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Regularity of the diffusion coefficient for lattice gas reversible under Bernoulli measures

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Abstract

We prove the smoothness of a diffusion coefficient with respect to the density of particles for a non-gradient type model. This fact gives a complete proof of the hydrodynamic equation for lattice gas reversible under Bernoulli measures.

Keywords: Reversible lattice gas; Regularity diffusion coefficient; Hydrodynamic equation

1. Introduction

In Funaki et al. (1996), proved an hydrodynamic limit for lattice gas reversible under Bernoulli measures. The model studied is an infinite system of particles interacting by exclusion. The jump rate between two sites x and y is a function of the configuration and gives a general model of non-gradient type. The generator $\mathscr{L}_N$ of the lattice gas on Γ_N, the d-dimensional periodic lattice $(\mathbb{Z}/N\mathbb{Z})^d$, is given by

$$(\mathscr{L}_N f)(\eta) = \frac{1}{2} \sum_{\substack{|x-y|=1 \\ x,y \in \Gamma_N}} c_{x,y}(\eta)(\pi_{x,y} f)(\eta),$$

where η denotes a configuration of the state space $\Omega_N = \{0,1\}^{\Gamma_N}$, f a function over $\Omega_N = \{0,1\}^{\Gamma_N}$ and $\pi_{x,y}$ the operator defined by

$$(\pi_{x,y} f)(\eta) = f(\eta^{x,y}) - f(\eta),$$

where, as usual, $\eta^{x,y}$ is the configuration obtained from η by exchange of the sites x and y. For simplicity, we assume jumps to be only nearest neighbors but this is not essential.

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The family of symmetric functions $\{c_{x,y}(\eta);\ (x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d\}$ is supposed to satisfy the following three conditions:

1. $c_{x,y}(\eta)>0$ and depends only on $\{\eta_z;\ |z-x|\leqslant r\}$ for some $r>0$.
2. $c_{x,y}(\eta)=\tau_x c_{0,y-x}$.
3. $c_{x,y}(\eta)$ is independent of η_x,η_y.

Here τ_x is the shift operator $(\tau_x g)(\eta)=g(\tau_x\eta)=g(\eta(x+\cdot))$. The third assumption is equivalent to the symmetry of $\mathscr{L}_N$ relative to the Bernoulli measure ν_ρ, $\rho\in[0,1]$, whose expectation is denoted by $\langle\cdot\rangle_\rho$.

In Funaki et al. (1996), the authors add a fourth condition, namely the smoothness of the diffusion coefficient $D(\rho)$, ρ being the density of particles. This matrix $D(\rho)$ is given by a Green–Kubo formula. If l is a fixed vector of $\mathbb{R}^d$ then we have

$$(l,D(\rho)l)$$
$$=\frac{1}{4\chi(\rho)}\inf_{F\in\mathscr{F}_0^d}\left\{\sum_{|x|=1}\left\langle c_{0,x}(\eta)\left(l,x(\eta_x-\eta_0)-\pi_{0,x}\left(\sum_{y\in\mathbb{Z}^d}\tau_y F\right)\right)^2\right\rangle_\rho\right\},\tag{1.1}$$

where $\mathscr{F}_0^d$ is the class of all local functions from $\Omega=\{0,1\}^{\mathbb{Z}^d}$ into $\mathbb{R}^d$ and $\chi(\rho)=\rho(1-\rho)$ is the compressibility. Remark that the term $\sum_y\tau_y F$ is formal but that, for a local function, $\pi_{0,x}(\sum_y\tau_y F)$ is just a way to write $\sum_y\pi_{0,x}(\tau_y F)$ which is a finite sum.

Let $\eta^N(t)=\{\eta_x^N(t);\ x\in\Gamma_N\}$ denote the Markov chain on Ω_N governed by the infinitesimal generator $\mathscr{L}_N$ speeded by N^2 and $\rho^N(t,\mathrm{d}\theta)$ the macroscopic empirical-mass distribution whose expression is given by

$$\rho^N(t,\mathrm{d}\theta)=\frac{1}{N^d}\sum_{x\in\Gamma_N}\eta_x^N(t)\delta_{x/N}(\mathrm{d}\theta),\quad \theta\in[0,1)^d.$$

Let us write f_t^N for the distribution of the actual time evolution of the microscopic system (i.e. the density of the distribution of η_t^N on Ω_N with respect to the uniform probability on Ω_N,ν^N). Let ρ_0 be a fixed C^∞ function on $[0,1]^d$ and let $\bar\lambda(\rho)=\ln(\rho/(1-\rho))$. We define $\bar\psi_0^N$ by $\bar\psi_0^N=Z_N^{-1}\exp\{\sum_{x\in\Gamma_N}\bar\lambda(\rho_0(x/N))\eta_x\}$ where Z_N is a constant normalization such that $\bar\psi_0^N(\eta)\,\mathrm{d}\nu^N$ is a probability measure on Ω_N.

It has been shown in Funaki et al. (1996) that if the relative entropy satisfies $H_N(f_0^N,\bar\psi_0^N)=o(N^d)$, then $\rho^N(t,\mathrm{d}\theta)$ converges to $\rho(t,\theta)\,\mathrm{d}\theta$ where the limit density $\rho(t,\theta)$ solves a non-linear diffusion equation

$$\frac{\partial\rho}{\partial t}(t,\theta)=\sum_{i,j=1}^{d}\frac{\partial}{\partial\theta_i}\left[D_{i,j}(\rho(t,\theta))\,\frac{\partial\rho}{\partial\theta_j}(t,\theta)\right]\tag{1.2}$$

with initial condition $\rho(0,\theta)=\rho_0(\theta)$.

The method used in the proof estimates the increase of relative entropy $H_N(f_t^N,\psi_t^N)$. Here ψ_t^N is a local equilibrium state of second-order approximation. To be precise, the local equilibrium state $\psi_t(\eta)\,\mathrm{d}\nu^N$ is defined by

$$\psi_t(\eta) = Z_t^{-1} \exp\left\{ \sum_{x\in\Gamma_N} \lambda(t,x/N)\eta_x + \frac{1}{N}\sum_{x\in\Gamma_N} (\partial\lambda(t,x/N), \tau_x F(\eta)) \right\},$$

where $F \in \mathscr{F}_0^d$ is a local function, Z_t a normalization constant and $\lambda(t,\theta)$ is a C^∞ function given by $\lambda(t,\theta) = \bar{\lambda}(\rho(t,\theta))$ where $\rho(t,\theta)$ is the solution of (1.2). The choice of λ is such that $H_N(f_t^N, \psi_t^N)$ divided by the microscopic total volume N^d remains arbitrarily small. Hence, the proof therefore requires smoothness of $\rho(t,\theta)$ and consequently smoothness of $D(\rho)$, which is why, Funaki et al. assume $D(\rho)$ to be C^∞ on $[0,1]$.

The object of this article is to prove the smoothness of the diffusion coefficient $D(\rho)$. We apply the method developed in Landim et al. (2001) which relies on the duality properties of the symmetric simple exclusion process. The main difference is the dependence on ρ of the coefficients of the generator.

Theorem 1.1. *The diffusion coefficient* $D(\rho)$ *given by* (1.1) *is a* C^∞ *function of* ρ *on* $[0,1]$.

Note that in Varadhan and Yau (1997), the authors prove, under certain mixing conditions on the invariant Gibbs measure, that the hydrodynamic behavior of a stochastic lattice gas is governed by a non-linear diffusion equation whose weak solution they assume to be unique. This uniqueness is indeed valid when the diffusion coefficient is Lipschitz continuous and satisfies uniform ellipticity bounds. Nevertheless, the following method cannot be used since it requires an invariant product measure.

For a local function $f : \Omega \to \mathbb{R}$, i.e. a function depending only on a finite number of coordinates, let us define

$$(\mathscr{L}f)(\eta) = \frac{1}{2} \sum_{\substack{x,y\in\mathbb{Z}^d \\ |x-y|=1}} c_{x,y}(\eta)(f(\eta^{x,y}) - f(\eta)).$$

In view of (1.1), the diffusion coefficient is given by the variational formula

$$(l, D(\rho)l) = \frac{1}{2\chi(\rho)} \left(\frac{1}{2}\sum_x (lx)^2 \langle c_{0,x}\rangle_\rho - 2\|J_l\|^2_{-1,\rho} \right),$$

where $J_l(\eta) = \sum_{|x|=1} (lx) c_{0,x}(\eta)(\eta(x) - \eta(0))$ is the current, and where

$$\|u\|^2_{-1,\rho} = \sup_g \{ 2 \ll u, g \gg_\rho - \ll g, -\mathscr{L}g \gg_\rho \}$$

for u a local function. Here, $\ll \cdot,\cdot \gg_\rho$ indicates the degenerate scalar product given by

$$\ll u, v \gg_\rho = \sum_{x\in\mathbb{Z}^d} (\langle u, \tau_x v\rangle_\rho - \langle u\rangle_\rho \langle v\rangle_\rho), \tag{1.3}$$

where $\langle\cdot,\cdot\rangle_\rho$ is the scalar product relative to the Bernoulli measure ν_ρ in $\mathbb{L}^2(\nu_\rho)$.

Let $\mathbb{L}^2_{\ll\cdot,\cdot\gg}(\nu_\rho)$ be the Hilbert space generated by the local functions and the scalar product $\ll\cdot,\cdot\gg_\rho$. We will denote the corresponding norm by $\|\cdot\|_{0,\rho}$.

In Section 2, we will introduce some notations and give the definition of the generalized duality. In Section 3, we will give the expression of the generator $\mathscr{L}$ in the basis $\{\Psi_A / A \in \mathscr{E}_*\}$ relative to the Hilbert space $\mathbb{L}^2_{\ll\cdot,\cdot\gg}(\nu_\rho)$. Section 4 will be devoted to the steps of the proof while Sections 5 and 6 will give the main estimates we use in the proof of the smoothness. In fact, the regularity of $D(\rho)$ at the boundary requires an extra argument which will be developed in Section 7.

2. Duality

Let us fix a density ρ in $(0,1)$. Let $\mathscr{E}$ (resp. $\mathscr{E}_*$) be the class of all finite subsets of $\mathbb{Z}^d$ (resp. $\mathbb{Z}^d_*$) and $\mathscr{E}_n$ (resp. $\mathscr{E}_{*,n}$) the subsets of $\mathbb{Z}^d$ (resp. $\mathbb{Z}^d_*$) with n points.

For each $A \in \mathscr{E}$, let Ψ_A be the local function

$$\Psi_A(\eta) = \prod_{x \in A} \frac{(\eta(x) - \rho)}{\sqrt{\chi(\rho)}}$$

and by convention $\Psi_\emptyset = 1$. It can easily be verified that $\{\Psi_A;\ A \in \mathscr{E}\}$ is a hilbertian basis of $\mathbb{L}^2(\nu_\rho)$. We will denote by $\mathscr{H}_n$ the subspace generated by $\{\Psi_A;\ A \in \mathscr{E}_n\}$ and by π_n the projection on $\mathscr{H}_n$. Functions belonging to $\mathscr{H}_n$ are called functions of degree n.

If $f = \sum_{A \in \mathscr{E}} \mathfrak{f}(A)\Psi_A$ and $g = \sum_{A \in \mathscr{E}} \mathfrak{g}(A)\Psi_A$ are two local functions then

$$\ll f, g \gg_\rho = \sum_{n \geqslant 0} \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \bar{\mathfrak{f}}(A)\bar{\mathfrak{g}}(A),$$

where, for a local function u, $\{\mathfrak{u}(A);\ A \in \mathscr{E}\}$ are the coefficients of u in the basis $\{\Psi_A;\ A \in \mathscr{E}\}$ and $\bar{\mathfrak{u}} : \mathscr{E}_* \to \mathbb{R}$ is defined by

$$\bar{\mathfrak{u}}(A) = \sum_{x \in \mathbb{Z}^d} \mathfrak{u}([A \cup \{0\}] + x).$$

We call $\mathfrak{H}$ the Hilbert space generated by finite supported functions on $\mathscr{E}$ whose scalar product $\ll\cdot,\cdot\gg$ is defined by

$$\ll \mathfrak{f}, \mathfrak{g} \gg = \sum_{n \geqslant 0} \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \bar{\mathfrak{f}}(A)\bar{\mathfrak{g}}(A).$$

$c_{0,x}$ can be decomposed in the basis $\{\Psi_A;\ A \in \mathscr{E}\}$ according to

$$c_{0,x} = \sum_{B \in \mathscr{C}} \alpha_x(B,\rho)\Psi_B,$$

where $\mathscr{C}$ is a finite subset of $\mathscr{E}$. Let $\mathscr{F}$ be the class of all subsets of elements of $\mathscr{C}$ and let n_0 be an integer such that $\mathscr{F} \subset \bigcup_{n \leqslant n_0} \mathscr{E}_n$.

Clearly, since $c_{0,x}$ is independent of η_0, η_x, we have

$$\alpha_x(B,\rho) = 0 \quad \text{if } 0 \in B \text{ or } x \in B. \tag{2.1}$$

And the symmetry of $c_{x,y}$ gives the following relations

$$\alpha_{y-x}(B - x, \rho) = \alpha_{x-y}(B - y, \rho) \quad \text{for each } B \in \mathscr{E}. \tag{2.2}$$

Let $B \in \mathscr{E}$ and $x \in \mathbb{Z}^d$. It is straightforward to show existence of a polynomial function $Q_x(B,.)$ of ρ such that $\alpha_x(B,\rho) = (\sqrt{\chi(\rho)})^{|B|} Q_x(B,\rho)$.

3. Expression of the generator

In this section, we give the expression of the generator $\mathscr{L}$ in the basis $\{\Psi_A;\ A\in\mathscr{E}\}$ for a $\rho\in(0,1)$ fixed. First of all, a few notations are required.

If A,S,P are subsets of $\mathbb{Z}^d$ (resp. $\mathbb{Z}^d_*$) such that $S\subset A$ and $P\cap A=\emptyset$, then $A_S^P=A\backslash S\cup P$. The class of sets (A,S,P) featuring these properties and verifying $|A|=n$, $n\in\mathbb{N}$, $|S|-|P|=k$, $k\in\mathbb{Z}$, is denoted by $\mathscr{U}_n^k$ (resp. $\mathscr{U}_{*,n}^k$). Note that $|A_S^P|=|A|-|S|+|P|=n-k$ for $A\in\mathscr{U}_n^k$ (resp. $\mathscr{U}_{*,n}^k$).

Let $\beta(\rho)=\dfrac{1-2\rho}{\sqrt{\chi(\rho)}}$.

For $x,y\in\mathbb{Z}^d$, $A\in\mathscr{E}$, $S\subset A$, $P\cap A=\emptyset$, we define $u_{x,y}(A,S,P,\rho)$ by

$$u_{x,y}(A,S,P,\rho)=\sum_{S\subset\Gamma\subset A}\alpha_{y-x}(P\cup\Gamma-x,\rho)\beta(\rho)^{|\Gamma|-|S|} \tag{3.1}$$

$$=(\sqrt{\chi(\rho)})^{|P|+|S|}\sum_{S\subset\Gamma\subset A}Q_{y-x}(P\cup\Gamma-x,\rho)(1-2\rho)^{|\Gamma|-|S|} \tag{3.2}$$

$$=(\sqrt{\chi(\rho)})^{|P|+|S|}v_{x,y}(A,S,P,\rho), \tag{3.3}$$

where $v_{x,y}(A,S,P,.)$ is C^∞ on $[0,1]$.

From here on, we will use the following properties of u:

$$u_{x,y}=u_{y,x}, \tag{3.4}$$

$$u_{x,y}(A,S,P,\rho)=0 \tag{3.5}$$

if x or y is in $S\cup P$ or if $S\notin\mathscr{F}$ or $P\notin\mathscr{F}$.

A straightforward computation shows that if $(A,S,P)\in\mathscr{U}_n^k$ then

$$u_{x,y}(A,S,P,\rho)=u_{x,y}(A_S^P,P,S,\rho). \tag{3.6}$$

Moreover, since α is of finite range, v and u are uniformly bounded by $\|\alpha\|_\infty|\mathscr{F}|$.

If $\mathfrak{f}:\mathscr{E}\to\mathbb{R}$ and if x,y are two points of $\mathbb{Z}^d$, we will note

$$(\Pi_{x,y}\mathfrak{f})(A)=\mathfrak{f}(A_{x,y})-\mathfrak{f}(A),$$

$$(\Phi_y\mathfrak{f})(A)=\mathfrak{f}(A-y)-\mathfrak{f}(A)\quad\text{for } y\notin A,$$

$$(\Delta_{x,y}\mathfrak{f})(A)=\mathfrak{f}([A\setminus\{x\}]-y)-\mathfrak{f}([A\setminus\{x\}]-x)\quad\text{for } x\in A,\ \ y\notin A.$$

Here, $A_{x,y}=A\backslash\{x\}\cup\{y\}$ if $x\in A$, $y\notin A$ and similarly $A_{x,y}=A\backslash\{y\}\cup\{x\}$ if $y\in A$, $x\notin A$ and otherwise, $A_{x,y}=A$.

Let f be a local function whose decomposition in $\mathbb{L}^2(\nu_\rho)$ is given by

$$f=\sum_A\mathfrak{f}(A)\Psi_A.$$

Then $\mathscr{L}f$ has the following decomposition:

$$\mathscr{L}f = \sum_A \mathfrak{L}f(A)\Psi_A,$$

where

$$\mathfrak{L} = \sum_{k\in\mathbb{Z}} \mathfrak{L}_k$$

with

$$\mathfrak{L}_k\mathfrak{f}(A) = \frac{1}{2}\sum_{|x-y|=1}\sum_{\substack{S\subset A, P\cap A=\emptyset\\ |S|-|P|=k}} u_{x,y}(A,S,P,\rho)(\Pi_{x,y}\mathfrak{f})(A_S^P).$$

In fact, if $|k|>n_0$ then $\mathfrak{L}_k = 0$. Therefore, the first sum is well defined. Moreover, $\mathfrak{L}_k$ maps any function of degree n onto a function of degree $n+k$.

An elementary yet rather long computation based on the properties of $\{\alpha_x(B,.);\ B\in\mathscr{E}\}$ shows that

$$\overline{\mathfrak{L}_k\mathfrak{f}} = \bar{\mathfrak{L}}_k\bar{\mathfrak{f}}$$

with

$$\bar{\mathfrak{L}}_k = \bar{\mathfrak{L}}_{k,\mathrm{ex}} + \bar{\mathfrak{L}}_{k,\tau} + \bar{\mathfrak{L}}_{k,\tau,\mathrm{ex}},$$

where

$$\bar{\mathfrak{L}}_{k,\mathrm{ex}}\bar{\mathfrak{f}}(A) = \frac{1}{2}\sum_{x,y}\sum_{\substack{S\subset A, P\cap A=\emptyset\\ |S|-|P|=k}} u_{x,y}(A\cup\{0\},S,P,\rho)(\Pi_{x,y}\bar{\mathfrak{f}})(A_S^P), \tag{3.7}$$

$$\bar{\mathfrak{L}}_{k,\tau}\bar{\mathfrak{f}}(A) = \sum_{\substack{S\subset A, P\cap A=\emptyset\\ |S|-|P|=k}}\sum_{y\notin A_S^P} u_{0,y}(A\cup\{0\},S,P,\rho)(\Phi_y\bar{\mathfrak{f}})(A_S^P), \tag{3.8}$$

$$\bar{\mathfrak{L}}_{k,\tau,\mathrm{ex}}\bar{\mathfrak{f}}(A) = \sum_{\substack{S\subset A, P\cap A=\emptyset\\ |S|-|P|=k-1}}\sum_{\substack{y\notin A_S^P\\ x\in A_S^P}} u_{x,y}(A\cup\{0\},S\cup\{0\},P,\rho)(\Delta_{x,y}\bar{\mathfrak{f}})(A_S^P). \tag{3.9}$$

In these sums, $x,y\neq 0$ and $S,P\in\mathscr{E}_*$. Actually, $\bar{\mathfrak{L}}_k, \bar{\mathfrak{L}}$ are operators which depend on ρ and as such, should be written as $\bar{\mathfrak{L}}_k(\rho), \bar{\mathfrak{L}}(\rho)$. But in this section we deal only with a fixed ρ and adopt a lighter notation.

Remark that $\bar{\mathfrak{L}}_{k,\mathrm{ex}}, \bar{\mathfrak{L}}_{k,\tau}, \bar{\mathfrak{L}}_{k,\tau,\mathrm{ex}}$ increase the degree by k if $k\geqslant 0$ and decrease it by $-k$ if $k\leqslant 0$. In particular, $\bar{\mathfrak{L}}_{0,\mathrm{ex}}, \bar{\mathfrak{L}}_{0,\tau}, \bar{\mathfrak{L}}_{0,\tau,\mathrm{ex}}$ preserve the degree.

The generator of the symmetric simple exclusion process on $\mathbb{Z}^d$ and his Dirichlet form will play a fundamental role in the sequel. Most important for us will be the fact that this generator presevers functions degrees and that his coefficients are independent of ρ.

This generator is given by

$$(\mathscr{L}_0 f)(\eta) = \frac{1}{2} \sum_{|x-y|=1} (f(\eta^{x,y}) - f(\eta)).$$

As before, it is not difficult to check that if f is a local function whose decomposition in $\mathbb{L}^2(v_\rho)$ is given by

$$f = \sum_A \mathfrak{f}(A)\Psi_A,$$

then $\mathscr{L}_0 f$ has the following decomposition:

$$\mathscr{L}_0 f = \sum_{A\in\mathscr{E}} \mathfrak{L}_0\mathfrak{f}(A)\Psi_A$$

and

$$\overline{\mathfrak{L}_0\mathfrak{f}} = \bar{\mathfrak{L}}_{\text{ex}}\bar{\mathfrak{f}} + \bar{\mathfrak{L}}_\tau\bar{\mathfrak{f}}$$

with

$$\bar{\mathfrak{L}}_{\text{ex}}\bar{\mathfrak{f}}(A) = \frac{1}{2} \sum_{\substack{x,y\neq 0\\ |y-x|=1}} (\bar{\mathfrak{f}}(A_{x,y}) - \bar{\mathfrak{f}}(A))$$

and

$$\bar{\mathfrak{L}}_\tau\bar{\mathfrak{f}}(A) = \sum_{\substack{x\notin A\\ |x|=1}} (\bar{\mathfrak{f}}(A-x) - \bar{\mathfrak{f}}(A)).$$

We define the Hilbert space $\mathfrak{H}_1$ associated to the operator $\bar{\mathfrak{L}}_{\text{ex}}$ for which the scalar product is given by

$$\ll \mathfrak{f},\mathfrak{g} \gg_1 = -\sum_{n\geqslant 0} \frac{1}{n+1} \sum_{A\in\mathscr{E}_{*,n}} \bar{\mathfrak{L}}_{\text{ex}}\bar{\mathfrak{f}}(A)\bar{\mathfrak{g}}(A).$$

Let $\mathfrak{H}_{-1}$ be the dual space of $\mathfrak{H}_1$ with respect to $\ll\cdot,\cdot\gg$. This is the Hilbert space generated by the finite supported functions and the norm $\|\cdot\|_{-1}$ defined by

$$\|\mathfrak{f}\|_{-1}^2 = \sup_{\mathfrak{g}}\{2 \ll \mathfrak{f},\mathfrak{g}\gg - \ll \mathfrak{g},\mathfrak{g}\gg_1\}.$$

In the next sections, we will use the following notations, for any $k \geqslant 1$:

$$\|\mathfrak{f}\|_{1,k}^2 = \sum_{n\geqslant 0} n^{2k}\|\pi_n\mathfrak{f}\|_1^2 \quad \|\mathfrak{f}\|_{-1,k}^2 = \sum_{n\geqslant 0} n^{2k}\|\pi_n\mathfrak{f}\|_{-1}^2.$$

We note the Dirichlet forms associated to $\bar{\mathfrak{L}}_{\text{ex}}$ and $\bar{\mathfrak{L}}_\tau$, respectively, by

$$\mathscr{D}_{\text{ex}}(\mathfrak{u}) = \sum_{n\geqslant 0} \frac{1}{n+1} \sum_{\substack{A\in\mathscr{E}_{*,n}\\ |x-y|=1}} (\mathfrak{u}(A_{x,y}) - \mathfrak{u}(A))^2,$$

$$\mathscr{D}_\tau(\mathfrak{u}) = \sum_{n\geqslant 0} \frac{1}{n+1} \sum_{\substack{A\in\mathscr{E}_{*,n}\\ y\notin A,|y|=1}} (\mathfrak{u}(A-y) - \mathfrak{u}(A))^2.$$

The independence of these norms from ρ is important and should be stressed here.

A fundamental inequality between these two forms is given by Lemma 5.1 in Landim et al. (2002).

Lemma 3.1. *There exists a finite constant C independent of n such that*

$$\mathscr{D}_\tau(\mathfrak{u}) \leqslant Cn\mathscr{D}_{\mathrm{ex}}(\mathfrak{u})$$

for all finite supported function $\mathfrak{u}$ of degree n.

Note that $-\mathscr{L}$ is a positive symmetric operator in $\mathbb{L}^2_{\ll\cdot,\cdot\gg}(v_\rho)$. Indeed, it is sufficient to prove that for all local functions f, g, we have

$$\ll \mathscr{L} f, g \gg_\rho = \ll \mathscr{L} g, f \gg_\rho. \tag{3.10}$$

Since $\langle \mathscr{L} f \rangle_\rho = 0$ (v_ρ is a reversible measure), we have $\ll \mathscr{L} f, g \gg = \sum_{x\in\mathbb{Z}^d} \langle \mathscr{L} f, \tau_x g\rangle_\rho$. But for $x \in \mathbb{Z}^d$, $(\tau_{-x}\eta)^{u,v} = \tau_{-x}(\eta^{u-x,v-x})$ and v_ρ is invariant by τ_{-x} so that

$$\begin{aligned}
\langle \mathscr{L} f, \tau_x g\rangle_\rho &= \frac{1}{2}\sum_{|u-v|=1} \langle c_{u,v}(\tau_{-x}\eta) f(\tau_{-x}(\eta^{u-x,v-x}))g(\eta)\rangle_\rho \\
&\quad - \frac{1}{2}\sum_{|u-v|=1} \langle c_{u,v}(\tau_{-x}\eta) f(\tau_{-x}\eta)g(\eta)\rangle_\rho \\
&= \frac{1}{2}\sum_{|u-v|=1} \langle c_{u-x,v-x}(\eta) f(\tau_{-x}(\eta^{u-x,v-x}))g(\eta)\rangle_\rho \\
&\quad - \frac{1}{2}\sum_{|u-v|=1} \langle c_{u-x,v-x}(\eta) f(\tau_{-x}\eta)g(\eta)\rangle_\rho \\
&= \frac{1}{2}\sum_{|u-v|=1} \langle c_{u-x,v-x}(\eta^{u-x,v-x}) f(\tau_{-x}\eta)g(\eta^{u-x,v-x})\rangle_\rho \\
&\quad - \frac{1}{2}\sum_{|u-v|=1} \langle c_{u-x,v-x}(\eta) f(\tau_{-x}\eta)g(\eta)\rangle_\rho \\
&= \frac{1}{2}\sum_{|u-v|=1} \langle c_{u-x,v-x}(\eta) f(\tau_{-x}\eta)g(\eta^{u-x,v-x})\rangle_\rho \\
&\quad - \frac{1}{2}\sum_{|u-v|=1} \langle c_{u-x,v-x}(\eta) f(\tau_{-x}\eta)g(\eta)\rangle_\rho.
\end{aligned}$$

Since v_ρ is invariant by $\eta \to \eta^{u-x,v-x}$ and $c_{u-x,v-x}(\eta)$ is independent of $\{\eta(u-x), \eta(v-x)\}$.

At this point, it is easy to conclude and obtain the formula (3.10).

Lemma 3.2. *The operator $\mathscr{L}$ may be extended to a self-adjoint unbounded operator in $\mathbb{L}^2_{\ll\cdot,\cdot\gg}(v_\rho)$.*

Proof. $\mathscr{L}$ is an unbounded symmetric operator whose domain is the subspace of local functions. Hence its domain is dense in $\mathbb{L}^2_{\ll\cdot,\cdot\gg}(v_\rho)$. Moreover, the operator $\mathscr{L}$ commutes with complex conjugation. By Theorem 18, XII.4.16, p. 1231, of Dunford and Schwarz (1964), there exists a self-adjoint extension of $\mathscr{L}$ in $\mathbb{L}^2_{\ll\cdot,\cdot\gg}(v_\rho)$. □

In the sequel, we will note $\mathscr{L}$ this extension. Solution to the resolvent equation, and spectral measure are defined with respect to this self-adjoint operator.

To $-\mathscr{L}$ we associate the $\mathscr{H}_1$-norm with respect to $\mathbb{L}^2_{\ll\cdot,\cdot\gg}(v_\rho)$ defined by

$$\|f\|^2_{1,\rho} = \ll -\mathscr{L}f, f\gg_\rho$$

$$= \frac{1}{4}\sum_{|x|=1}\left\langle c_{0,x}(\eta)\left|\sum_{y\in\mathbb{Z}^d}((\tau_y f)(\eta^{0,x}) - (\tau_y f)(\eta))\right|^2\right\rangle_\rho \tag{3.11}$$

and its dual-norm in $\mathbb{L}^2_{\ll\cdot,\cdot\gg}(v_\rho)$.

$$\|f\|^2_{-1,\rho} = \sup_g\{2\ll f,g\gg_\rho - \|g\|^2_{1,\rho}\}.$$

Our previous assumptions give us $\inf_{|x-y|=1}\{c_{x,y}\} > 0$.

Thus, there exists a constant $\gamma > 0$ independent of ρ such that

$$\mathscr{D}_{\rm ex}(\bar{\mathfrak{f}}) \leqslant \gamma\|f\|^2_{1,\rho}.$$

And consequently

$$\|f\|^2_{-1,\rho} \leqslant \gamma\|\mathfrak{f}\|^2_{-1}. \tag{3.12}$$

4. Steps of the proof

By the variational formula, the regularity of the diffusion coefficient is equivalent to the regularity of $\|I(\rho)\|^2_{-1,\rho}$ where $I(\rho) = \frac{1}{\sqrt{\chi(\rho)}}J_l$.

A simple computation shows that I is a local function of degree less than n_0 and that

$$\bar{\mathfrak{I}}(A,\rho) = 2\sum_{u\in A, x\notin A}(l,(u-x))\alpha_{u-x}(A^0_u - x,\rho) - 2\sum_{x\notin A}(l,x)\alpha_x(A,\rho).$$

In order to prove the regularity of $\|I(\rho)\|^2_{-1,\rho}$, we will need the derivatives of $\bar{\mathfrak{L}}(\rho)$, as a function of ρ to exist. For this reason we parameterize the density ρ by $\rho(t) = \sin^2(t)$ where $t\in[0,\pi/2]$ and we prove smoothness in t. Note that this parameterization is not a C^∞-diffeomorphism at the boundary. The behaviors at 0 and 1 will be dealt with separately in Section 7.

Nevertheless, this parameterization transforms $\alpha_x(B,.)$ into a C^∞ function of t since

$$\alpha_x(B,\rho(t)) = (\sin(t)\cos(t))^{|B|}Q_x(B,\rho(t)).$$

The same occurs with u and v. This shows that all derivatives of u as function of t are bounded.

When necessary, we will note the dependence on t of the operators. We call $R(t)$ the composed function $(I \circ \rho)(t)$.

We introduce the resolvent equation associated to I and denote, for $\lambda > 0$ and $\rho \in [0,1]$, by $g_\lambda(\rho)$ the solution of the equation

$$\lambda g_\lambda(\rho) - \mathscr{L} g_\lambda(\rho) = I(\rho). \tag{4.1}$$

If $t \in [0;\pi/2]$, we note $f_\lambda(t)$ the solution of the preceding resolvent equation for $\rho = \rho(t)$. Let $\mathfrak{f}_\lambda(A,t)$ be the coefficients of f_λ in the basis $\{\Psi_A;\ A \in \mathscr{E}\}$ which correspond to the parameter $\rho(t)$. We thus have $\mathfrak{f}_\lambda(.,t) = \mathfrak{g}_\lambda(.,\rho(t))$.

It is worth noticing that with this reparametrization, the coefficients of $\mathfrak{L}(t), \bar{\mathfrak{L}}(t)$ are C^∞.

It follows from Kipnis and Varadhan (1986) that

$$\|R\|^2_{-1,\rho(t)} = \lim_{\lambda \to 0} \langle\!\langle R, f_\lambda \rangle\!\rangle_{\rho(t)}.$$

Indeed, let us write μ for the spectral measure of R corresponding to the self-adjoint operator $\mathscr{L}$ in $\mathbb{L}^2_{\langle\!\langle \cdot,\cdot \rangle\!\rangle}(v_\rho)$. The following lemma shows that $\mathfrak{R}$ is in $\mathfrak{H}_{-1}$ and so is $\|R\|_{-1,\rho} < +\infty$ by (3.12). Consequently, we have

$$-\int_{-\infty}^{0} \frac{1}{y}\, \mathrm{d}\mu(y) < \infty.$$

It follows from the Beppo–Levy's theorem that

$$\begin{aligned}\lim_{\lambda \to 0} \langle\!\langle f_\lambda, R \rangle\!\rangle_\rho &= \lim_{\lambda \to 0} \int_{-\infty}^{0} \frac{1}{\lambda - y}\, \mathrm{d}\mu(y)\\ &= -\int_{-\infty}^{0} \frac{1}{y}\, \mathrm{d}\mu(y)\\ &= \|R\|^2_{-1,\rho}.\end{aligned}$$

Let $U_\lambda(t) = \langle\!\langle R, f_\lambda \rangle\!\rangle^2_{\rho(t)}$. To prove regularity, it is enough to show that there exists a subsequence $\lambda_k \downarrow 0$ such that $U_{\lambda_k}(t)$ converges uniformly to a C^∞ function. The existence of a such a subsequence can be inferred from Ascoli's theorem if we prove that $U_\lambda(t)$ are C^∞ in t for each $\lambda > 0$ and if we obtain bounds, independent of λ, on the L^∞-norm of the functions $U_\lambda^{(j)}$.

Lemma 4.1. *There exists a finite constant C independent of t such that for any local function f*

$$|\langle\!\langle R, f \rangle\!\rangle_{\rho(t)}| \leqslant C\sqrt{\mathscr{D}^{\mathrm{ex}}(\bar{\mathfrak{f}})}.$$

Proof. It is easy to verify that

$$\ll R, f \gg_\rho = \sum_{n=0}^{n_0} \frac{1}{n+1}(E_n(\bar{\mathfrak{f}}) + T_n(\bar{\mathfrak{f}})),$$

where

$$E_n(\bar{\mathfrak{f}}) = \frac{1}{2} \sum_{\substack{A\in\mathscr{E}_{*,n} \\ u\in A, x\notin A}} (l\cdot(u-x))\alpha_{u-x}(A_u^0 - x,\rho)[\bar{\mathfrak{f}}(A_{u,x},\rho) - \bar{\mathfrak{f}}(A,\rho)]$$

and

$$T_n(\bar{\mathfrak{f}}) = \frac{1}{2} \sum_{\substack{A\in\mathscr{E}_{*,n} \\ x\notin A}} (l\cdot x)\alpha_x(A,\rho)[\bar{\mathfrak{f}}(A,\rho) - \bar{\mathfrak{f}}(A-x,\rho)].$$

Using the Cauchy–Schwarz inequality, we have

$$|E_n(\bar{\mathfrak{f}})| \leqslant \frac{1}{2\sqrt{2}} \sqrt{\mathscr{G}_n}\sqrt{\mathscr{H}_n(\bar{\mathfrak{f}})},$$

where

$$\mathscr{G}_n = \sum_{\substack{A\in\mathscr{E}_{*,n} \\ u\in A, x\notin A}} [(l(u-x))\alpha_{u-x}(A_u^0 - x,\rho)]^2$$

and

$$\mathscr{H}_n(\bar{\mathfrak{f}}) = \sum_{\substack{A\in\mathscr{E}_{*,n} \\ |x-u|=1}} [\bar{\mathfrak{f}}(A_{u,x},\rho) - \bar{\mathfrak{f}}(A,\rho)]^2.$$

Remark that $\alpha_{u-x}(A_u^0 - x,\rho) = 0$ as soon as $|u-x| \neq 1$ or $A_u^0 - x \notin \mathscr{F}$ (recall that $\mathscr{F}$, defined at the end of Section 2, is the class of all subsets of elements of $\mathscr{C}$, which is the support of the functions α_x). In particular, $\alpha_{u-x}(A_u^0 - x,\rho) = 0$ if $\{-x\} \notin \mathscr{F}$. Therefore, we obtain

$$\sqrt{\mathscr{G}_n} \leqslant |l|\sqrt{2d|\mathscr{F}|^2}\|\alpha\|_\infty.$$

Since $n \leqslant n_0$ we have

$$\begin{aligned}\sum_{n=0}^{n_0} \frac{1}{n+1}|E_n(\bar{\mathfrak{f}})| &\leqslant C\sum_{n=0}^{n_0}\frac{1}{n+1}\sqrt{\sum_{\substack{A\in\mathscr{E}_{*,n} \\ |x-u|=1}} [\bar{\mathfrak{f}}(A_{u,x},\rho) - \bar{\mathfrak{f}}(A,\rho)]^2} \\ &\leqslant C\sum_{n=0}^{n_0}\frac{1}{\sqrt{n+1}}\sqrt{\mathscr{D}_{\text{ex}}(\bar{\mathfrak{f}})} \\ &\leqslant C'\sqrt{\mathscr{D}_{\text{ex}}(\bar{\mathfrak{f}})}.\end{aligned}$$

In a similar way, making use of the Cauchy–Schwarz inequality, and applying Lemma 5.1 in Landim et al. (2002), we can obtain a constant C such that

$$\sum_{n=0}^{n_0} \frac{1}{n+1} |T_n(\bar{\mathfrak{f}})| \leqslant C\sqrt{\mathscr{D}_{\text{ex}}(\bar{\mathfrak{f}})}.$$

These inequalities prove the lemma. □

Since the functions $\{\alpha_x(.,\rho(t));\ |x|=1\}$ are C^∞, by induction, we can find bounds for the derivatives $U_\lambda^{(n)}(t)$ of the following form: there exists constants C_n such that

$$|U_\lambda^{(n)}(t)| \leqslant C_n \sup_{1\leqslant j\leqslant n} \sqrt{\mathscr{D}_{\text{ex}}(\bar{\mathfrak{f}}_\lambda^{(j)})}.$$

Consequently, it is enough to obtain for each $j \geqslant 0$, the bound

$$\sup_{0<\lambda}\ \sup_{0\leqslant t\leqslant \pi/2} \mathscr{D}_{\text{ex}}(\bar{\mathfrak{f}}_\lambda^{(j)}) < \infty.$$

Lemma 4.2. *Let f be a function such that $\|\mathfrak{f}\|_{-1,k} < \infty$ for all $k \geqslant 1$. Let $\mathfrak{h}_\lambda$ be the solution of the resolvent equation*

$$\lambda\mathfrak{h}_\lambda - \mathfrak{L}(t)\mathfrak{h}_\lambda = \mathfrak{f}.$$

Then

$$\|\mathfrak{h}_\lambda\|_{1,k} \leqslant C(k)\|\mathfrak{f}\|_{-1,k}$$

for a finite constant $C(k)$ independent of t and λ.

Proof. See the next section. □

It is easy to check that $\|\mathfrak{R}\|_{-1,k}$ is finite for all $k \geqslant 1$ using Schwarz's inequality. Indeed, if $\mathfrak{g}$ is a finite supported function of degree $n \leqslant n_0$, as pointed above, $|\ll \mathfrak{g}, \pi_n\mathfrak{R} \gg| \leqslant C\|\mathfrak{g}\|_1$ where C is a constant independent of t and n. Putting this back in the variational formula, it is shown that $\|\pi_n\mathfrak{R}\|_{-1}$ is bounded by C and since $\pi_n\mathfrak{R}=0$ for $n > n_0$, we have the result.

We are now interested in the proof of the differentiability of $\mathfrak{h}_\lambda(\cdot)$ in $\mathfrak{H}$. We say that a function $\mathfrak{v}(t)$ with values in $\mathfrak{H}$ is differentiable at t if $\gamma^{-1}[\mathfrak{v}(t+\gamma) - \mathfrak{v}(t)]$ converges, as $\gamma \to 0$, strongly in $\mathfrak{H}$ to some function that we denote by $\mathfrak{v}'(t)$ or $\partial_t\mathfrak{v}$. Remark that for finite supported functions $\mathfrak{v}(t) \in \mathfrak{H}$, whose supports do not depend on t, differentiability is equivalent to differentiability for each $A \in \mathscr{E}_*$ of $\bar{\mathfrak{v}}(A,t)$. In particular, $\mathfrak{R}$ is differentiable and its derivatives are such that

$$\bar{\mathfrak{R}}^{(j)}(A,t) = 2\sum_{u\in A, x\notin A} (l,(u-x))\partial_t^j\alpha_{u-x}(A_u^0 - x,\rho) - 2\sum_{x\notin A}(l,x)\partial_t^j\alpha_x(A,\rho).$$

Lemma 4.3. *Suppose that $\mathfrak{f}(t)$ is a differentiable function of t and let $\mathfrak{h}_\lambda$ be the solution of the resolvent equation*

$$\lambda\mathfrak{h}_\lambda - \mathfrak{L}(t)\mathfrak{h}_\lambda = \mathfrak{f}(t).$$

Then $\mathfrak{h}_\lambda$ is a differentiable function of t and its derivative is the solution of

$$\lambda\mathfrak{h}'_\lambda - \mathfrak{L}(t)\mathfrak{h}'_\lambda = \mathfrak{f}'(t) + \mathfrak{L}'(t)\mathfrak{h}_\lambda.$$

Here, $\partial_t\mathfrak{L} = \mathfrak{L}'(t)$ denotes the formal derivative of $\mathfrak{L}(t)$ which means that we have derivated the coefficients in the differents sums of the operator

$$\mathfrak{L}'_k(t)\mathfrak{f}(A) = \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}} \partial_t u_{x,y}(A,S,P,\rho)(\Pi_{x,y}\mathfrak{f})(A_S^P)$$

and

$$\mathfrak{L}' = \sum_{k\in\mathbb{Z}} \mathfrak{L}'_k.$$

Of course, for a finite supported function $\mathfrak{f}$

$$\overline{\mathfrak{L}'_k\mathfrak{f}} = \bar{\mathfrak{L}}'_k\bar{\mathfrak{f}}$$

and

$$\bar{\mathfrak{L}}'_k = \bar{\mathfrak{L}}'_{k,\mathrm{ex}} + \bar{\mathfrak{L}}'_{k,\tau} + \bar{\mathfrak{L}}'_{k,\tau,\mathrm{ex}},$$

where

$$\bar{\mathfrak{L}}'_{k,\mathrm{ex}}\bar{\mathfrak{f}}(A) = \frac{1}{2}\sum_{x,y}\sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}} \partial_t u_{x,y}(A\cup\{0\},S,P,\rho)(\Pi_{x,y}\bar{\mathfrak{f}})(A_S^P),$$

$$\bar{\mathfrak{L}}'_{k,\tau}\bar{\mathfrak{f}}(A) = \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}}\sum_{y\notin A_S^P} \partial_t u_{0,y}(A\cup\{0\},S,P,\rho)(\Phi_y\bar{\mathfrak{f}})(A_S^P),$$

$$\bar{\mathfrak{L}}'_{k,\tau,\mathrm{ex}}\bar{\mathfrak{f}}(A) = \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k-1}}\sum_{\substack{y\notin A_S^P \\ x\in A_S^P}} \partial_t u_{x,y}(A\cup\{0\},S\cup\{0\},P,\rho)(\Delta_{x,y}\bar{\mathfrak{f}})(A_S^P).$$

Proof. The only thing that needs to be checked is that $\mathfrak{L}'(t)\mathfrak{h}_\lambda$ belongs in fact to $\mathfrak{H}$. This is proven by Lemma 6.1 and the rest is done on the usual way. □

For the moment, we know that $\mathfrak{f}_\lambda$ is differentiable (for λ fixed as function of t) and that the derivative $\mathfrak{f}'_\lambda$ satisfies the resolvent equation

$$\lambda\mathfrak{f}'_\lambda - \mathfrak{L}(t)\mathfrak{f}'_\lambda = \mathfrak{R}'(t) + \mathfrak{L}'(t)\mathfrak{f}_\lambda.$$

To iterate these argument and obtain the differentiability of higher order, we need to control the $\mathfrak{H}_{-p}$-norm, $\|\cdot\|_{-1,p}$, of the derivatives. For $\mathfrak{R}^{(j)}$, we use the same argument as for $\mathfrak{R}$ but we must replace $\alpha_x(.,\rho)$ by $\partial_t^j\alpha_x(.,\rho)$. For the other derivatives, we have the following lemma.

Lemma 4.4. *For each $j \geqslant 1$, there exists finite constants $\{C_{p,j},\ p \geqslant 1\}$ independent of t such that*

$$\|\mathfrak{L}_k^{(j)}(t)\mathfrak{h}\|_{-1,p} \leqslant C_{p,j}\|\mathfrak{h}\|_{1,p+1}$$

for $|k| \leqslant n_0$.

Proof. To keep notations simple, we just give the proof for $j = 1$. The proof for higher order derivatives is similar. Recall that the norm $\|\cdot\|_{-1}$ of a function $\mathfrak{f}$ is given by the variational formula

$$\|\mathfrak{f}\|_{-1}^2 = \sup_{\mathfrak{g}}\{2\ll \mathfrak{f}, \mathfrak{g} \gg - \ll \mathfrak{g}, \mathfrak{g} \gg_1\}.$$

Recall that $\bar{\mathfrak{L}}_k'$ increases the degree by k if $k > 0$, conserves the degree if $k = 0$ and decreases the degree by $-k$ if $k < 0$. If we now take a supported finite function $\mathfrak{h}$ of degree $n - k + 1$, to evaluate $\|\mathfrak{h}\|_{-1}$, we need only to consider the functions of degree $n + 1$ in the variational formula.

Let $\mathfrak{h}$ be a function of degree $n - k + 1$ and $\mathfrak{g}$ be a function of degree $n + 1$. We have to bound $\langle \bar{\mathfrak{L}}_{k,\mathrm{ex}}' \bar{\mathfrak{h}}, \bar{\mathfrak{g}} \rangle$ in term of $\|\mathfrak{g}\|_1$.

According to Remark 5.3, the following inequality holds:

$$\left| \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \overline{\partial_t \mathfrak{L}}_{k,\mathrm{ex}} \bar{\mathfrak{h}}(A)\bar{\mathfrak{g}}(A) \right|$$

$$= \frac{1}{4(n+1)} \left| \sum_{\substack{(A,S,P) \in \mathscr{U}_{*,n}^k \\ |y-x|=1}} \partial_t u_{x,y}(A \cup \{0\}, S, P, \rho)(\Pi_{x,y}\bar{\mathfrak{h}})(A_S^P)(\Pi_{x,y}\bar{\mathfrak{g}})(A) \right|$$

$$\leqslant \frac{1}{4(n+1)} \sqrt{\mathscr{I}_n(\bar{\mathfrak{g}})}\sqrt{\mathscr{J}_n(\bar{\mathfrak{h}})},$$

where

$$\mathscr{I}_n(\bar{\mathfrak{g}}) = \sum_{\substack{(A,S,P) \in \mathscr{U}_{*,n}^k \\ |y-x|=1}} |\partial_t u_{x,y}(A \cup \{0\}, S, P, \rho)|\,(\bar{\mathfrak{g}}(A_{x,y}) - \bar{\mathfrak{g}}(A))^2$$

and

$$\mathscr{J}_n(\bar{\mathfrak{h}}) = \sum_{\substack{(A,S,P) \in \mathscr{U}_{*,n}^k \\ |y-x|=1}} |\partial_t u_{x,y}(A \cup \{0\}, S, P, \rho)|\,(\bar{\mathfrak{h}}((A_S^P)_{x,y}) - \bar{\mathfrak{h}}(A_S^P))^2.$$

The principle is the same as in Lemma 5.1. We show that there exists some constant C such that

$$\mathscr{I}_n(\bar{\mathfrak{g}}) \leqslant Cn\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{g}}),$$

$$\mathscr{J}_n(\bar{\mathfrak{h}}) \leqslant Cn\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}}).$$

So that

$$\left|\frac{1}{n+1}\sum_{A\in\mathscr{E}_{*,n}}\overline{\partial_t\mathfrak{L}_{k,\mathrm{ex}}}\bar{\mathfrak{h}}(A)\bar{\mathfrak{g}}(A)\right| \leqslant C\sqrt{\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{h}})}\sqrt{\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{g}})}.$$

Using this in the variational formula, we obtain the first inequality.

By Remark 5.3, we have for $\overline{\partial_t\mathfrak{L}_{k,\tau}}$ the following estimate:

$$\left|\frac{1}{n+1}\sum_{A\in\mathscr{E}_{*,n}}\overline{\partial_t\mathfrak{L}_{k,\tau}}\bar{\mathfrak{h}}(A)\bar{\mathfrak{g}}(A)\right| \leqslant \frac{1}{n+1}\sqrt{\mathscr{P}_n(\bar{\mathfrak{g}})}\sqrt{\mathscr{Q}_n(\bar{\mathfrak{h}})},$$

where

$$\mathscr{Q}_n(\bar{\mathfrak{h}}) = \sum_{(A,S,P)\in\mathscr{U}^k_{*,n}}\ \sum_{\substack{|y|=1\\ y\notin A\cup P}} |\partial_t u_{0,y}(A\cup\{0\},S,P,\rho)|\,(\bar{\mathfrak{h}}(A^P_S - y) - \bar{\mathfrak{h}}(A^P_S))^2$$

and

$$\mathscr{P}_n(\bar{\mathfrak{g}}) = \sum_{(A,S,P)\in\mathscr{U}^k_{*,n}}\ \sum_{\substack{|y|=1\\ y\notin A\cup P}} |\partial_t u_{0,y}(A\cup\{0\},S,P,\rho)|\,(\bar{\mathfrak{g}}(A-y) - \bar{\mathfrak{g}}(A))^2.$$

As in Lemma 5.1, it is not hard to show that

$$\left|\frac{1}{n+1}\sum_{A\in\mathscr{E}_{*,n}}\overline{\partial_t\mathfrak{L}_{k,\tau}}\bar{\mathfrak{h}}(A)\bar{\mathfrak{g}}(A)\right| \leqslant C\sqrt{\mathscr{D}_{\tau}(\bar{\mathfrak{h}})}\sqrt{\mathscr{D}_{\tau}(\bar{\mathfrak{g}})} \leqslant Cn\sqrt{\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{h}})}\sqrt{\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{g}})}$$

by Lemma 5.1 in Landim et al. (2002).

The third operator $\overline{\partial_t\mathfrak{L}_{k,\tau,\mathrm{ex}}}$ can be evaluated by a similar method (see Lemma 5.1 for more details) and we have

$$\left|\frac{1}{n+1}\sum_{A\in\mathscr{E}_{*,n}}\overline{\partial_t\mathfrak{L}_{k,\tau,\mathrm{ex}}}\bar{\mathfrak{h}}(A)\bar{\mathfrak{g}}(A)\right| \leqslant C\sqrt{\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{h}})}\sqrt{\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{g}})}.$$

And with this, we are done. □

By now, we have all that is needed in order to prove the regularity of $\|R\|_{-1,\rho(.)}$ on a similar way to Landim et al. (2001). The main difference being that here, the coefficients of R depend on t and the $\mathfrak{H}$-norm of $\overline{\partial_t\mathfrak{L}}\bar{\mathfrak{f}}_\lambda$ is bounded by $\|\bar{\mathfrak{f}}_\lambda\|_{1,1}$.

We have shown that $\{\mathfrak{f}_\lambda(t),\ \lambda>0\}$ is a family of C^∞ functions. Its derivatives satisfy for each $k\geqslant 0$

$$\sup_{0<\lambda}\ \sup_{0\leqslant t\leqslant\pi/2}\|\mathfrak{f}'_\lambda(t)\|_{1,k}<\infty$$

in view of the preceding lemma, Lemmas 4.3 and 4.4.

To iterate the argument, it is enough to check by induction the existence of constants $(a_{n,i})_{0\leqslant i<n}$ such that

$$\lambda\mathfrak{f}^{(j)}_\lambda - \mathfrak{L}(t)\mathfrak{f}^{(j)}_\lambda = \sum_{i=0}^{j-1} a_{j,i}\mathfrak{L}^{(j-i)}(t)\mathfrak{f}^{(i)}_\lambda + \mathfrak{R}^{(j)}.$$

Then, the functions $\mathfrak{f}_\lambda(t)$ are C^∞ functions on $[0,1]$ with their derivatives having the uniforms bounds

$$\sup_{0<\lambda} \sup_{0\leqslant t\leqslant \pi/2} \mathscr{D}_{\rm ex}(\bar{\mathfrak{f}}_\lambda^{(j)}) < \infty.$$

5. Proof of Lemma 4.2

In order to keep notations as simple as possible, we note $\mathscr{L}$ for $\mathscr{L}(t)$ and ρ for $\rho(t)$. We can write a n_0-diagonal block matrix $\mathscr{L}_{i,j}$ corresponding to the decomposition $\mathbb{L}^2(v_{\rho(t)}) = \oplus \mathscr{H}_j$. Let $u = \sum_j u_j$ be the decomposition of the local function u in this orthogonal sum. We consider an operator $\mathbb{T}$ acting on $\mathscr{H}_j$ as scalar multiplication by $t(j) > 0$ and we suppose that $t(j)$ is constant for $j \leqslant n_1$ and for $j \geqslant n_2$. The commutator $[\mathbb{T}, \mathscr{L}] = \mathbb{T}\mathscr{L} - \mathscr{L}\mathbb{T}$ can be computed

$$[\mathbb{T}, \mathscr{L}](u) = \sum_{k=-n_0}^{n_0} \sum_j [t(j) - t(j-k)]\mathscr{L}_{j-k,j} u_{j-k}.$$

We let $t(j)u_j = v_j$ and $s_k(j) = t(j) - t(j-k)$. We thus have

$$\begin{aligned}
\ll [\mathbb{T}, \mathscr{L}]u, \mathbb{T}u \gg_\rho &= \sum_{k=-n_0}^{n_0} \sum_j \frac{s_k(j)}{t(j-k)} \ll \mathscr{L}_{j-k,j} v_{j-k}, v_j \gg_\rho \\
&= \sum_{k=1}^{n_0} \sum_j \frac{s_k(j)}{t(j-k)} \ll \mathscr{L}_{j-k,j} v_{j-k}, v_j \gg_\rho \\
&\quad - \sum_{k=1}^{n_0} \sum_j \frac{s_k(j+k)}{t(j+k)} \ll \mathscr{L}_{j+k,j} v_{j+k}, v_j \gg_\rho \\
&= \sum_{k=1}^{n_0} \sum_j \frac{s_k(j)}{t(j-k)} \ll \mathscr{L}_{j-k,j} v_{j-k}, v_j \gg_\rho \\
&\quad - \sum_{k=1}^{n_0} \sum_j \frac{s_k(j+k)}{t(j+k)} \ll \mathscr{L}_{j,j+k} v_j, v_{j+k} \gg_\rho
\end{aligned}$$

because $\mathscr{L}$ is symmetric. Then

$$\begin{aligned}
&\ll [\mathbb{T}, \mathscr{L}]u, \mathbb{T}u \gg_\rho \\
&= \sum_{k=1}^{n_0} \sum_j \frac{s_k(j)^2}{t(j-k)t(j)} \ll \mathscr{L}_{j-k,j} v_{j-k}, v_j \gg_\rho \\
&= \sum_{k=1}^{n_0} \sum_j \frac{s_k(j)^2}{t(j-k)t(j)} \ll \mathscr{L} v_{j-k}, v_j \gg_\rho
\end{aligned}$$

$$\leqslant \sum_{k=1}^{n_0}\sum_j \frac{s_k(j)^2}{t(j-k)t(j)}\sqrt{\ll -\mathscr{L}v_{j-k}, v_{j-k}\gg_\rho}\sqrt{\ll -\mathscr{L}v_j, v_j\gg_\rho}$$

$$\leqslant \frac{1}{2}\sum_{k=1}^{n_0}\sum_j \frac{s_k(j)^2}{t(j-k)t(j)}\{\ll -\mathscr{L}v_{j-k}, v_{j-k}\gg_\rho + \ll -\mathscr{L}v_j, v_j\gg_\rho\}$$

$$=\frac{1}{2}\sum_{k=1}^{n_0}\sum_j \frac{s_k(j)^2}{t(j-k)t(j)}\{\ll -\mathfrak{L}\mathfrak{v}_{j-k}, \mathfrak{v}_{j-k}\gg + \ll -\mathfrak{L}\mathfrak{v}_j, \mathfrak{v}_j\gg\}$$

because $-\mathfrak{L}$ is positive and symmetric.

Lemma 5.1. Let $\mathfrak{f}$ *be a finite supported function of degree* $n+1$. *Then the following inequality holds*:

$$\ll -\mathfrak{L}\mathfrak{f},\mathfrak{f}\gg \leqslant Cn\mathscr{D}_{\rm ex}(\bar{\mathfrak{f}}),$$

where C *is some constant independent of* $\mathfrak{f}$ *and* ρ.

Proof. There are three operators which are degree-conservative $\bar{\mathfrak{L}}_{0,\rm ex}, \bar{\mathfrak{L}}_{0,\tau}, \bar{\mathfrak{L}}_{0,\tau,\rm ex}$. We examine each of them separately. For the first, Lemma 5.2 gives

$$\left|\frac{1}{n+1}\sum_{A\in\mathscr{E}_{*,n}} \bar{\mathfrak{L}}_{0,\rm ex}\bar{\mathfrak{f}}(A)\bar{\mathfrak{f}}(A)\right|$$

$$=\frac{1}{4(n+1)}\sum_{\substack{(A,S,P)\in\mathscr{U}^0_{*,n}\\ |y-x|=1}} |u_{x,y}(A\cup\{0\},S,P,\rho)|\,|(\Pi_{x,y}\bar{\mathfrak{f}})(A^P_S)(\Pi_{x,y}\bar{\mathfrak{f}})(A)|$$

$$\leqslant \frac{1}{4(n+1)}\sqrt{\mathscr{I}_n(\bar{\mathfrak{f}})}\sqrt{\mathscr{J}_n(\bar{\mathfrak{f}})},$$

where

$$\mathscr{I}_n(\bar{\mathfrak{f}}) = \sum_{\substack{(A,S,P)\in\mathscr{U}^0_{*,n}\\ |y-x|=1}} |u_{x,y}(A\cup\{0\},S,P,\rho)|\,|\bar{\mathfrak{f}}(A_{x,y})-\bar{\mathfrak{f}}(A)|^2$$

and

$$\mathscr{J}_n(\bar{\mathfrak{f}}) = \sum_{\substack{(A,S,P)\in\mathscr{U}^0_{*,n}\\ |y-x|=1}} |u_{x,y}(A\cup\{0\},S,P,\rho)|\,|\bar{\mathfrak{f}}((A^P_S)_{x,y})-\bar{\mathfrak{f}}(A^P_S)|^2.$$

To estimate $\mathscr{J}_n$, recall that u is uniformly bounded so that we have to evaluate, for fixed $B\in\mathscr{E}_{*,n}$, $x\in B$, $y\notin B$, the cardinal $\Delta(B,x,y)$ of the set

$$\{(A,S,P)\in\mathscr{U}^0_{*,n}/A^P_S=B, S-x\in\mathscr{F}, P-x\in\mathscr{F}\}.$$

Since x is fixed, at most $|\mathscr{F}|$ choices remain for S and $|\mathscr{F}|$ choices for P and then $A=B^S_P$ is fixed. Therefore, $\Delta(B,x,y)$ is less than K where K is a constant independent of B,x,y.

In consequence, there exists one constant C such that

$$\mathscr{J}_n(\bar{\mathfrak{f}}) \leqslant Cn\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}}).$$

Estimation of $\mathscr{I}_n$ can be computed in a same way and it can be shown that

$$\mathscr{I}_n(\bar{\mathfrak{f}}) \leqslant Cn\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}}).$$

Finally, we obtain

$$\left| \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \bar{\mathfrak{L}}_{0,\mathrm{ex}} \bar{\mathfrak{f}}(A) \bar{\mathfrak{f}}(A) \right| \leqslant C\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}}).$$

For $\bar{\mathfrak{L}}_{0,\tau}$, we just indicate the mains steps. From Lemma 5.2, we have

$$\left| \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \bar{\mathfrak{L}}_{0,\tau} \bar{\mathfrak{f}}(A) \bar{\mathfrak{f}}(A) \right| \leqslant \frac{1}{2(n+1)} \sqrt{\mathscr{P}_n(\bar{\mathfrak{f}})} \sqrt{\mathscr{Q}_n(\bar{\mathfrak{f}})},$$

where

$$\mathscr{Q}_n(\bar{\mathfrak{f}}) = \sum_{(A,S,P) \in \mathscr{U}^0_{*,n}} \sum_{\substack{|y|=1 \\ y \notin A \cup P}} |u_{0,y}(A \cup \{0\}, S, P, \rho)| \, (\bar{\mathfrak{f}}(A^P_S - y) - \bar{\mathfrak{f}}(A^P_S))^2$$

and

$$\mathscr{P}_n(\bar{\mathfrak{f}}) = \sum_{(A,S,P) \in \mathscr{U}^0_{*,n}} \sum_{\substack{|y|=1 \\ y \notin A \cup P}} |u_{0,y}(A \cup \{0\}, S, P, \rho)| \, (\bar{\mathfrak{f}}(A - y) - \bar{\mathfrak{f}}(A))^2.$$

Using the fact that u is uniformly bounded, we can check that

$$\left| \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \bar{\mathfrak{L}}_{0,\tau} \bar{\mathfrak{f}}(A) \bar{\mathfrak{f}}(A) \right| \leqslant C\mathscr{D}_{\tau}(\bar{\mathfrak{f}})$$

$$\leqslant C' n\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}})$$

by Lemma 5.1 in Landim et al. (2002).

The third operator $\bar{\mathfrak{L}}_{0,\tau,\mathrm{ex}}$ remains alone. By Lemma 5.2

$$\left| \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \mathfrak{L}_{0,\tau,\mathrm{ex}} \bar{\mathfrak{f}}(A) \bar{\mathfrak{f}}(A) \right| \leqslant \frac{1}{2(n+1)} \sqrt{\mathscr{H}_n(\bar{\mathfrak{f}})} \sqrt{\mathscr{G}_n(\bar{\mathfrak{f}})},$$

where

$$\mathscr{H}_n(\bar{\mathfrak{f}}) = \sum_{(A,S,P) \in \mathscr{U}^{-1}_{*,n}} \sum_{\substack{|y-x|=1 \\ x \in A^P_S, y \notin A^P_S}} |u_{x,y}(A \cup \{0\}, S \cup \{0\}, P, \rho)| \, ((\Delta_{x,y} \bar{\mathfrak{f}})(A^P_S))^2$$

and

$$\mathscr{G}_n(\bar{\mathfrak{f}}) = \sum_{(A,S,P) \in \mathscr{U}^{-1}_{*,n}} \sum_{\substack{|y-x|=1 \\ x \in A^P_S, y \notin A^P_S}} |u_{x,y}(A \cup \{0\}, S \cup \{0\}, P, \rho)| \, (\bar{\mathfrak{f}}(A_{x,y}) - \bar{\mathfrak{f}}(A))^2.$$

To evaluate $\mathscr{H}_n(\bar{\mathfrak{f}})$, since u is uniformly bounded, we must, for each $B \in \mathscr{E}_{*,n+1}$, $x \in B$, $y \notin B$, estimate the cardinal $\Gamma(B,x,y)$ of the set

$$\{(A,S,P) \in \mathscr{U}_{*,n}^{-1} / B = A_S^P, S - x \in \mathscr{F}, P - x \in \mathscr{F}\},$$

which is bounded by a constant independent of B, x and y.

Consequently, we have

$$\begin{aligned}\mathscr{H}_n(\bar{\mathfrak{f}}) &\leqslant C \sum_{B \in \mathscr{E}_{*,n+1}} \sum_{\substack{|y-x|=1 \\ x \in B, y \notin B}} [\bar{\mathfrak{f}}((B \backslash \{x\}) - y) - \bar{\mathfrak{f}}((B \backslash \{x\}) - x)]^2 \\ &\leqslant C \sum_{A \in \mathscr{E}_{*,n}} \sum_{\substack{|y-x|=1 \\ x \notin A, y \notin A}} [\bar{\mathfrak{f}}(A - y) - \bar{\mathfrak{f}}(A - x)]^2 \\ &\leqslant 2Cd \sum_{\substack{A \in \mathscr{E}_{*,n} \\ x \notin A}} [\bar{\mathfrak{f}}(A - y) - \bar{\mathfrak{f}}(A)]^2 \\ &\leqslant Kn^2 \mathscr{D}_{\text{ex}}(\bar{\mathfrak{f}}) \text{ by Lemma 5.1 in [4].}\end{aligned}$$

The evaluation of $\mathscr{G}_n(\bar{\mathfrak{f}})$ is not difficult and we have

$$\mathscr{G}_n(\bar{\mathfrak{f}}) \leqslant C \mathscr{D}_{\text{ex}}(\bar{\mathfrak{f}}).$$

Combining these two results, we have proven the third inequality

$$\left| \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \bar{\mathfrak{L}}_{0,\tau,\text{ex}} \bar{\mathfrak{f}}(A) \bar{\mathfrak{f}}(A) \right| \leqslant C \mathscr{D}_{\text{ex}}(\bar{\mathfrak{f}}),$$

where C is a constant.

Those three inequalities give the desired result. □

Lemma 5.2. *If* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are, respectively, finite supported functions of degree* $n-k+1$ *and* $n+1$ *then we have*

1.

$$\begin{aligned}&-\frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \bar{\mathfrak{L}}_{k,\text{ex}} \bar{\mathfrak{f}}(A) \bar{\mathfrak{g}}(A) \\ &\quad = \frac{1}{4(n+1)} \sum_{\substack{(A,S,P) \in \mathscr{U}_{*,n}^k \\ |y-x|=1}} u_{x,y}(A \cup \{0\}, S, P, \rho)(\Pi_{x,y}\bar{\mathfrak{f}})(A_S^P)(\Pi_{x,y}\bar{\mathfrak{g}})(A),\end{aligned}$$

2.

$$\begin{aligned}&-\frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \bar{\mathfrak{L}}_{k,\tau} \bar{\mathfrak{f}}(A) \bar{\mathfrak{g}}(A) \\ &\quad = \frac{1}{2(n+1)} \sum_{\substack{(A,S,P) \in \mathscr{U}_{*,n}^k \\ |y|=1, y \notin A}} u_{0,y}(A \cup \{0\}, S, P, \rho)(\Phi_y\bar{\mathfrak{f}})(A_S^P)(\Phi_y\bar{\mathfrak{g}})(A),\end{aligned}$$

3.

$$-\frac{1}{n+1}\sum_{A\in\mathscr{E}_{*,n}}\bar{\mathfrak{L}}_{k,\tau,\mathrm{ex}}\bar{\mathfrak{f}}(A)\bar{\mathfrak{g}}(A)$$

$$=\frac{1}{2(n+1)}\sum_{\substack{(A,S,P)\in\mathscr{U}^{k-1}_{*,n}\\ |x-y|=1\\ x\in A^P_S, y\notin A^P_S}} u_{x,y}(A\cup\{0\},S\cup\{0\},P,\rho)(\Delta_{x,y}\bar{\mathfrak{f}})(A^P_S)(\Pi_{x,y}\bar{\mathfrak{g}})(A),$$

where C is a constant independent of $\mathfrak{f},\mathfrak{g}$.

Proof. We just prove the first point. The others may be established similarly. First, recall the definition of u given by (3.1) and its main properties given by (3.4)–(3.6). Now

$$-\sum_{A\in\mathscr{E}_{*,n}}\bar{\mathfrak{L}}_{k,\mathrm{ex}}\bar{\mathfrak{f}}(A)\bar{\mathfrak{g}}(A) \tag{5.1}$$

$$=\frac{1}{2}\sum_{\substack{(A,S,P)\in\mathscr{U}^{k}_{*,n}\\ |y-x|=1}} u_{x,y}(A\cup\{0\},S,P,\rho)(\Pi_{x,y}\bar{\mathfrak{f}})(A^P_S)(\Pi_{x,y}\bar{\mathfrak{g}})(A) \tag{5.2}$$

$$-\frac{1}{2}\sum_{\substack{(A,S,P)\in\mathscr{U}^{k}_{*,n}\\ |y-x|=1}} u_{x,y}(A\cup\{0\},S,P,\rho)(\Pi_{x,y}\bar{\mathfrak{f}})(A^P_S)\bar{\mathfrak{g}}(A_{x,y}). \tag{5.3}$$

According to u's properties cited above, and from $x,y\notin S\cup P$, $(A^P_S)_{x,y}=(A_{x,y})^P_S$, we can rewrite the second term as

$$\frac{1}{2}\sum_{\substack{(A,S,P)\in\mathscr{U}^{k}_{*,n}\\ |y-x|=1}} u_{x,y}(A\cup\{0\},S,P,\rho)(\Pi_{x,y}\bar{\mathfrak{f}})(A^P_S)\bar{\mathfrak{g}}(A_{x,y}) \tag{5.4}$$

$$=-\frac{1}{2}\sum_{\substack{(A,S,P)\in\mathscr{U}^{k}_{*,n}\\ |y-x|=1}} u_{x,y}(A\cup\{0\},S,P,\rho)(\Pi_{x,y}\bar{\mathfrak{f}})((A_{x,y})^P_S)\bar{\mathfrak{g}}(A_{x,y}) \tag{5.5}$$

and so (5.4) is equal to

$$-\frac{1}{2}\sum_{\substack{|y-x|=1\\ (B_{x,y},S,P)\in\mathscr{U}^{k}_{*,n}}} u_{x,y}(B_{x,y}\cup\{0\},S,P,\rho)(\Pi_{x,y}\bar{\mathfrak{f}})(B^P_S)(\bar{\mathfrak{g}})(B)$$

by the change of variables $B=A_{x,y}$ for a fixed x,y. Remark now that $u_{x,y}(B_{x,y}\cup\{0\},S,P,\rho)=u_{x,y}(B\cup\{0\},S,P,\rho)$ by (2.1). Eq. (3.5) allows indexation to be replaced by $|y-x|=1$, $(B,S,P)\in\mathscr{U}^k_{*,n}$. Indeed, fix $B\in\mathscr{E}_{*,n}$, $x,y\in\mathbb{Z}^d_*$ and remark that $(B_{x,y},S,P)\in\mathscr{U}^k_{*,n}$ if and only if $(B,S,P)\in\mathscr{U}^k_{*,n}$ otherwise $u_{x,y}(B\cup\{0\},S,P,\rho)=0$. Along with (5.1), this completes the proof. □

Remark 5.3. This lemma remains true if $\bar{\mathfrak{L}}_{k,\mathrm{ex}},\bar{\mathfrak{L}}_{k,\tau}\bar{\mathfrak{L}}_{k,\tau,\mathrm{ex}}$ is replaced by $\partial_t^j\bar{\mathfrak{L}}_{k,\mathrm{ex}},\partial_t^j\bar{\mathfrak{L}}_{k,\tau}$, $\partial_t^j\bar{\mathfrak{L}}_{k,\tau,\mathrm{ex}}$, where $j\geqslant 0$, since the derivatives of $u(\cdot,\cdot,\cdot,\rho(t))$ verify (3.4)–(3.6).

We conclude the proof as in Landim et al. (2002). We have

$$\ll[\mathbb{T},\mathscr{L}]\mathbb{T}^{-1}u,\mathbb{T}u\gg_\rho\leqslant\sum_j\sum_{k=1}^{n_0}\frac{C_0 j s_k(j)^2}{t(j-k)t(j)}[\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{v}}_{j-k})+\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{v}}_j)].$$

Let us suppose that

$$\sup_j\frac{C_0 j s_k(j)^2}{t(j-k)t(j)}\leqslant\frac{\delta}{2\gamma n_0}\quad\text{for } k\in\mathbb{N}_{n_0},$$

where γ is a constant (independent of ρ) such that $\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{w}})\leqslant\gamma\ll-\mathscr{L}w,w\gg_\rho$.

Then

$$\ll[\mathbb{T},\mathscr{L}]\mathbb{T}^{-1}\mathbb{T}u,\mathbb{T}u\gg_\rho\leqslant\frac{\delta}{\gamma}\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{v}})\leqslant\delta\ll-\mathscr{L}v,v\gg_\rho=\delta\ll-\mathscr{L}\mathbb{T}u,\mathbb{T}u\gg_\rho.$$

Since h_λ is the solution of the resolvent equation

$$\lambda h_\lambda-\mathscr{L}h_\lambda=f$$

operating by $\mathbb{T}$ and taking inner product with $k_\lambda=\mathbb{T}h_\lambda$, we get

$$\begin{aligned}\lambda\ll k_\lambda,k_\lambda\gg_\rho-\ll\mathscr{L}k_\lambda,k_\lambda\gg_\rho&=\ll[\mathbb{T},\mathscr{L}]\mathbb{T}^{-1}k_\lambda,k_\lambda\gg_\rho+\ll\mathbb{T}f,k_\lambda\gg_\rho\\&\leqslant\delta\ll-\mathscr{L}k_\lambda,k_\lambda\gg_\rho+\ll\mathbb{T}f,k_\lambda\gg_\rho.\end{aligned}$$

Since $\|w\|^2_{-1,\rho}\leqslant\gamma\|\mathfrak{w}\|^2_{-1}$ by (3.12), this implies the estimate

$$\begin{aligned}\|k_\lambda\|^2_{1,\rho}=\ll-\mathscr{L}k_\lambda,k_\lambda\gg_\rho&\leqslant(1-\delta)^{-1}|\ll\mathbb{T}f,k_\lambda\gg_\rho|\\&\leqslant(1-\delta)^{-1}\|\mathbb{T}f\|_{-1,\rho}\|k_\lambda\|_{1,\rho}\\&\leqslant\frac{\sqrt{\gamma}}{1-\delta}\|\mathbb{T}\mathfrak{f}\|_{-1}\|k_\lambda\|_{1,\rho}\end{aligned}$$

and in particular

$$\sup_{\lambda>0}\|\mathbb{T}h_\lambda\|_1\leqslant\frac{\gamma}{1-\delta}\|\mathbb{T}\mathfrak{f}\|_{-1}.$$

If we take $t(j)=\mathrm{e}^{c\sqrt{j}}$, with a good c, we can obtain

$$\sup_j\frac{C_0 j s_k(j)^2}{t(j-k)t(j)}\leqslant\frac{\delta}{2\gamma n_0}\quad\text{for } k\in\mathbb{N}_{n_0}.$$

It is then easy to deduce that for any function f with $\mathfrak{H}_{-p}$-norm finite, there exists a constant $C(p)$

$$\|\mathfrak{h}_\lambda\|_{1,p}\leqslant C(p)\|\mathfrak{f}\|_{-1,p}.$$

6. Estimation of the $\mathfrak{H}$-norm of $\mathfrak{L}'(t)\bar{\mathfrak{h}}_\lambda$

Lemma 6.1. *Let f be a local function. The following inequalities are valid*:

$$\|\overline{\partial_t \mathfrak{L}}_{k,\mathrm{ex}}\bar{\mathfrak{f}}\|_0 \leqslant C\|\mathfrak{f}\|_{1,1},$$

$$\|\overline{\partial_t \mathfrak{L}}_{k,\tau}\bar{\mathfrak{f}}\|_0 \leqslant C\|\mathfrak{f}\|_{1,1},$$

$$\|\overline{\partial_t \mathfrak{L}}_{k,\tau,\mathrm{ex}}\bar{\mathfrak{f}}\|_0 \leqslant C\|\mathfrak{f}\|_{1,1},$$

where C is a constant.

Proof. Let f be a local function of degree $n-k+1$. We have

$$4\sum_{A\in\mathscr{E}_{*,n}} (\overline{\partial_t \mathfrak{L}}_{k,\mathrm{ex}}\bar{\mathfrak{f}})^2(A)$$

$$= \sum_{A\in\mathscr{E}_{*,n}} \left[\sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}} \sum_{\substack{x\in A\cup P \\ y\notin A\cup P}} \partial_t u_{x,y}(A\cup\{0\},S,P,\rho)(\Pi_{x,y}\bar{\mathfrak{f}})(A_S^P) \right]^2$$

$$\leqslant \sum_{A\in\mathscr{E}_{*,n}} \mathscr{I}_n(A,t)\mathscr{J}_n(A,t),$$

where

$$\mathscr{I}_n(A,t) = \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}} \sum_{\substack{x\in A\cup P \\ y\notin A\cup P}} |\partial_t u_{x,y}(A\cup\{0\},S,P,\rho)|$$

and

$$\mathscr{J}_n(A,t)$$

$$= \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}} \sum_{\substack{x\in A\cup P \\ y\notin A\cup P}} |\partial_t u_{x,y}(A\cup\{0\},S,P,\rho)|\,(\bar{\mathfrak{f}}((A_S^P)_{x,y}) - \bar{\mathfrak{f}}(A_S^P))^2.$$

Using the fact that $u_{x,y}(A\cup\{0\},S,P,\rho)=0$ if $S\notin\mathscr{F}+x$ or $P\notin\mathscr{F}+x$, it is straightforward to show that $\mathscr{I}_n$ is uniformly bounded by Cn where C is a constant independent of k (since we can always suppose $|k|\leqslant n_0$), A and t.

As it has been done in Section 5, we can obtain a bound for

$$\sum_{A\in\mathscr{E}_{*,n}} \mathscr{J}_n(A,t)$$

of the form $Cn\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}})$.

Consequently, there exists a constant C such that

$$\sum_{A\in\mathscr{E}_{*,n}} (\overline{\partial_t \mathfrak{L}_{k,\mathrm{ex}}\bar{\mathfrak{f}}})^2(A) \leqslant Cn^2 \mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}})$$

and we obtain the first inequality.

For the second inequality, we have

$$\sum_{A\in\mathscr{E}_{*,n}} (\overline{\partial_t \mathfrak{L}_{k,\tau}\bar{\mathfrak{f}}})^2(A)$$

$$= \sum_{A\in\mathscr{E}_{*,n}} \left[\sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}} \sum_{y\notin A\cup P} \partial_t u_{0,y}(A\cup\{0\}, S, P, \rho)(\Phi_y \bar{\mathfrak{f}})(A_S^P) \right]^2$$

$$\leqslant \sum_{A\in\mathscr{E}_{*,n}} \mathscr{P}_n(A,t)\mathscr{Q}_n(A,t),$$

where

$$\mathscr{P}_n(A,t) = \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}} \sum_{y\notin A\cup P} |\partial_t u_{0,y}(A\cup\{0\}, S, P, \rho)|$$

and

$$\mathscr{Q}_n(A,t) = \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k}} \sum_{y\notin A\cup P} |\partial_t u_{0,y}(A\cup\{0\}, S, P, \rho)| \, (\bar{\mathfrak{f}}(A_S^P - y) - \bar{\mathfrak{f}}((A_S^P))^2.$$

$\mathscr{P}_n$ is bounded by a constant independent of A and t. This gives the bound

$$\sum_{A\in\mathscr{E}_{*,n}} \mathscr{P}_n(A,t).$$

Since $u_{0,y}(A\cup\{0\}, S, P, \rho) = 0$ if $|y| \neq 0$ or if S or P are not in $\mathscr{F}$

$$\sum_{A\in\mathscr{E}_{*,n}} \mathscr{P}_n(A,t) \leqslant C\mathscr{D}_\tau(\bar{\mathfrak{f}}) \leqslant C' n^2 \mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}})$$

by Lemma 5.1 in Landim et al. (2002).

Similarly, we obtain the following estimates:

$$\sum_{A\in\mathscr{E}_{*,n}} (\overline{\partial_t \mathfrak{L}_{k,\tau,\mathrm{ex}}\bar{\mathfrak{f}}})^2(A)$$

$$= \sum_{A\in\mathscr{E}_{*,n}} \left[\sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k-1}} \sum_{\substack{x\in A\cup P \\ y\notin A\cup P}} \partial_t u_{x,y}(A\cup\{0\}, S\cup\{0\}, P, \rho)(\Delta_{x,y}\bar{\mathfrak{f}})(A_S^P) \right]^2$$

$$\leqslant \sum_{A\in\mathscr{E}_{*,n}} \mathscr{G}_n(A,t)\mathscr{H}_n(A,t),$$

where

$$\mathscr{G}_n(A,t) = \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k-1}} \sum_{\substack{x\in A\cup P \\ y\notin A\cup P}} |\partial_t u_{x,y}(A\cup\{0\}, S\cup\{0\}, P, \rho)|$$

and

$$\mathscr{H}_n(A,t) = \sum_{\substack{S\subset A, P\cap A=\emptyset \\ |S|-|P|=k-1}} \sum_{\substack{x\in A\cup P \\ y\notin A\cup P}} |\partial_t u_{x,y}(A\cup\{0\}, S\cup\{0\}, P, \rho)|\,((\Delta_{x,y}\bar{\mathfrak{f}})(A_S^P))^2.$$

The term $\mathscr{G}_n$ is uniformly bounded by Cn for a constant C. The term $\sum_{A\in\mathscr{E}_{*,n}} \mathscr{H}_n(A,t)$ is uniformly bounded by $Cn^2\mathscr{D}_{\mathrm{ex}}(\bar{\mathfrak{f}})$. □

Remark 6.2. Same estimates hold for the derivatives of higher order.

7. Regularity at the boundary

Up to now, we have shown that $\bar{\mathfrak{f}}_\lambda(t)$ is a sequence of C^∞ functions on $[0,1]$ with uniformly bounded derivatives in norm k

$$\sup_{0<\lambda}\sup_{0\leqslant t\leqslant\pi/2} \|f_\lambda^{(j)}(t)\|_{1,k} \leqslant \infty$$

for each $j\geqslant 0$ and $k\geqslant 1$. A first consequence of this is the regularity of $\|R\|_{-1,\rho}$ on $(0,1)$.

Lemma 7.1. *For each $j\geqslant 0$, we have*

$$\bar{\mathfrak{f}}_\lambda^{(2j)}(A,0) = 0 \qquad \textit{if } |A| \textit{ is odd},$$
$$\bar{\mathfrak{f}}_\lambda^{(2j+1)}(A,0) = 0 \quad \textit{if } |A| \textit{ is even}.$$

Proof. Remark that

$$\bar{\mathfrak{R}}(A,t) = 2(\sin t\cos t)^{|A|}$$

$$\times\left(\sum_{\substack{x\in A\\ u\notin A}}(l,(u-x))Q_{u-x}(A_u^0-x,\sin^2(t))-2\sum_{x\notin A}(l,x)Q_x(A,\sin^2(t))\right).$$

Therefore $\bar{\mathfrak{R}}(A,.)$ is an even function if $|A|$ is even, and an odd function if $|A|$ is odd. In particular, we have

$$\bar{\mathfrak{R}}^{(2j)}(A,0)=0 \quad \text{if } |A| \text{ is odd,}$$
$$\bar{\mathfrak{R}}^{(2j+1)}(A,0)=0 \quad \text{if } |A| \text{ is even.}$$

A second remark is that $\bar{\mathfrak{L}}_k(t)$ is an even function of t if k is even and an odd function of t if k is odd. In particular, we have

$$\bar{\mathfrak{L}}_k^{(2j+1)}(0)=0 \quad \text{if } k \text{ is even,}$$
$$\bar{\mathfrak{L}}_k^{(2j)}(0)=0 \quad \text{if } k \text{ is odd.}$$

λ is now fixed and we denote by $f(t)$ the solution of the resolvent equation

$$\lambda f-\mathscr{L}(t)f=R(t).$$

We prove the claim by induction on j.
For all $A\in\mathscr{E}_*$ such that $|A|$ is odd, we have

$$\lambda\bar{\mathfrak{f}}(A,0)-\sum_{k\in\mathbb{Z}}\bar{\mathfrak{L}}_{2k}(0)\bar{\mathfrak{f}}(A,0)=\bar{\mathfrak{R}}(A,0)=0$$

and for all $A\in\mathscr{E}_*$ such that $|A|$ is even

$$\lambda\bar{\mathfrak{f}}(A,0)-\sum_{k\in\mathbb{Z}}\bar{\mathfrak{L}}_{2k}(0)\bar{\mathfrak{f}}(A,0)=\bar{\mathfrak{R}}(A,0).$$

The first equation concerns only the $\bar{\mathfrak{f}}(A,0)$ with $|A|$ odd and the second the $\bar{\mathfrak{f}}(A,0)$ with $|A|$ even. Consequently, we have $\bar{\mathfrak{f}}(A,0)=0$ if $|A|$ is odd.
Similarly, we have for all A such that $|A|$ is even

$$\lambda\bar{\mathfrak{f}}'(A,0)-\sum_{k\in\mathbb{Z}}\bar{\mathfrak{L}}_{2k}(0)\bar{\mathfrak{f}}'(A,0)=\bar{\mathfrak{R}}'(A,0)+\sum_{k\in\mathbb{Z}}\bar{\mathfrak{L}}'_{2k+1}(0)\bar{\mathfrak{f}}(A,0)=0$$

and for all A such that $|A|$ is odd

$$\lambda\bar{\mathfrak{f}}'(A,0)-\sum_{k\in\mathbb{Z}}\bar{\mathfrak{L}}_{2k}(0)\bar{\mathfrak{f}}'(A,0)=\bar{\mathfrak{R}}'(A,0)+\sum_{k\in\mathbb{Z}}\bar{\mathfrak{L}}'_{2k+1}(0)\bar{\mathfrak{f}}(A,0).$$

The first equation concerns only the sets whose cardinal is even and the second only sets whose cardinal is odd. Consequently, we have $\bar{\mathfrak{f}}'(A,0)=0$ if $|A|$ is even.
The induction is ended by similar arguments, since we have

$$\lambda\bar{\mathfrak{f}}^{(2j)}(A,0)-\sum_{k\in\mathbb{Z}}\bar{\mathfrak{L}}_{2k}(0)\bar{\mathfrak{f}}^{(2j)}(A,0)=\bar{\mathfrak{R}}^{(2j)}(A,0)+\bar{\mathfrak{U}}_{2j}(A,0),$$

where

$$\bar{\mathfrak{U}}_{2j}(A,0) = \sum_{k\in\mathbb{Z}} \left\{ \sum_{p=0}^{j-1} C_{2j}^{2p} \bar{\mathfrak{L}}_{2k}^{2(j-p)}(0) \bar{\mathfrak{f}}^{(2p)}(A,0) \right.$$

$$\left. + \sum_{p=0}^{j} C_{2j}^{2p-1} \bar{\mathfrak{L}}_{2k+1}^{2(j-p)-1}(0) \bar{\mathfrak{f}}^{(2p-1)}(A,0) \right\}.$$

If $|A|$ is odd, $\bar{\mathfrak{R}}^{(2j)}(A,0)+\bar{\mathfrak{U}}_{2j}(A,0)=0$ and since the equation can be decomposed into two parts, the first concerning the $\bar{\mathfrak{f}}^{(2j)}(A,0)$ where $|A|$ is odd, the second concerning the $\bar{\mathfrak{f}}^{(2j)}(A,0)$ where $|A|$ is even, we have

$$\bar{\mathfrak{f}}^{(2j)}(A,0) = 0 \quad \text{if } |A| \text{ is odd.}$$

The same method gives the result for $\bar{\mathfrak{f}}^{(2j+1)}$. □

Let $U(t) = \ll R(t), f(t) \gg^2_{\rho(t)}$. It is easy to check that

$$U^{(2j+1)}(0) = \sum_{p=0}^{2j+1} C_{2j+1}^{p} \ll R^{(p)}(0), f^{(2j+1-p)}(0) \gg .$$

And since p and $2j+1-p$ do not have the same parity, $U^{(2j+1)}(0) = 0$. Elementary analytic considerations show that $U(t)$ is in fact a C^∞ function of t^2 and so of $\sin^2(t)$.

Eventually, we have shown regularity at the boundary.

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