldots,p_\nu} \) of genus \( g \) with \( s \) punctures and \( \nu \) cyclic boundary components containing \( p_i > 0 \) marked points respectively.

Theorem (Penner, Igusa, B-K)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \cong \bigcup_G \mathcal{M}_G \simeq |\text{rb}_{g,s}^{p_1,\ldots,p_\nu}| \] where the flagged ribbon graphs \( G \) are of type \((g,s;p_1,\ldots,p_\nu)\) and at least trivalent.

Proof sketch (via doubling construction).

(bordered R. surface with \( \chi < 0 \)) \( \mapsto \) (involutive hyperbolic surface) (flagged ribbon graph with \( \chi < 0 \)) \( \mapsto \) (involutive ribbon graph) involution = orientation-reversing with separating fixpoint set
Definition (bordered case)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \] moduli space of hyperbolic metrics on a surface \( S_{g,s}^{p_1,\ldots,p_\nu} \) of genus \( g \) with \( s \) punctures and \( \nu \) cyclic boundary components containing \( p_i > 0 \) marked points respectively.

Theorem (Penner, Igusa, B-K)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \cong \bigcup_G \mathcal{M}_G \cong |\text{rb}_{g,s}^{p_1,\ldots,p_\nu}| \] where the flagged ribbon graphs \( G \) are of type \((g, s; p_1, \ldots, p_\nu)\) and at least trivalent.

Proof sketch (via doubling construction).

(bordered R. surface with \( \chi < 0 \)) \( \rightsquigarrow \) (involutive hyperbolic surface)
(involutive ribbon graph with \( \chi < 0 \)) \( \rightsquigarrow \) (involutive ribbon graph)
involution = orientation-reversing with separating fixpoint set
**Definition (bordered case)**

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \] moduli space of hyperbolic metrics on a surface \( S_{g,s}^{p_1,\ldots,p_\nu} \) of genus \( g \) with \( s \) punctures and \( \nu \) cyclic boundary components containing \( p_i > 0 \) marked points respectively.

**Theorem (Penner, Igusa, B-K)**

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \simeq \bigcup_G \mathcal{M}_G \simeq |\text{rb}_{g,s}^{p_1,\ldots,p_\nu}| \] where the flagged ribbon graphs \( G \) are of type \((g,s;p_1,\ldots,p_\nu)\) and at least trivalent.

**Proof sketch (via doubling construction).**

(bordered R. surface with \( \chi < 0 \)) \( \hookrightarrow \) (involutive hyperbolic surface)  
(flagged ribbon graph with \( \chi < 0 \)) \( \hookrightarrow \) (involutive ribbon graph)

*involution = orientation-reversing with separating fixpoint set*
Definition (bordered case)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \] moduli space of hyperbolic metrics on a surface \( S_{g,s}^{p_1,\ldots,p_\nu} \) of genus \( g \) with \( s \) punctures and \( \nu \) cyclic boundary components containing \( p_i > 0 \) marked points respectively.

Theorem (Penner, Igusa, B-K)

\[ \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \cong \bigcup_G \mathcal{M}_G \cong |r_{b_{g,s}}^{p_1,\ldots,p_\nu}| \] where the flagged ribbon graphs \( G \) are of type \((g,s;p_1,\ldots,p_\nu)\) and at least trivalent.

Proof sketch (via doubling construction).

(bordered R. surface with \( \chi < 0 \)) \( \Longleftrightarrow \) (involutive hyperbolic surface)

(flagged ribbon graph with \( \chi < 0 \)) \( \Longleftrightarrow \) (involutive ribbon graph)

involution = orientation-reversing with separating fixpoint set
Feynman categories, derived modular envelopes and moduli spaces
Moduli space of bordered Riemann surfaces

Remark (dual point of view: Harer, Kaufmann-Penner)

- \( \text{nerve}(rb_{g,n}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}} \)
- \( \text{nerve}(rb_{g,s}^{p_1, \ldots, p_\nu}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1, \ldots, p_\nu})^{\text{op}} \)

Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

\[
J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads})
\]

induces homotopy equivalences

\[
L_J((g,s; p_1, \ldots, p_\nu)) \cong M_{g,s}^{p_1, \ldots, p_\nu}
\]
Remark (dual point of view: Harer, Kaufmann-Penner)

- \( \text{nerve}(rb_{g,n}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}} \)
- \( \text{nerve}(rb_{g,s}^{p_1,\ldots,p_\nu}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1,\ldots,p_\nu})^{\text{op}} \)

Purpose of the talk

- Define surface-modular operads (cf. Markl)
- Show that the functor
  \[
  J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads})
  \]
  induces homotopy equivalences
  \[
  L_J([1],g,s,p_1,\ldots,p_\nu) \cong M_{g,s}^{p_1,\ldots,p_\nu}
  \]
Remark (dual point of view: Harer, Kaufmann-Penner)

- \( \text{nerve} \left( \text{rb}_{g,n} \right) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}} \)
- \( \text{nerve} \left( \text{rb}_{g,s}^{p_1, \ldots, p_\nu} \right) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1, \ldots, p_\nu})^{\text{op}} \)

Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

\[ J : (\text{planar-cyclic operads}) \to (\text{surface-modular operads}) \]

induces homotopy equivalences

\[ L_1 J (1) (g, n; p_1, \ldots, p_\nu) \cong M_{g,s}^{p_1, \ldots, p_\nu} \]
Feynman categories, derived modular envelopes and moduli spaces
Moduli space of bordered Riemann surfaces

Remark (dual point of view: Harer, Kaufmann-Penner)

- $\text{nerve}(r_{b_{g,n}}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}}$
- $\text{nerve}(r_{b_{g,s}^{p_1,\ldots,p_\nu}}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1,\ldots,p_\nu})^{\text{op}}$

Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

$$J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads})$$

induces homotopy equivalences

$$LJ(1)(g,s;p_1,\ldots,p_\nu) \simeq M_{g,s}^{p_1,\ldots,p_\nu}$$
Remark (dual point of view: Harer, Kaufmann-Penner)

- \(\text{nerve}(rb_{g,n}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}}\)
- \(\text{nerve}(rb_{g,s}^{p_1,\ldots,p_\nu}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1,\ldots,p_\nu})^{\text{op}}\)

Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

\[ J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads}) \]

induces homotopy equivalences

\[ \mathbb{L}J_!(1)(g, s; p_1, \ldots, p_\nu) \cong \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \]
Remark (dual point of view: Harer, Kaufmann-Penner)

- \text{nerve}(rb_{g,n}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}}
- \text{nerve}(rb_{g,s}^{p_1,\ldots,p_\nu}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1,\ldots,p_\nu})^{\text{op}}

Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

\[ J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads}) \]

induces homotopy equivalences

\[ L \omega (1)(g, s; p_1, \ldots, p_\nu) \cong \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \]
Feynman categories, derived modular envelopes and moduli spaces
Moduli space of bordered Riemann surfaces

Remark (dual point of view: Harer, Kaufmann-Penner)

\bullet \text{nerve}(\text{rb}_{g,n}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}}
\bullet \text{nerve}(\text{rb}_{g,s}^{p_1,\cdots,p_\nu}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1,\cdots,p_\nu})^{\text{op}}

Purpose of the talk

\bullet \text{define surface-modular operads (cf. Markl)}
\bullet \text{show that the functor}

\[ J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads}) \]
induces homotopy equivalences

\[ \mathbb{L}J!(1)(g,s;p_1,\ldots,p_\nu) \simeq \mathcal{M}_{g,s}^{p_1,\cdots,p_\nu} \]

(cf. Costello, Giansiracusa)
Remark (dual point of view: Harer, Kaufmann-Penner)

- $\text{nerve}(\text{rb}_{g,n}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}}$
- $\text{nerve}(\text{rb}_{g,s}^{p_1,\ldots,p_\nu}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1,\ldots,p_\nu})^{\text{op}}$

Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

$$J : (\text{planar-cyclic operads}) \to (\text{surface-modular operads})$$

induces homotopy equivalences

$$\mathbb{L}J!(1)(g, s; p_1, \ldots, p_\nu) \simeq \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu}$$

(cf. Costello, Giansiracusa)
Remark (dual point of view: Harer, Kaufmann-Penner)

- $\text{nerve}(rb_{g,n}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\text{op}}$
- $\text{nerve}(rb_{g,s}^{p_1,\ldots,p_\nu}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1,\ldots,p_\nu})^{\text{op}}$

Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

\[ J : (\text{planar-cyclic operads}) \rightarrow (\text{surface-modular operads}) \]

induces homotopy equivalences

\[ \mathbb{L} J!(1)(g, s; p_1, \ldots, p_\nu) \simeq \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu} \]

(cf. Costello, Giansiracusa)
Proposition (May-Thomason, Elmendorf-Mandell, Hermida)
Each coloured operad \( \mathcal{O}(i_1, \ldots, i_k; i) \) induces a symmetric monoidal category \( \mathcal{F}_\mathcal{O} \) having as objects ordered sequences of colours and as morphisms ordered sequences of operations.

Remark (framed symmetric monoidal categories)
\( \mathcal{F}_\mathcal{O} \) contains the invertible unary operations of \( \mathcal{O} \) as subgroupoid \( \mathcal{V}_\mathcal{O} \) such that \( (\mathcal{V}_\mathcal{O}) \otimes \simeq \text{Iso}(\mathcal{F}_\mathcal{O}) \) (we call \( \mathcal{V}_\mathcal{O} \) a *framing* of \( \mathcal{F}_\mathcal{O} \)).

Proposition (Getzler, B-K, Batanin-Kock-Weber)
Coloured operads are *coreflective* inside framed symmetric monoidal categories. The essential image consists of *Feynman categories*.

Definition (Kaufmann-Ward)
A Feynman category \( \mathcal{F} \) is a symmetric monoidal category with framing \( \mathcal{V} \otimes \simeq \text{Iso}(\mathcal{F}) \) such that hereditary and size conditions are satisfied.
Feynman categories, derived modular envelopes and moduli spaces

Feynman categories

**Proposition (May-Thomason, Elmendorf-Mandell, Hermida)**

Each coloured operad $O(i_1, \ldots, i_k; i)$ induces a symmetric monoidal category $\mathcal{F}_O$ having as objects ordered sequences of colours and as morphisms ordered sequences of operations.

**Remark (framed symmetric monoidal categories)**

$\mathcal{F}_O$ contains the invertible unary operations of $O$ as subgroupoid $\mathcal{V}_O$ such that $(\mathcal{V}_O) \otimes \simeq \text{Iso}(\mathcal{F}_O)$ (we call $\mathcal{V}_O$ a framing of $\mathcal{F}_O$).

**Proposition (Getzler, B-K, Batanin-Kock-Weber)**

Coloured operads are coreflective inside framed sym. monoidal categories. The essential image consists of Feynman categories.

**Definition (Kaufmann-Ward)**

A Feynman category $\mathcal{F}$ is a sym. mon. cat. with framing $\mathcal{V} \otimes \simeq \text{Iso}(\mathcal{F})$ such that hereditary and size conditions are satisfied.
Proposition (May-Thomason, Elmendorf-Mandell, Hermida)

Each coloured operad $O(i_1, \ldots, i_k; i)$ induces a symmetric monoidal category $\mathcal{F}_O$ having as objects ordered sequences of colours and as morphisms ordered sequences of operations.

Remark (framed symmetric monoidal categories)

$\mathcal{F}_O$ contains the invertible unary operations of $O$ as subgroupoid $\mathcal{V}_O$ such that $(\mathcal{V}_O) \otimes \simeq \text{Iso}(\mathcal{F}_O)$ (we call $\mathcal{V}_O$ a framing of $\mathcal{F}_O$).

Proposition (Getzler, B-K, Batanin-Kock-Weber)

Coloured operads are coreflective inside framed sym. monoidal categories. The essential image consists of Feynman categories.

Definition (Kaufmann-Ward)

A Feynman category $\mathcal{F}$ is a sym. mon. cat. with framing $\mathcal{V} \otimes \simeq \text{Iso}(\mathcal{F})$ such that hereditary and size conditions are satisfied.
Proposition (May-Thomason, Elmendorf-Mandell, Hermida)

Each coloured operad $\mathcal{O}(i_1, \ldots, i_k; i)$ induces a symmetric monoidal category $\mathcal{F}_\mathcal{O}$ having as objects ordered sequences of colours and as morphisms ordered sequences of operations.

Remark (framed symmetric monoidal categories)

$\mathcal{F}_\mathcal{O}$ contains the invertible unary operations of $\mathcal{O}$ as subgroupoid $\mathcal{V}_\mathcal{O}$ such that $(\mathcal{V}_\mathcal{O})^\otimes \simeq \text{Iso}(\mathcal{F}_\mathcal{O})$ (we call $\mathcal{V}_\mathcal{O}$ a framing of $\mathcal{F}_\mathcal{O}$).

Proposition (Getzler, B-K, Batanin-Kock-Weber)

Coloured operads are coreflective inside framed sym. monoidal categories. The essential image consists of Feynman categories.

Definition (Kaufmann-Ward)

A Feynman category $\mathcal{F}$ is a sym. mon. cat. with framing $\mathcal{V}^\otimes \simeq \text{Iso}(\mathcal{F})$ such that hereditary and size conditions are satisfied.
Feynman categories, derived modular envelopes and moduli spaces

Feynman categories

**Proposition (May-Thomason, Elmendorf-Mandell, Hermida)**

Each coloured operad $\mathcal{O}(i_1, \ldots, i_k; i)$ induces a symmetric monoidal category $\mathcal{F}_\mathcal{O}$ having as objects ordered sequences of colours and as morphisms ordered sequences of operations.

**Remark (framed symmetric monoidal categories)**

$\mathcal{F}_\mathcal{O}$ contains the invertible unary operations of $\mathcal{O}$ as subgroupoid $\mathcal{V}_\mathcal{O}$ such that $(\mathcal{V}_\mathcal{O})^\otimes \simeq \text{Iso}(\mathcal{F}_\mathcal{O})$ (we call $\mathcal{V}_\mathcal{O}$ a framing of $\mathcal{F}_\mathcal{O}$).

**Proposition (Getzler, B-K, Batanin-Kock-Weber)**

Coloured operads are coreflective inside framed sym. monoidal categories. The essential image consists of *Feynman categories*.

**Definition (Kaufmann-Ward)**

A Feynman category $\mathcal{F}$ is a sym. mon. cat. with framing $\mathcal{V}^\otimes \simeq \text{Iso}(\mathcal{F})$ such that hereditary and size conditions are satisfied.
Feynman categories, derived modular envelopes and moduli spaces

**Feynman categories**

**Lemma (\(O\)-algebra=\(\mathcal{F}_O\)-operad)**

Any \(O\)-algebra extends to a strong sym. mon. functor \(\mathcal{F}_O \to \text{Sets}\).

**Proposition (Kaufmann-Ward)**

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) induces an adjunction

\[
j_! : \mathcal{F}\text{-operads} \longrightarrow \mathcal{F}'\text{-operads} : j^*
\]

such that the left adjoint is given by pointwise left Kan extension

\[
(j_! P)(A') = \text{colim}_{j(-) \downarrow A'} P(-).
\]

**Proposition (B-K, cf. Street-Walters' comprehensive factorisation)**

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) factors essentially uniquely as a connected Feynman functor followed by a covering where \(j\) is connected (resp. a covering) iff \(j_!(1) = 1\) (resp. \(\mathcal{F} \cong \text{el}_{\mathcal{F}'}(j_!(1))\)).
Lemma (\(\mathcal{O}\)-algebra=\(\mathcal{F}_\mathcal{O}\)-operad)

Any \(\mathcal{O}\)-algebra extends to a strong sym. mon. functor \(\mathcal{F}_\mathcal{O} \to \text{Sets}\).

Proposition (Kaufmann-Ward)

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) induces an adjunction

\[ j! : \mathcal{F}\text{-operads} \to \mathcal{F}'\text{-operads} : j^* \]

such that the left adjoint is given by pointwise left Kan extension

\[ (j! P)(A') = \text{colim}_{j(-) \downarrow A'} P(-). \]

Proposition (B-K, cf. Street-Walters’ comprehensive factorisation)

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) factors essentially uniquely as a connected Feynman functor followed by a covering where \(j\) is connected (resp. a covering) iff \(j!(1) = 1\) (resp. \(\mathcal{F} \cong \text{el}_{\mathcal{F}'}(j!(1))\)).
Lemma (\(\mathcal{O}\)-algebra=\(\mathcal{F}_\mathcal{O}\)-operad)

Any \(\mathcal{O}\)-algebra extends to a strong sym. mon. functor \(\mathcal{F}_\mathcal{O} \to \text{Sets}\).

Proposition (Kaufmann-Ward)

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) induces an adjunction

\[
j_! : \mathcal{F}\text{-operads} \longrightarrow \mathcal{F}'\text{-operads} : j^*
\]

such that the left adjoint is given by pointwise left Kan extension

\[
(j_! P)(A') = \text{colim}_{j(-) \downarrow A'} P(-).
\]

Proposition (B-K, cf. Street-Walters’ comprehensive factorisation)

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) factors essentially uniquely as a connected Feynman functor followed by a covering where \(j\) is connected (resp. a covering) iff \(j_!(1) = 1\) (resp. \(\mathcal{F} \cong \mathcal{F}'(j_!(1))\)).
Lemma (\(\mathcal{O}\)-algebra=\(\mathcal{F}_\mathcal{O}\)-operad)

Any \(\mathcal{O}\)-algebra extends to a strong sym. mon. functor \(\mathcal{F}_\mathcal{O} \to \text{Sets}\).

Proposition (Kaufmann-Ward)

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) induces an adjunction

\[
j_! : \mathcal{F}\text{-operads} \longrightarrow \mathcal{F}'\text{-operads} : j^*
\]

such that the left adjoint is given by pointwise left Kan extension

\[
(j_! P)(A') = \colim_{j(-)(A')} P(-).
\]

Proposition (B-K, cf. Street-Walters’ comprehensive factorisation)

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) factors essentially uniquely as a connected Feynman functor followed by a covering where \(j\) is connected (resp. a covering) iff \(j_!(1) = 1\) (resp. \(\mathcal{F} \cong \text{el}_{\mathcal{F}'}(j_!(1))\)).
Lemma (\(\mathcal{O}\)-algebra=\(\mathcal{F}_\mathcal{O}\)-operad)

Any \(\mathcal{O}\)-algebra extends to a strong sym. mon. functor \(\mathcal{F}_\mathcal{O} \to \text{Sets}\).

Proposition (Kaufmann-Ward)

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) induces an adjunction

\[
j_! : \mathcal{F}\text{-operads} \to \mathcal{F}'\text{-operads} : j^*
\]

such that the left adjoint is given by pointwise left Kan extension

\[
(j_! P)(A') = \text{colim}_{j(-) \downarrow A'} P(-).
\]

Proposition (B-K, cf. Street-Walters’ comprehensive factorisation)

Any Feynman functor \(j : \mathcal{F} \to \mathcal{F}'\) factors essentially uniquely as a connected Feynman functor followed by a covering where \(j\) is connected (resp. a covering) iff \(j_!(1) = 1\) (resp. \(\mathcal{F} \cong \text{el}_{\mathcal{F}'}(j_!(1))\)).
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $S$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_S = \mathcal{F}_{\text{sym}}$ has

- as objects disjoint unions of rooted corollas
- as morphisms disjoint unions of rooted trees
- composition induced by rooted tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{\text{cyc}}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $S$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_S = \mathcal{F}_{\text{sym}}$ has
- as objects disjoint unions of rooted corollas
- as morphisms disjoint unions of rooted trees
- composition induced by rooted tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{\text{cyc}}$ for cyclic operads has
- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $S$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_S = \mathcal{F}_{sym}$ has

- as objects disjoint unions of *rooted* corollas
- as morphisms disjoint unions of *rooted* trees
- composition induced by *rooted* tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{cyc}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $S$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_S = \mathcal{F}_{sym}$ has

- as objects disjoint unions of *rooted* corollas
- as morphisms disjoint unions of *rooted* trees
- composition induced by *rooted* tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{cyc}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $S$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_S = \mathcal{F}_{sym}$ has

- as objects disjoint unions of *rooted* corollas
- as morphisms disjoint unions of *rooted* trees
- composition induced by *rooted* tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{cyc}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $S$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_S = \mathcal{F}_{sym}$ has

- as objects disjoint unions of *rooted* corollas
- as morphisms disjoint unions of *rooted* trees
- composition induced by *rooted* tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{cyc}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $S$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_S = \mathcal{F}_{sym}$ has

- as objects disjoint unions of rooted corollas
- as morphisms disjoint unions of rooted trees
- composition induced by rooted tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{cyc}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $\mathcal{S}$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_\mathcal{S} = \mathcal{F}_{sym}$ has

- as objects disjoint unions of *rooted* corollas
- as morphisms disjoint unions of *rooted* trees
- composition induced by *rooted* tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{cyc}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $\mathcal{S}$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_\mathcal{S} = \mathcal{F}_{\text{sym}}$ has

- as objects disjoint unions of rooted corollas
- as morphisms disjoint unions of rooted trees
- composition induced by rooted tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{\text{cyc}}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $S$ whose algebras are symmetric operads. Its associated Feynman category $\mathcal{F}_S = \mathcal{F}_{sym}$ has

- as objects disjoint unions of *rooted* corollas
- as morphisms disjoint unions of *rooted* trees
- composition induced by *rooted* tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathcal{F}_{cyc}$ for cyclic operads has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of trees
- composition induced by tree insertion
Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors $\mathcal{F}_{\text{sym}} \rightarrow \mathcal{F}_{\text{cyc}} \rightarrow \mathcal{F}_{\text{ctd}}$ where $\mathcal{F}_{\text{ctd}}$ has
- as objects disjoint unions of corollas
- as morphisms disjoint unions of connected graphs
- composition induced by graph insertion.

Proposition (Getzler-Kapranov)

The Feynman functor $h : \mathcal{F}_{\text{cyc}} \rightarrow \mathcal{F}_{\text{ctd}}$ factors as connected functor
$j : \mathcal{F}_{\text{cyc}} \rightarrow \mathcal{F}_{\text{mod}}$ followed by a covering $k : \mathcal{F}_{\text{mod}} \rightarrow \mathcal{F}_{\text{ctd}}$
where $\mathcal{F}_{\text{mod}}$ is the Feynman category for modular operads.

Corollary (B-K)

The $\mathcal{F}_{\text{ctd}}$-operad $h_!(1)$ is “genus-labeling” and $j_!(1) = 1$. 
Lemma (Borisov-Manin, Kaufmann-Ward)
There are Feynman functors $\mathcal{F}_{\text{sym}} \to \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{ctd}}$ where $\mathcal{F}_{\text{ctd}}$ has
- as objects disjoint unions of corollas
- as morphisms disjoint unions of connected graphs
- composition induced by graph insertion

Proposition (Getzler-Kapranov)
The Feynman functor $h : \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{ctd}}$ factors as connected functor $j : \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{mod}}$ followed by a covering $k : \mathcal{F}_{\text{mod}} \to \mathcal{F}_{\text{ctd}}$ where $\mathcal{F}_{\text{mod}}$ is the Feynman category for modular operads.

Corollary (B-K)
The $\mathcal{F}_{\text{ctd}}$-operad $h!(1)$ is “genus-labeling” and $j!(1) = 1$. 
Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors \( \mathcal{F}_{\text{sym}} \to \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{ctd}} \) where \( \mathcal{F}_{\text{ctd}} \) has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of connected graphs
- composition induced by graph insertion

Proposition (Getzler-Kapranov)

The Feynman functor \( h : \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{ctd}} \) factors as connected functor \( j : \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{mod}} \) followed by a covering \( k : \mathcal{F}_{\text{mod}} \to \mathcal{F}_{\text{ctd}} \) where \( \mathcal{F}_{\text{mod}} \) is the Feynman category for modular operads.

Corollary (B-K)

The \( \mathcal{F}_{\text{ctd}} \)-operad \( h_!(1) \) is “genus-labeling” and \( j!(1) = 1 \).
Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors $\mathcal{F}_{\text{sym}} \to \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{ctd}}$ where $\mathcal{F}_{\text{ctd}}$ has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of connected graphs
- composition induced by graph insertion

Proposition (Getzler-Kapranov)

The Feynman functor $h: \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{ctd}}$ factors as connected functor $j: \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{mod}}$ followed by a covering $k: \mathcal{F}_{\text{mod}} \to \mathcal{F}_{\text{ctd}}$ where $\mathcal{F}_{\text{mod}}$ is the Feynman category for modular operads.

Corollary (B-K)

The $\mathcal{F}_{\text{ctd}}$-operad $h_!(1)$ is “genus-labeling” and $j_!(1) = 1$. 
Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors $\mathcal{F}_{sym} \rightarrow \mathcal{F}_{cyc} \rightarrow \mathcal{F}_{ctd}$ where $\mathcal{F}_{ctd}$ has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of connected graphs
- composition induced by graph insertion

Proposition (Getzler-Kapranov)

The Feynman functor $h : \mathcal{F}_{cyc} \rightarrow \mathcal{F}_{ctd}$ factors as connected functor $j : \mathcal{F}_{cyc} \rightarrow \mathcal{F}_{mod}$ followed by a covering $k : \mathcal{F}_{mod} \rightarrow \mathcal{F}_{ctd}$ where $\mathcal{F}_{mod}$ is the Feynman category for modular operads.

Corollary (B-K)

The $\mathcal{F}_{ctd}$-operad $h! (1)$ is “genus-labeling” and $j! (1) = 1$. 
Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors $\mathcal{F}_{\text{sym}} \rightarrow \mathcal{F}_{\text{cyc}} \rightarrow \mathcal{F}_{\text{ctd}}$ where $\mathcal{F}_{\text{ctd}}$ has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of connected graphs
- composition induced by graph insertion

Proposition (Getzler-Kapranov)

The Feynman functor $h : \mathcal{F}_{\text{cyc}} \rightarrow \mathcal{F}_{\text{ctd}}$ factors as connected functor $j : \mathcal{F}_{\text{cyc}} \rightarrow \mathcal{F}_{\text{mod}}$ followed by a covering $k : \mathcal{F}_{\text{mod}} \rightarrow \mathcal{F}_{\text{ctd}}$

where $\mathcal{F}_{\text{mod}}$ is the Feynman category for modular operads.

Corollary (B-K)

The $\mathcal{F}_{\text{ctd}}$-operad $h!(1)$ is “genus-labeling” and $j!(1) = 1$. 
Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors $\mathcal{F}_{sym} \to \mathcal{F}_{cyc} \to \mathcal{F}_{ctd}$ where $\mathcal{F}_{ctd}$ has

- as objects disjoint unions of corollas
- as morphisms disjoint unions of connected graphs
- composition induced by graph insertion

Proposition (Getzler-Kapranov)

The Feynman functor $h : \mathcal{F}_{cyc} \to \mathcal{F}_{ctd}$ factors as connected functor $j : \mathcal{F}_{cyc} \to \mathcal{F}_{mod}$ followed by a covering $k : \mathcal{F}_{mod} \to \mathcal{F}_{ctd}$ where $\mathcal{F}_{mod}$ is the Feynman category for modular operads.

Corollary (B-K)

The $\mathcal{F}_{ctd}$-operad $h_!(1)$ is “genus-labeling” and $j_!(1) = 1.$
Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors $\mathcal{F}_{\text{sym}} \to \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{ctd}}$ where $\mathcal{F}_{\text{ctd}}$ has
- as objects disjoint unions of corollas
- as morphisms disjoint unions of connected graphs
- composition induced by graph insertion

Proposition (Getzler-Kapranov)

The Feynman functor $h : \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{ctd}}$ factors as connected functor $j : \mathcal{F}_{\text{cyc}} \to \mathcal{F}_{\text{mod}}$ followed by a covering $k : \mathcal{F}_{\text{mod}} \to \mathcal{F}_{\text{ctd}}$

where $\mathcal{F}_{\text{mod}}$ is the Feynman category for modular operads.

Corollary (B-K)

The $\mathcal{F}_{\text{ctd}}$-operad $h_!(1)$ is “genus-labeling” and $j_!(1) = 1$. 
where vertical arrows are coverings, and $j, J$ are connected.

- $\tau_{assoc}$ is the $\mathcal{F}_{sym}$-operad for associative monoids
- $\tau_{planar}$ is the $\mathcal{F}_{cyc}$-operad for planar structures
- $i^*(\tau_{planar}) = \tau_{assoc}$ ($\tau_{planar}$ is the "cyclic" version of $\tau_{assoc}$)
- $\tau_{ribbon} = j!(\tau_{planar})$
Feynman categories, derived modular envelopes and moduli spaces
Non-symmetric, planar-cyclic and surface-modular operads

\[
\begin{array}{ccc}
\text{\tilde{F}}_{\text{non-sym}} & \xrightarrow{l} & \text{\tilde{F}}_{\text{plan-cyc}} & \xrightarrow{J} & \text{\tilde{F}}_{\text{surf-mod}} \\
\downarrow p(\tau_{\text{assoc}}) & & \downarrow p(\tau_{\text{planar}}) & & \downarrow p(\tau_{\text{ribbon}}) \\
\text{\tilde{F}}_{\text{sym}} & \xrightarrow{i} & \text{\tilde{F}}_{\text{cyc}} & \xrightarrow{j} & \text{\tilde{F}}_{\text{mod}} \\
\downarrow & & \downarrow h & & \downarrow \\
& & \text{\tilde{F}}_{\text{ctd}} & & \\
\end{array}
\]

where vertical arrows are coverings, and \( j, J \) are connected.

- \( \tau_{\text{assoc}} \) is the \( \tilde{F}_{\text{sym}} \)-operad for associative monoids
- \( \tau_{\text{planar}} \) is the \( \tilde{F}_{\text{cyc}} \)-operad for planar structures
- \( i^*(\tau_{\text{planar}}) = \tau_{\text{assoc}} \) (\( \tau_{\text{planar}} \) is the “cyclic” version of \( \tau_{\text{assoc}} \))
- \( \tau_{\text{ribbon}} = j!(\tau_{\text{planar}}) \)
Feynman categories, derived modular envelopes and moduli spaces
Non-symmetric, planar-cyclic and surface-modular operads

\[ \mathcal{F}_{\text{non-sym}} \xrightarrow{I} \mathcal{F}_{\text{plan-cyc}} \xrightarrow{J} \mathcal{F}_{\text{surf-mod}} \]
\[ \mathcal{F}_{\text{sym}} \xrightarrow{i} \mathcal{F}_{\text{cyc}} \xrightarrow{j} \mathcal{F}_{\text{mod}} \]
\[ k = p(\tau_{\text{genus}}) \]

where vertical arrows are coverings, and \( j, J \) are connected.

- \( \tau_{\text{assoc}} \) is the \( \mathcal{F}_{\text{sym}} \)-operad for associative monoids
- \( \tau_{\text{planar}} \) is the \( \mathcal{F}_{\text{cyc}} \)-operad for planar structures
- \( i^*(\tau_{\text{planar}}) = \tau_{\text{assoc}} \) (\( \tau_{\text{planar}} \) is the “cyclic” version of \( \tau_{\text{assoc}} \))
- \( \tau_{\text{ribbon}} = j!(\tau_{\text{planar}}) \)
\( F_{\text{non-sym}} \xrightarrow{I} F_{\text{plan-cyc}} \xrightarrow{J} F_{\text{surf-mod}} \)

\[ \downarrow_{p(\tau_{\text{assoc}})} \downarrow_{p(\tau_{\text{planar}})} \downarrow_{p(\tau_{\text{ribbon}})} \]

\( F_{\text{sym}} \xrightarrow{i} F_{\text{cyc}} \xrightarrow{j} F_{\text{mod}} \)

\[ \downarrow_{i} \downarrow_{j} \]

\[ \downarrow_{h} \downarrow_{k=p(\tau_{\text{genus}})} \]

\( \Rightarrow F_{\text{ctd}} \)

where vertical arrows are coverings, and \( j, J \) are connected.

- \( \tau_{\text{assoc}} \) is the \( F_{\text{sym}} \)-operad for associative monoids
- \( \tau_{\text{planar}} \) is the \( F_{\text{cyc}} \)-operad for planar structures
- \( i^*(\tau_{\text{planar}}) = \tau_{\text{assoc}} \) (\( \tau_{\text{planar}} \) is the “cyclic” version of \( \tau_{\text{assoc}} \))
- \( \tau_{\text{ribbon}} = j^!(\tau_{\text{planar}}) \)
\[ \begin{array}{ccl}
\mathcal{F}_{\text{non-sym}} & \xrightarrow{I} & \mathcal{F}_{\text{plan-cyc}} \\
\downarrow p(\tau_{\text{assoc}}) & & \downarrow p(\tau_{\text{planar}}) \\
\mathcal{F}_{\text{sym}} & \xrightarrow{i} & \mathcal{F}_{\text{cyc}} \\
\downarrow & & \downarrow j \\
\downarrow h & & \downarrow k = p(\tau_{\text{genus}}) \\
\mathcal{F}_{\text{surf-mod}} & & \mathcal{F}_{\text{ctd}}
\end{array} \]

where vertical arrows are coverings, and \(j, J\) are connected.

- \(\tau_{\text{assoc}}\) is the \(\mathcal{F}_{\text{sym}}\)-operad for associative monoids
- \(\tau_{\text{planar}}\) is the \(\mathcal{F}_{\text{cyc}}\)-operad for planar structures
- \(i^*(\tau_{\text{planar}}) = \tau_{\text{assoc}}\) (\(\tau_{\text{planar}}\) is the “cyclic” version of \(\tau_{\text{assoc}}\))
- \(\tau_{\text{ribbon}} = j!(\tau_{\text{planar}})\)
\[
\begin{align*}
\mathcal{F}_{\text{non-sym}} & \xrightarrow{l} \mathcal{F}_{\text{plan-cyc}} \xrightarrow{J} \mathcal{F}_{\text{surf-mod}} \\
\mathcal{F}_{\text{sym}} & \xrightarrow{i} \mathcal{F}_{\text{cyc}} \xrightarrow{j} \mathcal{F}_{\text{mod}} \\
& \xrightarrow{h} \mathcal{F}_{\text{ctd}}
\end{align*}
\]

where vertical arrows are coverings, and \(j, J\) are connected.

- \(\tau_{\text{assoc}}\) is the \(\mathcal{F}_{\text{sym}}\)-operad for associative monoids
- \(\tau_{\text{planar}}\) is the \(\mathcal{F}_{\text{cyc}}\)-operad for planar structures
- \(i^*(\tau_{\text{planar}}) = \tau_{\text{assoc}}\) (\(\tau_{\text{planar}}\) is the “cyclic” version of \(\tau_{\text{assoc}}\))
- \(\tau_{\text{ribbon}} = j!(\tau_{\text{planar}})\)
Proposition (Doubek, B-K)

The set $j! (\tau_{planar})(\gamma, n)$ is in bijection with either
- equ. cl. of one-vertex ribbon graphs with $\gamma$ loops and $n$ flags
- $\{(g, s; p_1, \ldots, p_\nu) \mid n = p_1 + \cdots + p_\nu$ and $1 - 2g = \nu + s - \gamma\}$
- topological types of bordered oriented surfaces of genus $g$ with $s$ punctures and $\nu$ boundaries having $p_i$ marked points each

Corollary (Markl, B-K)

The morphisms of the Feynman category $\mathcal{F}_{surf-mod}$ can be considered as genus-labeled “polycyclic” graphs and $J(1) = 1$.

Proposition (B-K)

$$J \downarrow (g, s; p_1, \ldots, p_\nu) \simeq rb_{g, s}^{p_1, \ldots, p_\nu}$$
Proposition (Doubek, B-K)

The set $j_! (\tau_{planar})(\gamma, n)$ is in bijection with either

- equ. cl. of one-vertex ribbon graphs with $\gamma$ loops and $n$ flags
- $\{(g, s; p_1, \ldots, p_\nu) | n = p_1 + \cdots + p_\nu \text{ and } 1 - 2g = \nu + s - \gamma\}$
- topological types of bordered oriented surfaces of genus $g$ with $s$ punctures and $\nu$ boundaries having $p_i$ marked points each

Corollary (Markl, B-K)

The morphisms of the Feynman category $\mathcal{F}_{surf-mod}$ can be considered as genus-labeled “polycyclic” graphs and $J(1) = 1$.

Proposition (B-K)

$J \downarrow (g, s; p_1, \ldots, p_\nu) \cong rb_{g,s}^{p_1,\ldots,p_\nu}$
Proposition (Doubek, B-K)

The set $j! (\tau_{planar}) (\gamma, n)$ is in bijection with either

- equ. cl. of one-vertex ribbon graphs with $\gamma$ loops and $n$ flags
- $\{(g, s; p_1, \ldots, p_\nu) | n = p_1 + \cdots + p_\nu$ and $1 - 2g = \nu + s - \gamma}\$
- topological types of bordered oriented surfaces of genus $g$ with $s$ punctures and $\nu$ boundaries having $p_i$ marked points each

Corollary (Markl, B-K)

The morphisms of the Feynman category $\mathcal{F}_{surf-mod}$ can be considered as genus-labeled “polycyclic” graphs and $J(1) = 1$.

Proposition (B-K)

$J \downarrow (g, s; p_1, \ldots, p_\nu) \simeq rb_{g,s}^{p_1,\ldots,p_\nu}$
Proposition (Doubek, B-K)

The set \( j! (\tau_{\text{planar}})(\gamma, n) \) is in bijection with either
- equ. cl. of one-vertex ribbon graphs with \( \gamma \) loops and \( n \) flags
- \( \{(g, s; p_1, \ldots, p_\nu) \mid n = p_1 + \cdots + p_\nu \text{ and } 1 - 2g = \nu + s - \gamma\} \)
- topological types of bordered oriented surfaces of genus \( g \) with \( s \) punctures and \( \nu \) boundaries having \( p_i \) marked points each

Corollary (Markl, B-K)

The morphisms of the Feynman category \( \mathcal{F}_{\text{surf-mod}} \) can be considered as genus-labeled “polycyclic” graphs and \( J(1) = 1 \).

Proposition (B-K)

\( J \downarrow (g, s; p_1, \ldots, p_\nu) \cong \text{rb}^{p_1, \ldots, p_\nu}_{g, s} \)
Feynman categories, derived modular envelopes and moduli spaces
Non-symmetric, planar-cyclic and surface-modular operads

**Proposition (Doubek, B-K)**

The set \( j!(\tau_{planar})(\gamma, n) \) is in bijection with either

- equ. cl. of one-vertex ribbon graphs with \( \gamma \) loops and \( n \) flags
- \( \{(g, s; p_1, \ldots, p_\nu) | n = p_1 + \cdots + p_\nu \text{ and } 1 - 2g = \nu + s - \gamma\} \)
- topological types of bordered oriented surfaces of genus \( g \) with \( s \) punctures and \( \nu \) boundaries having \( p_i \) marked points each

**Corollary (Markl, B-K)**

The morphisms of the Feynman category \( \mathcal{F}_{surf-mod} \) can be considered as genus-labeled “polycyclic” graphs and \( J(1) = 1 \).

**Proposition (B-K)**

\( J \downarrow (g, s; p_1, \ldots, p_\nu) \cong \text{rb}^{p_1, \ldots, p_\nu}_{g, s} \)
Proposition (Doubek, B-K)

The set $j!(\tau_{planar})(\gamma, n)$ is in bijection with either

- equ. cl. of one-vertex ribbon graphs with $\gamma$ loops and $n$ flags
- $\{(g, s; p_1, \ldots, p_\nu) \mid n = p_1 + \cdots + p_\nu \text{ and } 1 - 2g = \nu + s - \gamma\}$
- topological types of bordered oriented surfaces of genus $g$ with $s$ punctures and $\nu$ boundaries having $p_i$ marked points each

Corollary (Markl, B-K)

The morphisms of the Feynman category $\mathcal{F}_{surf-mod}$ can be considered as genus-labeled “polycyclic” graphs and $J(1) = 1$.

Proposition (B-K)

$J(\downarrow (g, s; p_1, \ldots, p_\nu) \cong \text{rb}_{g,s}^{p_1,\ldots,p_\nu}$
Proposition (Doubek, B-K)

The set $j_!(\tau_{\text{planar}})(\gamma, n)$ is in bijection with either

- equ. cl. of one-vertex ribbon graphs with $\gamma$ loops and $n$ flags
- $\{(g, s; p_1, \ldots, p_\nu) \mid n = p_1 + \cdots + p_\nu$ and $1 - 2g = \nu + s - \gamma\}$
- topological types of bordered oriented surfaces of genus $g$ with $s$ punctures and $\nu$ boundaries having $p_i$ marked points each

Corollary (Markl, B-K)

The morphisms of the Feynman category $\mathcal{F}_{\text{surf-mod}}$ can be considered as genus-labeled “polycyclic” graphs and $J(1) = 1$.

Proposition (B-K)

$J \downarrow (g, s; p_1, \ldots, p_\nu) \simeq \text{rb}_{g, s}^{p_1, \ldots, p_\nu}$
Definition (Kaufmann-Ward)

A Feynman category $\mathcal{F}$ is *cubical* if there is a degree function $\text{deg} : \text{Mor}(\mathcal{F}) \to \mathbb{N}_0$ such that

- $\text{deg}(\phi \circ \psi) = \text{deg}(\phi) + \text{deg}(\psi)$
- $\text{deg}(\phi \otimes \psi) = \text{deg}(\phi) + \text{deg}(\psi)$
- Degree 0 morphisms are invertible
- Each degree $n$ morphism factors (up to iso) in $n!$ ways into degree 1 morphisms “compatibly with composition”

Remark

In the non-unital case without constants, the Feynman categories $\mathcal{F}_{\text{sym}}, \mathcal{F}_{\text{cyc}}, \mathcal{F}_{\text{mod}}, \mathcal{F}_{\text{non-sym}}, \mathcal{F}_{\text{plan-cyc}}, \mathcal{F}_{\text{surf-mod}}$ are cubical. The degree of $\phi$ is the number of edges of the representing graph $\Gamma_\phi$. 
Definition (Kaufmann-Ward)

A Feynman category $\mathcal{F}$ is \textit{cubical} if there is a degree function $\deg : \text{Mor}(\mathcal{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- Degree 0 morphisms are invertible
- Each degree $n$ morphism factors (up to iso) in $n!$ ways into degree 1 morphisms “compatibly with composition”

Remark

In the non-unital case without constants, the Feynman categories $\mathcal{F}_{\text{sym}}, \mathcal{F}_{\text{cyc}}, \mathcal{F}_{\text{mod}}, \mathcal{F}_{\text{non-sym}}, \mathcal{F}_{\text{plan-cyc}}, \mathcal{F}_{\text{surf-mod}}$ are cubical. The degree of $\phi$ is the number of edges of the representing graph $\Gamma_\phi$. 
Definition (Kaufmann-Ward)

A Feynman category $\mathcal{F}$ is *cubical* if there is a degree function $\text{deg} : \text{Mor}(\mathcal{F}) \to \mathbb{N}_0$ such that

- $\text{deg}(\phi \circ \psi) = \text{deg}(\phi) + \text{deg}(\psi)$
- $\text{deg}(\phi \otimes \psi) = \text{deg}(\phi) + \text{deg}(\psi)$

- Degree 0 morphisms are invertible
- Each degree $n$ morphism factors (up to iso) in $n!$ ways into degree 1 morphisms “compatibly with composition”

Remark

In the non-unital case without constants, the Feynman categories $\mathcal{F}_{\text{sym}}, \mathcal{F}_{\text{cyc}}, \mathcal{F}_{\text{mod}}, \mathcal{F}_{\text{non-sym}}, \mathcal{F}_{\text{plan-cyc}}, \mathcal{F}_{\text{surf-mod}}$ are cubical. The degree of $\phi$ is the number of edges of the representing graph $\Gamma_\phi$. 
Definition (Kaufmann-Ward)

A Feynman category $\mathcal{F}$ is *cubical* if there is a degree function $\text{deg} : \text{Mor}(\mathcal{F}) \to \mathbb{N}_0$ such that

- $\text{deg}(\phi \circ \psi) = \text{deg}(\phi) + \text{deg}(\psi)$
- $\text{deg}(\phi \otimes \psi) = \text{deg}(\phi) + \text{deg}(\psi)$
- Degree 0 morphisms are invertible
- Each degree $n$ morphism factors (up to iso) in $n!$ ways into degree 1 morphisms “compatibly with composition”

Remark

In the non-unital case without constants, the Feynman categories $\mathcal{F}_{\text{sym}}, \mathcal{F}_{\text{cyc}}, \mathcal{F}_{\text{mod}}, \mathcal{F}_{\text{non-sym}}, \mathcal{F}_{\text{plan-cyc}}, \mathcal{F}_{\text{surf-mod}}$ are cubical. The degree of $\phi$ is the number of edges of the representing graph $\Gamma_{\phi}$. 
Definition (Kaufmann-Ward)

A Feynman category \( \mathcal{F} \) is \textit{cubical} if there is a degree function \( \deg : \text{Mor}(\mathcal{F}) \rightarrow \mathbb{N}_0 \) such that

- \( \deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi) \)
- \( \deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi) \)
- Degree 0 morphisms are invertible
- Each degree \( n \) morphism factors (up to iso) in \( n! \) ways into degree 1 morphisms “compatibly with composition”

Remark

In the non-unital case without constants, the Feynman categories \( \mathcal{F}_{\text{sym}}, \mathcal{F}_{\text{cyc}}, \mathcal{F}_{\text{mod}}, \mathcal{F}_{\text{non-sym}}, \mathcal{F}_{\text{plan-cyc}}, \mathcal{F}_{\text{surf-mod}} \) are cubical. The degree of \( \phi \) is the number of edges of the representing graph \( \Gamma_\phi \).
Definition (Kaufmann-Ward)

A Feynman category $\mathcal{F}$ is *cubical* if there is a degree function $\text{deg} : \text{Mor}(\mathcal{F}) \rightarrow \mathbb{N}_0$ such that

- $\text{deg}(\phi \circ \psi) = \text{deg}(\phi) + \text{deg}(\psi)$
- $\text{deg}(\phi \otimes \psi) = \text{deg}(\phi) + \text{deg}(\psi)$
- Degree 0 morphisms are invertible
- Each degree $n$ morphism factors (up to iso) in $n!$ ways into degree 1 morphisms “compatibly with composition”

Remark

In the non-unital case without constants, the Feynman categories $\mathcal{F}_{sym}, \mathcal{F}_{cyc}, \mathcal{F}_{mod}, \mathcal{F}_{non-sym}, \mathcal{F}_{plan-cyc}, \mathcal{F}_{surf-mod}$ are cubical. The degree of $\phi$ is the number of edges of the representing graph $\Gamma_\phi$. 
**Definition (\( W_\mathfrak{F} \)-construction)**

Let \( P \) be an operad over a cubical Feynman category \( \mathfrak{F} \). Put

\[
(W_\mathfrak{F}P)(B) = \left( \bigsqcup_{\phi \in \mathfrak{F}(A,B)} P(A) \times \text{Aut}_\mathfrak{F}(\phi) [0, 1]^{\text{deg}(\phi)} \right) / \sim
\]

where identifications are on faces of \([0, 1]^{\text{deg}(\phi)}\) according to coarser factorisations of \( \phi \). \( \text{Aut}_\mathfrak{F}(\phi) \) acts on both sides.

For “graphical” Feynman categories: \( \text{Aut}_\mathfrak{F}(\phi) \cong \text{Aut}(\Gamma_\phi) \).

**Proposition (Kaufmann-Ward, cf. Boardman-Vogt, B-Moerdijk)**

For any cubical Feynman category \( \mathfrak{F} \), the category of topological \( \mathfrak{F} \)-operads admits a *transferred model structure*. If \( P \) has an underlying cofibrant \( \mathcal{V} \)-collection then \( W_\mathfrak{F}P \) is a *cofibrant \( \mathfrak{F} \)-operad*. 
Definition \((W_{\mathcal{F}}\text{-construction})\)

Let \(P\) be an operad over a cubical Feynman category \(\mathcal{F}\). Put

\[
(W_{\mathcal{F}}P)(B) = \left( \bigsqcup_{\phi \in \mathcal{F}(A,B)} P(A) \times_{\text{Aut}_{\mathcal{F}}(\phi)} [0, 1]^{\text{deg}(\phi)} \right) / \sim
\]

where identifications are on faces of \([0, 1]^{\text{deg}(\phi)}\) according to coarser factorisations of \(\phi\). \(\text{Aut}_{\mathcal{F}}(\phi)\) acts on both sides.

For “graphical” Feynman categories: \(\text{Aut}_{\mathcal{F}}(\phi) \cong \text{Aut}(\Gamma_{\phi})\).

Proposition (Kaufmann-Ward, cf. Boardman-Vogt, B-Moerdijk)

For any cubical Feynman category \(\mathcal{F}\), the category of topological \(\mathcal{F}\)-operads admits a transferred model structure. If \(P\) has an underlying cofibrant \(\mathcal{V}\)-collection then \(W_{\mathcal{F}}P\) is a cofibrant \(\mathcal{F}\)-operad.
Definition \((W_{\mathcal{F}}\text{-construction})\)

Let \(P\) be an operad over a cubical Feynman category \(\mathcal{F}\). Put

\[
(W_{\mathcal{F}}P)(B) = \left( \bigsqcup_{\phi \in \mathcal{F}(A,B)} P(A) \times_{\text{Aut}_\mathcal{F}(\phi)} [0, 1]^{\deg(\phi)} \right) / \sim
\]

where identifications are on faces of \([0, 1]^{\deg(\phi)}\) according to coarser factorisations of \(\phi\). \(\text{Aut}_\mathcal{F}(\phi)\) acts on both sides.

For “graphical” Feynman categories: \(\text{Aut}_\mathcal{F}(\phi) \cong \text{Aut}(\Gamma_\phi)\).

Proposition (Kaufmann-Ward, cf. Boardman-Vogt, B-Moerdijk)

For any cubical Feynman category \(\mathcal{F}\), the category of topological \(\mathcal{F}\)-operads admits a transferred model structure. If \(P\) has an underlying cofibrant \(\mathcal{V}\)-collection then \(W_{\mathcal{F}}P\) is a cofibrant \(\mathcal{F}\)-operad.
Definition ($W_{\mathcal{F}}$-construction)

Let $P$ be an operad over a cubical Feynman category $\mathcal{F}$. Put

$$(W_{\mathcal{F}}P)(B) = \left( \coprod_{\phi \in \mathcal{F}(A,B)} P(A) \times_{\text{Aut}_{\mathcal{F}}(\phi)} [0,1]^{\deg(\phi)} \right) / \sim$$

where identifications are on faces of $[0,1]^{\deg(\phi)}$ according to coarser factorisations of $\phi$. $\text{Aut}_{\mathcal{F}}(\phi)$ acts on both sides.

For “graphical” Feynman categories: $\text{Aut}_{\mathcal{F}}(\phi) \cong \text{Aut}(\Gamma_{\phi})$.

Proposition (Kaufmann-Ward, cf. Boardman-Vogt, B-Moerdijk)

For any cubical Feynman category $\mathcal{F}$, the category of topological $\mathcal{F}$-operads admits a transferred model structure. If $P$ has an underlying cofibrant $\mathcal{V}$-collection then $W_{\mathcal{F}}P$ is a cofibrant $\mathcal{F}$-operad.
Example (cubically subdivided convex polytopes)

- $W_{\text{sym}}(T_{\text{assoc}})(\text{rooted corolla}) = \text{associahedron}$
- $W_{\text{cyc}}(T_{\text{planar}})(\text{corolla}) = \text{cyclohedron}$

Proposition (B-K)

Let $\phi : F \to F'$ be a functor of cubical Feynman categories.

- $(W_{\text{sym}})(B) \simeq |\text{nerve}(F \downarrow B)|$
- $\phi_!(W_{\text{sym}})(B') \simeq |\text{nerve}(\phi \downarrow B')|$

Theorem (B-K)

$J!(W_{\text{plan-cyc}})(g, s; p_1, \ldots, p_\nu) \simeq |\text{rb}_{g, s_{p_1, \ldots, p_\nu}}| \simeq M_{g, s_{p_1, \ldots, p_\nu}}$
Example (cubically subdivided convex polytopes)

- $W_{\text{sym}}(\tau_{\text{assoc}})(\text{rooted corolla}) = \text{associahedron}$
- $W_{\text{cyc}}(\tau_{\text{planar}})(\text{corolla}) = \text{cyclohedron}$

Proposition (B-K)

Let $\phi : \mathcal{F} \to \mathcal{F}'$ be a functor of cubical Feynman categories.

- $(W_{\text{cyc}})(B) \simeq |\text{nerve}(\mathcal{F} \downarrow B)|$
- $\phi_!(W_{\text{cyc}})(B') \simeq |\text{nerve}(\phi \downarrow B')|$}

Theorem (B-K)

$$J!(W_{\text{plan-cyc}})(g, s; p_1, \ldots, p_\nu) \simeq |\text{rb}_{g,s}^{p_1,\ldots,p_\nu}| \simeq M_{g,s}^{p_1,\ldots,p_\nu}$$
Example (cubically subdivided convex polytopes)

- $W_{\text{sym}}(\tau_{\text{assoc}})(\text{rooted corolla}) = \text{associahedron}$
- $W_{\text{cyc}}(\tau_{\text{planar}})(\text{corolla}) = \text{cyclohedron}$

Proposition (B-K)

Let $\phi : \mathcal{F} \to \mathcal{F}'$ be a functor of cubical Feynman categories.

- $(W_{\text{sym}})(B) \simeq \vert \text{nerve}(\mathcal{F} \downarrow B) \vert$
- $\phi_!(W_{\text{sym}})(B') \simeq \vert \text{nerve}(\phi \downarrow B') \vert$

Theorem (B-K)

$J_!(W_{\text{plan-cyc}})(g, s; p_1, \ldots, p_\nu) \simeq \vert \mathfrak{rb}_{g, s}^{p_1, \ldots, p_\nu} \vert \simeq M_{g, s}^{p_1, \ldots, p_\nu}$
Example (cubically subdivided convex polytopes)

- $W_{\text{sym}}(\tau_{\text{assoc}})(\text{rooted corolla}) = \text{associahedron}$
- $W_{\text{cyc}}(\tau_{\text{planar}})(\text{corolla}) = \text{cyclohedron}$

Proposition (B-K)

Let $\phi : \mathcal{F} \to \mathcal{F}'$ be a functor of cubical Feynman categories.

- $(W_{\mathcal{F}})(B) \simeq |\text{nerve}(\mathcal{F} \downarrow B)|$
- $\phi_!(W_{\mathcal{F}})(B') \simeq |\text{nerve}(\phi \downarrow B')|$

Theorem (B-K)

$$J_!(W_{\text{plan-cyc}})(g, s; p_1, \ldots, p_\nu) \simeq |\text{rb}_{g,s}^{p_1,\ldots,p_\nu}| \simeq \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu}$$
Example (cubically subdivided convex polytopes)

- $W_{\text{sym}}(\tau_{\text{assoc}})(\text{rooted corolla}) = \text{associahedron}$
- $W_{\text{cyc}}(\tau_{\text{planar}})(\text{corolla}) = \text{cyclohedron}$

Proposition (B-K)

Let $\phi : \mathcal{F} \to \mathcal{F}'$ be a functor of cubical Feynman categories.

- $(W_{\mathbf{3}1})(B) \simeq |\text{nerve}(\mathcal{F} \downarrow B)|$
- $\phi!(W_{\mathbf{3}1})(B') \simeq |\text{nerve}(\phi \downarrow B')|$

Theorem (B-K)

$$J!(W_{\text{plan-cyc}1})(g, s; p_1, \ldots, p_\nu) \simeq |\text{rb}^{p_1, \ldots, p_\nu}_{g,s} | \simeq \mathcal{M}^{p_1, \ldots, p_\nu}_{g,s}$$
Example (cubically subdivided convex polytopes)

- $W_{\text{sym}}(\tau_{\text{assoc}})(\text{rooted corolla}) = \text{associahedron}$
- $W_{\text{cyc}}(\tau_{\text{planar}})(\text{corolla}) = \text{cyclohedron}$

Proposition (B-K)

Let $\phi : \mathfrak{F} \to \mathfrak{F}'$ be a functor of cubical Feynman categories.

- $(W_{\mathfrak{F}})_1(B) \simeq |\text{nerve}(\mathfrak{F} \downarrow B)|$
- $\phi_!(W_{\mathfrak{F}})_1(B') \simeq |\text{nerve}(\phi \downarrow B')|$

Theorem (B-K)

$$J_!(W_{\text{plan-cyc}})_1(g, s; p_1, \ldots, p_\nu) \simeq |\text{rb}_{g,s}^{p_1,\ldots,p_\nu}| \simeq \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu}$$
Example (cubically subdivided convex polytopes)
- $W_{\text{sym}}(\tau_{\text{assoc}})$ (rooted corolla) = associahedron
- $W_{\text{cyc}}(\tau_{\text{planar}})$ (corolla) = cyclohedron

Proposition (B-K)
Let $\phi : \mathcal{F} \to \mathcal{F}'$ be a functor of cubical Feynman categories.
- $(W_{\mathcal{F}}(\mathcal{1}))(B) \simeq |\text{nerve}(\mathcal{F} \downarrow B)|$
- $\phi!(W_{\mathcal{F}}(\mathcal{1}))(B') \simeq |\text{nerve}(\phi \downarrow B')|$

Theorem (B-K)
$$J!(W_{\text{plan-cyc}})(g, s; p_1, \ldots, p_\nu) \simeq \left| \text{rb}^{p_1, \ldots, p_\nu}_{g,s} \right| \simeq \mathcal{M}_{g,s}^{p_1, \ldots, p_\nu}$$
Example (cubically subdivided convex polytopes)

- $W_{\text{sym}}(\tau_{\text{assoc}})(\text{rooted corolla}) = \text{associahedron}$
- $W_{\text{cyc}}(\tau_{\text{planar}})(\text{corolla}) = \text{cyclohedron}$

Proposition (B-K)

Let $\phi : \mathcal{F} \to \mathcal{F}'$ be a functor of cubical Feynman categories.

- $(W_{\mathcal{F}} 1)(B) \simeq |\text{nerve}(\mathcal{F} \downarrow B)|$
- $\phi!(W_{\mathcal{F}} 1)(B') \simeq |\text{nerve}(\phi \downarrow B')|$

Theorem (B-K)

$$J!(W_{\text{plan-cyc}} 1)(g, s; p_1, \ldots, p_\nu) \simeq \left| rb_{g,s}^{p_1,\ldots,p_\nu} \right| \simeq \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu}$$
Feynman categories, derived modular envelopes and moduli spaces
Perspectives and open problems

- $J!(W_{plan-cyc} \mathbb{1}) = (\mathbb{L}J!)(\mathbb{1})$ ?
  - no for transferred \textit{projective} model structure, but yes for transferred \textit{equivariant} model structure, cf. Vogt.

- Since $p!(\mathbb{1}_{plan-cyc}) = \tau_{planar}$, $j!(W\tau_{planar})$ decomposes according to $p!J!(W_{plan-cyc} \mathbb{1})$. What about derived modular envelopes of other cyclic operads ?

- $\chi_{orbi}(\mathcal{M}_{g,1}) = \chi_{orbi}(rb_{g,1}) = \zeta(1 - 2g) = \zeta(\chi(S_{g,1}))$ (Harer-Zagier).
  - What about $\chi_{orbi}(rb^{p_1,\ldots,p_\nu}_{g,s})$ ?
  - Relationship with multi-zeta functions ?

- Relationship with Kontsevich’s graph homology or a flagged version of it ?
Feynman categories, derived modular envelopes and moduli spaces
Perspectives and open problems

- $J!(W_{plan-cyc} \mathbf{1}) = (\mathbb{L}J!)(\mathbf{1})$?
  no for transferred projective model structure, but yes for transferred equivariant model structure, cf. Vogt.

- Since $p!(1_{plan-cyc}) = \tau_{planar}$, $j!(W_{\tau_{planar}})$ decomposes according to $p!J!(W_{plan-cyc} \mathbf{1})$. What about derived modular envelopes of other cyclic operads?

- $\chi_{orbi}(M_{g,1}) = \chi_{orbi}(rb_{g,1}) = \zeta(1 - 2g) = \zeta(\chi(S_{g,1}))$ (Harer-Zagier).
  What about $\chi_{orbi}(rb_{g,s}^{p_1,...,p_\nu})$?
  Relationship with multi-zeta functions?

- Relationship with Kontsevich’s graph homology or a flagged version of it?
• $J!(W_{plan-cyc}) = (\mathbb{L}J!)(1)$?
no for transferred projective model structure, but yes for transferred equivariant model structure, cf. Vogt.

• Since $p!(1_{plan-cyc}) = \tau_{planar}, J!(W_{\tau_{planar}})$ decomposes according to $p!J!(W_{plan-cyc})$. What about derived modular envelopes of other cyclic operads?

• $\chi_{orbi}(M_{g,1}) = \chi_{orbi}(rb_{g,1}) = \zeta(1 - 2g) = \zeta(\chi(S_{g,1}))$ (Harer-Zagier).
What about $\chi_{orbi}(rb_{g,s}^{p_1,...,p_\nu})$?
Relationship with multi-zeta functions?

• Relationship with Kontsevich’s graph homology or a flagged version of it?
Feynman categories, derived modular envelopes and moduli spaces
Perspectives and open problems

- $J!(\mathcal{W}_{\text{plan-cyc}}) = (\mathbb{L}J!)(1)$?
  - no for transferred *projective* model structure, but yes for transferred *equivariant* model structure, cf. Vogt.

- Since $p!(1_{\text{plan-cyc}}) = \tau_{\text{planar}}$, $j!(\mathcal{W}_{\tau_{\text{planar}}})$ decomposes according to $p!J!(\mathcal{W}_{\text{plan-cyc}})$. What about derived modular envelopes of other cyclic operads?

- $\chi_{\text{orbi}}(\mathcal{M}_{g,1}) = \chi_{\text{orbi}}(\text{rb}_{g,1}) = \zeta(1 - 2g) = \zeta(\chi(S_{g,1}))$ (Harer-Zagier).
  - What about $\chi_{\text{orbi}}(\text{rb}_{g,1}^{p_1,\ldots,p_\nu})$?
  - Relationship with multi-zeta functions?

- Relationship with Kontsevich’s graph homology or a flagged version of it?
\( J!(\mathcal{W}_{plan-cyc}1) = (\mathbb{L}J!)(1) \) ?
no for transferred \textit{projective} model structure, but yes for transferred \textit{equivariant} model structure, cf. Vogt.

Since \( p!(1_{plan-cyc}) = \tau_{planar}, J!(\mathcal{W}_{\tau_{planar}}) \) decomposes according to \( p!J!(\mathcal{W}_{plan-cyc}1) \). What about derived modular envelopes of other cyclic operads ?

\( \chi_{orbi}(\mathcal{M}_{g,1}) = \chi_{orbi}(rb_{g,1}) = \zeta(1-2g) = \zeta(\chi(S_{g,1})) \) (Harer-Zagier).
What about \( \chi_{orbi}(rb_{g,s}^{p_1,...,p_\nu}) \) ?
Relationship with multi-zeta functions ?

Relationship with Kontsevich’s graph homology or a flagged version of it ?
\( J!(W_{plan-cyc} \mathbf{1}) = (\mathbb{L} J!)(\mathbf{1}) \) ?
no for transferred \textit{projective} model structure, but yes for transferred \textit{equivariant} model structure, cf. Vogt.

Since \( p!(1_{plan-cyc}) = \tau_{planar}, j!(W_{\tau_{planar}}) \) decomposes according to \( p! J!(W_{plan-cyc} \mathbf{1}) \). What about derived modular envelopes of other cyclic operads?

\( \chi_{orbi}(\mathcal{M}_{g,1}) = \chi_{orbi}(rb_{g,1}) = \zeta(1 - 2g) = \zeta(\chi(S_{g,1})) \) (Harer-Zagier).
What about \( \chi_{orbi}(rb_{g,s}^{p_1,\ldots,p_\nu}) \)?
Relationship with multi-zeta functions?

Relationship with Kontsevich’s graph homology or a flagged version of it?