Available online at www.sciencedirect.com
ScienceDirect
Stochastic Processes and their Applications 117 (2007) 487–513
stochastic processes and their applications
www.elsevier.com/locate/spa

# Hydrodynamics for a system of harmonic oscillators perturbed by a conservative noise

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Received 19 October 2005; received in revised form 20 July 2006; accepted 28 August 2006
Available online 18 September 2006

**Abstract**

We consider a heat conduction model for solids. Nearest neighbour atoms interact as coupled oscillators exchanging velocities in such a way that the total energy is conserved. The system is considered under periodic boundary conditions. We will show that the system has a hydrodynamic limit given by the solution of the heat equation and we discuss some aspects of the model.


*MSC:* primary 60K35; secondary 80A20

*Keywords:* Hydrodynamic limit; Hamiltonian system

## 1. Introduction

A microscopic description of heat conduction in solids sufficiently rich to explain phenomenological constitutive equations is one of the goals of theoretical physics (cf. [11] for a review). For the moment, at least from a mathematically rigorous point of view, this program is far from being accomplished. In this article, we introduce a simple stochastic model which we hope will respond (partially) to this attempt. In a perfect crystal, the equilibrium positions of atoms form a perfect regular configuration. The real positions are subject to fluctuations around equilibrium. The vibrations of the crystal yield the most important contribution to heat conduction in a solid. Since the pioneering works of Einstein and Debye to explain the behavior of the heat capacity at low temperature, it is usual to describe the $N$ atoms of the crystal

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doi:10.1016/j.spa.2006.08.006

as coupled oscillators. The first and last of them can interact with temperature baths. Let $m_\ell$ and $x_\ell$ be the mass and the position of the atom $\ell$. To simplify, we consider only nearest neighbour interactions. We denote by $p_\ell$ the momenta of atom $\ell$. The Hamiltonian of the system is

$$\mathcal{H} = \sum_{\ell=1}^{N} \left[ \frac{p_\ell^2}{2m_\ell} + V(x_{\ell+1} - x_\ell) \right].$$

Here $V$ is the potential associated to the interaction between atoms.

For the moment, the derivation of hydrodynamical equations for deterministic microscopic models is a completely open problem, probably outside the range of mathematical techniques. Formally, it is possible to derive them from Hamiltonian microscopic dynamics but under the condition of local equilibrium (cf. [3]). This last results from *ergodic properties* which should be the consequence of instabilities of the microscopic dynamics. If we assume that we are close to equilibrium, the potential can be expanded up to second order (since the first order vanishes) and we get the harmonic Hamiltonian

$$\mathcal{H}_{\text{harmonic}} = \sum_{\ell=1}^{N} \left[ \frac{p_\ell^2}{2m_\ell} + \{(x_{\ell+1} - x_\ell) - a\}^2 \right] \tag{1.1}$$

where $a$ is the equilibrium spacing between atoms. In the sequel, we will only work with this approximation. Of course, the harmonic model above, proposed by Debye one century ago, is elementary and explicitly computable. But harmonic chains have infinitely many conservation quantities and have hence very poor ergodic properties. On the other hand, harmonic systems (cf. [13] nevertheless) are the only models where an explicit analysis is possible from a mathematically rigorous point of view (cf. [11]). In such a situation, the introduction of some artificial sufficiently mixing noise for harmonic chains is helpful. We add hence to the Hamiltonian $\mathcal{H}_{\text{harmonic}}$ a stochastic noise in order to stimulate instabilities and assure good ergodic properties. This approach has been carried out in [4] and in [12] where the authors consider an Hamiltonian system with a noise term. In [4], Fritz et al. show that all translation invariant stationary states with finite entropy per unit volume are microcanonical Gibbs states. This result can be used to prove hydrodynamic behavior of these systems. In [12], Olla et al. prove that, in the scaling limit, the time conserved quantities satisfy the Euler equation of conservation laws up to a fixed time $t$ provided that the Euler equation has a smooth solution with a given initial data up to time $t$. The model presented here and the model given in [12] present two important differences for us (even if some other difficulties present in [12] are absent here). The first is that in their model, the strength of the noise term chosen is very small so that it disappears in the scaling limit. In our model, it appears still at the macroscopic level. The second difference is more technical. In [12], because of a lack of effective truncation techniques, the authors modify the kinetic energy by cutting off the large velocities. In this paper, we work with large velocities and a new microcanonical approach is then necessary (see Section 6). To my knowledge, such a problem is new in the hydrodynamic limit literature. Even for Ginzburg-Landau dynamics or for the zero range process, where the state space is not compact, some exponential moments (that we have not here) are very helpful. In fact, this difficulty appears still in a heat conduction model introduced in [9], but because of the so-called striking duality, which is very particular to the system considered, the problem of large energies can be avoided.

We will now give a precise description of the model. Atoms are labeled by $x \in \mathbb{T}_N$ where $\mathbb{T}_N = \{0, \dots, N-1\}$ is the lattice torus of length $N$. This corresponds to periodic boundary conditions. The configuration space is denoted by $\Omega^N = (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$ and a typical configuration is then $\omega = (p_x, r_x)_{x \in \mathbb{T}^N}$ where $r_x$ represents the distance between particle $x$ and particle $x+1$ and $p_x$ is the velocity of particle $x$.

The dynamics are described by the following coupled stochastic differential equations

$$\begin{cases} \mathrm{d}p_x(t) = (r_x - r_{x-1})\mathrm{d}t + p_{x-1}\mathrm{d}W_{x-1,x}(t) - p_{x+1}\mathrm{d}W_{x,x+1}(t) - p_x\mathrm{d}t \\ \mathrm{d}r_x(t) = (p_{x+1} - p_x)\mathrm{d}t \end{cases} \tag{1.2}$$

for $x \in \mathbb{T}_N$. Here $\{W_{x,x+1}\}_{x \in \mathbb{T}_N}$ are independent standard Brownian motions.

Let $\mathcal{L}_N$ be the generator of the system. A core for $\mathcal{L}_N$ is given by the space $C^\infty(\Omega^N)$ of smooth functions on $\Omega^N$ endowed with the product topology. On $C^\infty(\Omega^N)$, the generator is defined by

$$\mathcal{L}_N = \mathcal{A}_N + \mathcal{S}_N$$

where

$$\mathcal{A}_N = \sum_{x \in \mathbb{T}_N} \left\{ (p_{x+1} - p_x)\partial_{r_x} + (r_x - r_{x-1})\partial_{p_x} \right\}$$

and

$$\mathcal{S}_N = \frac{1}{2} \sum_{x \in \mathbb{T}_N} X_{x,x+1}^2$$

with $X_{x,x+1} = \left(p_{x+1}\partial_{p_x} - p_x\partial_{p_{x+1}}\right)$.

$\mathcal{A}_N$ is the Liouville operator of a chain of interacting harmonic oscillators. $\mathcal{S}_N$ is a diffusion operator corresponding to the noise part of Eq. (1.2). It acts only on velocities and couples the velocities of neighbouring atoms in such a way that the total energy of the chain is preserved.

We will denote by $(\omega^N(t))_{t \geq 0} = (r^N(t), p^N(t))_{t \geq 0}$ the process on the torus $\mathbb{T}_N$ whose evolution time is given by the generator $N^2\mathcal{L}_N$. Here, the factor $N^2$ corresponds to the acceleration of time by $N^2$ in the previous stochastic differential equations. The associated semi-group is denoted by $(S_t^N)_{t \geq 0}$. The law of the process $(\omega_t^N)_{t \geq 0}$ starting from $\mu^N$ is denoted by $\mathbb{P}_{\mu^N}$ (and its expectation by $\mathbb{E}_{\mu^N}$). When there is no chance of confusion, the index $N$ will be dropped from the notation.

Let $\omega = \{(r_x, p_x)\}_{x \in \mathbb{T}_N}$ be a configuration of $\Omega^N$. We denote by $\mathcal{R}^N(\omega)$ the deformation between particle 0 and particle $N-1$ which we call the total deformation. $\mathcal{R}^N(\omega)$ is given by

$$\mathcal{R}^N(\omega) = \sum_{x \in \mathbb{T}_N} r_x.$$

We note $\mathcal{E}_x = \mathcal{E}_x(\omega) = \frac{r_x^2 + p_x^2}{2}$ the contribution of atom $x$ to the total energy of the configuration $\omega$. The total energy of the configuration $\omega$ is then defined by

$$\mathcal{E}^N(\omega) = \sum_{x \in \mathbb{T}_N} \mathcal{E}_x(\omega) = \frac{1}{2} \sum_{x \in \mathbb{T}_N} (r_x^2 + p_x^2).$$

Deformation field $\mathcal{R}^N(\omega(t))$ is denoted by $\mathcal{R}^N(t)$ and energy field $\mathcal{E}^N(\omega(t))$ by $\mathcal{E}^N(t)$. It is easy to check that $(\mathcal{L}_N\mathcal{R}^N)(\omega) = 0$ and $(\mathcal{L}_N\mathcal{E}^N)(\omega) = 0$. Therefore the deformation field and the energy field are constants in the time.

The deformation and the energy define a family of invariant measures depending on two parameters. For $\beta > 0$ and $u \in \mathbb{R}$, the Gaussian product measure $\mu^N_{\beta,u}$ on $\Omega^N$ with marginal density (w. r. t. Lebesgue measure) $g_{\beta,u}(r, p)$ given by

$$g_{\beta,u}(r, p) = \frac{\beta}{2\pi} \exp\left\{ -\frac{\beta}{2}((r-u)^2 + p^2) \right\}$$

is invariant for the process. Let us denote by $\mathbb{L}^2(\mu^N_{\beta,u})$ the Hilbert space of functions $f$ on $\Omega^N$ such that $\mu^N_{\beta,u}(f^2) < +\infty$. On $\mathbb{L}^2(\mu^N_{\beta,u})$, the Liouville operator $\mathcal{A}_N$ is the antisymmetric part of the generator and $\mathcal{S}^N$ the symmetric part. In particular, the system is not reversible with respect to the Gaussian measures defined above.

Hence, our heat conduction model has a family of measures indexed with two parameters which are the inverse temperature $\beta$ and the deformation $u$. The measure $\mu^N_{1,0}$ is called the reference measure and will be denoted by $\mu_*$. The expectation with respect to $\mu^N_{\beta,u}$ will be noted $\langle \cdot \rangle_{\beta,u}$ and the expectation with respect to $\mu_*$ will be written $\langle \cdot \rangle_*$.

In this article, we prove the hydrodynamical behavior of the system introduced above in a diffusive scale. In order to state "properly" the theorem, we need to introduce notations and definitions.

The signed deformation empirical measure $\Pi^N(\omega, \mathrm{d}v)$ on the torus $\mathbb{T} = [0, 1)$ associated to the configuration $\omega$ is defined by:

$$\Pi^N(\omega, \mathrm{d}v) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} r_x \delta_{x/N}(\mathrm{d}v) \tag{1.3}$$

and the positive energy empirical measure $\Phi^N(\omega, \mathrm{d}v)$ on $\mathbb{T}$ by:

$$\Phi^N(\omega, \mathrm{d}v) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \mathcal{E}_x \delta_{x/N}(\mathrm{d}v). \tag{1.4}$$

Here, $\delta_a(\mathrm{d}v)$ is the Dirac measure localized on the point $a \in \mathbb{T}$. In order to simplify the notations, we note

$$\Pi^N_t = \Pi^N(\mathcal{R}^N(\omega^N_t)) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} r_x(t) \delta_{x/N}(\mathrm{d}v)$$

$$\Phi^N_t = \Phi^N(\mathcal{E}^N(\omega^N_t)) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \mathcal{E}_x(t) \delta_{x/N}(\mathrm{d}v).$$

We will assume that initially the system is close to a local equilibrium. More precisely, let us give the following definition.

**Definition 1.1.** A sequence $(\mu^N)_{N \geq 1}$ of probability measures on $\Omega^N$ is associated to a deformation profile $u_0 : \mathbb{T} \to \mathbb{R}$ and an energy profile $e_0 : \mathbb{T} \to [0, +\infty)$ if for every continuous function $G : \mathbb{T} \to \mathbb{R}$ and for every $\delta > 0$ we have

$$\lim_{N \to +\infty} \mu^N \left[ \left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} G(x/N) r_x - \int_{\mathbb{T}} G(v) u_0(v) \mathrm{d}v \right| > \delta \right] = 0$$

and

$$\lim_{N\to+\infty}\mu^N\left[\left|\frac{1}{N}\sum_{x\in\mathbb{T}_N}G(x/N)\mathcal{E}_x-\int_{\mathbb{T}}G(v)e_0(v)\mathrm{d}v\right|>\delta\right]=0.$$

Of course, such profiles satisfy the inequality $e_0 \geq u_0^2/2$. To understand this notion, it is more intuitive to consider a product measure $\mu^N$ whose marginals have the form

$$\mu^N_{\beta_0,u_0}(\mathrm{d}r_x,\mathrm{d}p_x)=\left[\frac{\beta_0(x/N)}{2\pi}\exp\left\{-\frac{\beta_0(x/N)}{2}\left[(r_x-u_0(x/N))^2+p_x^2\right]\right\}\right]\mathrm{d}r_x\mathrm{d}p_x \tag{1.5}$$

where $\beta_0 > 0$ and $u_0$ are continuous functions. Such a sequence $(\mu^N)_N$ is associated to the deformation profile $u_0$ and energy profile $e_0 = (\beta_0^{-1} + u_0^2/2)$.

Here is our main theorem.

**Theorem 1.2.** *Let $(\mu^N)_N$ be a sequence of probability measures on $\Omega^N$ associated to a bounded energy profile $e_0$ and a deformation profile $u_0$. We assume that there exists a positive constant $K_0$ such that the relative entropy $H(\mu^N|\mu_*^N)$ (cf. (2.1)) of $\mu^N$ with respect to the reference measure $\mu_*^N$ is bounded by $K_0N$:*

$$H(\mu^N|\mu_*^N)\leq K_0N \tag{1.6}$$

*and that there exists a positive constant $E_0$ such that*

$$\limsup_{N\to\infty}\mu^N\left[\frac{1}{N}\sum_{x\in\mathbb{T}_N}\mathcal{E}_x^2\right]\leq E_0. \tag{1.7}$$

*Then for every $t \geq 0$, the sequence of random measures $(\Pi_t^N, \Phi_t^N)_N$ converges in probability to the absolutely continuous measure $(\pi(t,\mathrm{d}v), \phi(t,\mathrm{d}v)) = (u(t,v)\mathrm{d}v, e(t,v)\mathrm{d}v)$ whose density is the solution of the heat equation*

$$\begin{cases}\partial_t u=\Delta u\\ \partial_t e=\Delta e\end{cases} \tag{1.8}$$

*with initial conditions $u(0,v) = u_0(v)$, $e(0,v) = e_0(v)$.*

The first hypothesis is very common in the literature of hydrodynamic limits. The second is less so and is of a technical nature (cf. Lemma 5.1). Even if the hydrodynamic equation obtained is linear, essentially, the model is not. In fact, consider a slight modification of the system by introducing a coupling parameter $\gamma > 0$ between the Hamiltonian and the noise term. The Markov process $(\omega_\gamma^N(t))_{t\geq 0}$ now has for generator $N^2\mathcal{L}_N^\gamma$ where:

$$\mathcal{L}_N^\gamma=\mathcal{A}_N+\gamma\mathcal{S}_N.$$

We expect that the hydrodynamical behavior of $\omega_\gamma^N$ is described by the following non linear equation

$$\begin{cases}\partial_t u=\gamma^{-1}\Delta u\\ \partial_t e=\dfrac{1}{2}(\gamma+\gamma^{-1})\Delta e+\dfrac{1}{4}(\gamma^{-1}-\gamma)\Delta(u^2).\end{cases} \tag{1.9}$$

We are not able to prove this behavior for $\gamma \neq 1$ for a technical reason explained in Section 6. One of the main difficulties in proving hydrodynamic limit is to establish a fluctuation–dissipation relation (cf. [10]), i.e. a decomposition of the currents of the conserved quantities into a dissipative part (a spatial gradient) and a fluctuating part (a time derivative). For any $\gamma > 0$, the noise introduced in the Hamiltonian dynamics permits us to write an exact fluctuation–dissipation relation (cf. [2] for the general case and Eq. (4.5) for the cases $\gamma = 1$). The (heuristic) derivation of this hydrodynamic equation is then obtained from similar computations of the present article. The equilibrium fluctuations of $\omega_\gamma^N$ may also be studied (cf. [5]) and when the system is in contact with thermal baths, Fourier's law and the energy profile may be established (cf. [2]). Other modifications of the noise have been considered in [1] and give some light on anomalous conductivity in low dimension.

The article is organized as follows. In Section 2, the entropy inequality and its consequences are recalled. In Section 3, we establish the hydrodynamic limit for the deformation field. Section 4 is devoted to the scheme for the proof of the hydrodynamic limit of the energy field. The technical parts, such as the $\mathbb{L}^2$ estimate, are postponed to Section 5. In Section 6, we discuss the technical problem mentioned before.

## 2. The entropy inequality

This section recall some basic facts concerning the entropy. Let $(X, \mathcal{F})$ be a measurable space and $\mu$, $\nu$ two probability measures on $(X, \mathcal{F})$. The entropy $H(\mu|\nu)$ of the probability measure $\mu$ with respect to the probability measure $\nu$ is defined by

$$H(\mu|\nu) = \sup_f \left\{ \int_X f \mathrm{d}\mu - \log \left( \int_X \mathrm{e}^f \mathrm{d}\nu \right) \right\}. \tag{2.1}$$

In this formula the supremum is carried over all bounded measurable functions $f$ on $X$. In Theorem 1.2, we assume that

$$H(\mu^N|\mu_N^*) \leq K_0 N.$$

This hypothesis is natural since a straightforward computation shows that $\mu^N$ defined by (1.5) satisfies this condition.

Let $(x_t)_{t\geq 0}$ be a Markov process whose semi-group is denoted by $(P_t)_{t\geq 0}$. Then $(x_t)_{t\geq 0}$ satisfies the following property of decreasing of entropy ([8], pp. 340–341)

$$\forall t \geq 0, \quad H(\mu P_t|\nu) \leq H(\mu|\nu).$$

By definition of the entropy, we have for any $\alpha > 0$ and any positive measurable function $f$,

$$\int f \mathrm{d}\mu \leq \alpha^{-1} \left\{ \log \left( \int \mathrm{e}^{\alpha f} \mathrm{d}\nu \right) + H(\mu|\nu) \right\}.$$

This inequality, known as "the entropy inequality", permits us hence to estimate expectation with respect to $\mu$ in terms of expectation with respect to $\nu$. It is particulary useful if the expectations with respect to $\nu$ are easy to estimate. For example, using (1.6) and the entropy inequality, one can easily establish the following lemma.

**Lemma 2.1.** *Assume that $H(\mu^N|\mu_N^*) \leq K_0 N$. Let $P(p, q)$ be a polynomial function with degree less than 2. There exists a positive constant $C_P$ depending only on $P$ such that for every* *time* $t \geq 0$,

$$\mathbb{E}_{\mu^N}\left[\frac{1}{N}\sum_{x\in\mathbb{T}_N}|P(r_x(t),p_x(t))|\right] \leq C_P.$$

*In particular, there exists* $C > 0$ *such that for any* $t \geq 0$,

$$\mathbb{E}_{\mu^N}\left[\frac{1}{N}\sum_{x\in\mathbb{T}_N}\mathcal{E}_x(t)\right] \leq C.$$

Of course, since the total energy is a conserved quantity, this lemma can be established without (1.6) using only (1.7) and the Schwartz inequality. We will see in the sequel (Section 4) that in the derivation of the hydrodynamic limit it is necessary to prove that

$$\lim_{N\to\infty}\mathbb{E}_{\mu^N}\left[\int_0^t \mathrm{d}s\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}p_x^4(s)\right] = 0. \tag{2.2}$$

A simple computation shows that the application of the entropy inequality, as is done in the preceding lemma, is not sufficient to have the control (2.2). This explains why the second hypothesis (1.7) is added. The control of this quantity is proved in Lemma 5.1 using the entropy inequality but in the microcanonical description.

Let us introduce some notations used in the sequel. For any function $w : \mathbb{T}_N \to \mathbb{R}$, we denote $\nabla w$ the discrete gradient of $w$ defined by

$$(\nabla w)(x) = w(x) - w(x-1).$$

$\nabla^*$ is the adjoint of the gradient $\nabla$ in $\mathbb{L}^2(\mathbb{T}_N)$ endowed with the standard inner product:

$$(\nabla^* w)(x) = w(x) - w(x+1).$$

The discrete Laplacian is $\Delta = -\nabla\nabla^*$. For a discrete function $w$, $\Delta w$ is given by

$$(\Delta w)(x) = w(x+1) + w(x-1) - 2w(x).$$

Moreover, if $G$ is a smooth local function on $\mathbb{T}$ and $x \in \mathbb{T}_N$, we note

$$(\nabla_N G)(x/N) = N\left[G\left(\frac{x}{N}\right) - G\left(\frac{x-1}{N}\right)\right] = (\nabla G)(x/N) + o(N^{-1})$$

and

$$(\Delta_N G)(x/N) = N^2\left[G\left(\frac{x+1}{N}\right) + G\left(\frac{x-1}{N}\right) - 2G\left(\frac{x}{N}\right)\right] = (\Delta G)(x/N) + o(N^{-1}).$$

In these formulas, $\nabla G$ and $\Delta G$ are respectively the continuous gradient and continuous Laplacian of the smooth function $G$.

The hydrodynamic limit is described thanks to empirical measures. $\mathcal{M}_a$ denotes the set of signed measures $\nu$ on $\mathbb{T}$ with finite mass less than $a$: $|\nu(\mathbb{T})| \leq a$. $\mathcal{M}_a^+ \subset \mathcal{M}_a$ is the set of positive measures with finite mass less than $a$. In view of Lemma 2.1, there exists some constant $a > 0$ now fixed throughout the article such that for any $t \geq 0$,

$$\Pi_t^N \in \mathcal{M}_a, \qquad \Phi_t^N \in \mathcal{M}_a^+. \tag{2.3}$$

The process $(\omega_t^N)_{t\geq 0}$ induces a Markov process $(\Pi_t^N, \Phi_t^N)_{t\geq 0}$ on $\mathcal{M}_a \times \mathcal{M}_a^+$. $\mathcal{M}_a$ is endowed with the weak topology and $\mathcal{M}_a^+$ is closed in $\mathcal{M}_a$. If $\nu$ is a measure on $\mathbb{T}$ and $G : \mathbb{T} \to \mathbb{R}$ an integrable function with respect to $\nu$, we use the notation

$$\langle \nu, G\rangle = \int_{\mathbb{T}} G(v)\mathrm{d}\nu(v).$$

We recall that the law of the process $(\omega_t^N)_{t\geq 0}$ starting from $\mu^N$ is denoted by $\mathbb{P}_{\mu^N}$ (and its expectation by $\mathbb{E}_{\mu^N}$). Moreover, we will denote by $\mathbb{Q}_{\Pi}^N$ (resp. $\mathbb{Q}_{\Phi}^N$) the law on $C([0,T], \mathcal{M}_a)$ (resp. $C([0,T], \mathcal{M}_a^+)$) of the empirical measure process $(\Pi_t^N)_{t\geq 0}$ (resp. $(\Phi_t^N)_{t\geq 0}$) starting from $\mu^N(\Pi^N)^{-1}$ (resp. $\mu^N(\Phi^N)^{-1}$).

## 3. Hydrodynamic limit of the deformation field

In this section, we prove the hydrodynamic behavior for the deformation field $\mathcal{R}_t^N$. Let $G \in C^2(\mathbb{T})$. An elementary computation gives:

$$\langle \Pi_t^N, G\rangle = \langle \Pi_0^N, G\rangle + \frac{1}{N^2}\sum_x (\nabla_N G)((x+1)/N)(p_x(t) - p_x(0)) \tag{3.1}$$

$$+ \frac{1}{N}\sum_x \int_0^t (\Delta_N G)(x/N) r_x(s)\mathrm{d}s + M^N(t) \tag{3.2}$$

where $M^N$ is a martingale whose quadratic variation is

$$\begin{aligned}\langle M^N\rangle(t) &= \frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\int_0^t \{(\nabla_N G)((x+1)/N)p_x(s) - (\nabla_N G)(x/N)p_{x+1}(s)\}^2\,\mathrm{d}s\\ &\leq \frac{C(G)}{N^2}\sum_{x\in\mathbb{T}_N}\int_0^t \mathcal{E}_x(s)\mathrm{d}s\end{aligned}$$

where $C(G)$ is a positive constant depending only on $G$.

**Lemma 3.1.** *The sequence $(\mathbb{Q}_{\Pi}^N)_{N\geq 1}$ is relatively compact.*

**Proof.** It suffices to prove that the sequence of laws of the real processes $(\langle \Pi_t^N, G\rangle)_{t\geq 0}$ is relatively compact for any fixed $G$ in $C^2(\mathbb{T})$ (see [8], Proposition 1.7, p.54). Let us denote by $\mathbb{Q}_{\Pi}^{N,G}$ the probability $\mathbb{Q}_{\Pi}^N G^{-1}$ on $C([0,T], \mathbb{R})$.

We recall here the well known criterion for relative compactness of probabilities on $C([0,T], \mathbb{R})$. For any function $x \in C([0,T], \mathbb{R})$ and any $\gamma > 0$, we define the modulus of continuity of $x$ by $w(x,\gamma) = \sup\{|x(s) - x(t)|;\ s,t \in [0,T],\ |s-t| \leq \gamma\}$. $\square$

**Lemma 3.2.** *Let $\mathbb{P}^N$ be a sequence of probability measures on $C([0,T], \mathbb{R})$ such that*

- $\forall t \in [0,T], \forall \varepsilon > 0, \exists A = A(t,\varepsilon) > 0,\ \sup_N \mathbb{P}^N[|x_t| \geq A] \leq \varepsilon$
- $\limsup_{\gamma\to 0}\limsup_{N\to\infty} \mathbb{P}^N[w(x,\gamma) > \varepsilon] = 0$

*Then the sequence $\{\mathbb{P}^N\}_N$ is relatively compact.*

The first condition of the lemma is satisfied thanks to Lemma 2.1 with function $P(a,b) = a$ and the Chebichev inequality. The second condition is satisfied if it is for each term appearing in (3.1).

Using Lemma 2.1, for any $0 \le \tau \le \tau + \theta \le T$, we get

$$
\begin{aligned}
&\mathbb{P}_{\mu^N}\left[\left|\frac{1}{N}\sum_x \int_\tau^{\tau+\theta} (\Delta_N G)(x/N) r_x(s)\mathrm{d}s\right| \ge \delta\right] \\
&\quad \le \mathbb{P}_{\mu^N}\left[\frac{1}{N}\sum_x \int_\tau^{\tau+\theta} |r_x(s)|\mathrm{d}s \ge C(G)\delta\right] \\
&\quad \le \frac{1}{C(G)\delta}\mathbb{E}_{\mu^N}\left[\frac{1}{N}\sum_x \int_\tau^{\tau+\theta} |r_x(s)|\mathrm{d}s\right] \\
&\quad \le C(G,\delta)\theta. 
\end{aligned} \tag{3.3}
$$

For the martingale term, notice that

$$
\mathbb{E}_{\mu^N}\left[(M^N(\tau+\theta) - M^N(\tau))^2\right] \le \frac{C(G)}{N^2}\sum_{x\in\mathbb{T}_N} \mathbb{E}_{\mu^N}\left[\int_\tau^{\tau+\theta} \mathcal{E}_x(s)\mathrm{d}s\right]. \tag{3.4}
$$

This last term can be bounded by $\frac{C'(G)\theta}{N}$ thanks to Lemma 2.1. The Chebichev inequality yields the wanted estimate. For the last term, we have

$$
\begin{aligned}
&\mathbb{P}_{\mu^N}\left[\left|\frac{1}{N^2}\sum_x (\nabla_N G)((x+1)/N)(p_x(\tau+\theta) - p_x(\tau))\right| \ge \delta\right] \\
&\quad \le \frac{C(G,\delta)}{N}\mathbb{E}_{\mu^N}\left[\frac{1}{N}\sum_{x\in\mathbb{T}_N} (|p_x(\tau+\theta)| + |p_x(\tau)|)\right] \\
&\quad \le \frac{C(G,\delta)}{N}.
\end{aligned}
$$

This concludes the proof that the sequence $\mathbb{Q}_\Pi^N$ is relatively compact. $\square$

**Lemma 3.3.** *Let $\mathbb{Q}_\Pi^*$ be a limit point of the sequence $(\mathbb{Q}_\Pi^N)_{N\ge 1}$. Then $\mathbb{Q}_\Pi^*$ is concentrated on trajectories $\pi_t \in C([0,T], \mathcal{M}_a)$ satisfying*

$$
\langle \pi_t, G\rangle = \langle \pi_0, G\rangle + \int_0^t \langle \pi_s, \Delta G\rangle \mathrm{d}s.
$$

**Proof.** Let $\mathbb{Q}_\Pi^*$ be a limit point and let $\mathbb{Q}_\Pi^{N_k}$ be a sub-sequence converging to $\mathbb{Q}_\Pi^*$. Since $(\Delta_N G)(x/N) = (\Delta G)(x/N) + o(N^{-1})$ uniformly in $N$, we can replace the discrete Laplacian $\Delta_N$ by the continuous Laplacian $\Delta$ in formula (3.1). We fix $t \in [0,T]$. The application from $C([0,T], \mathcal{M}_a)$ to $\mathbb{R}$ that associates to a path $\{\pi_t; 0 \le t \le T\}$ the number

$$
\left|\langle \pi_t, G\rangle - \langle \pi_0, G\rangle - \int_0^t \langle \pi_s, \Delta G\rangle \mathrm{d}s\right|
$$

is continuous so that

$$
\begin{aligned}
&\liminf_{k\to\infty} \mathbb{Q}_\Pi^{N_k}\left(\left|\langle \pi_t, G\rangle - \langle \pi_0, G\rangle - \int_0^t \langle \pi_s, \Delta G\rangle \mathrm{d}s\right| > \epsilon\right) \\
&\quad \ge \mathbb{Q}_\pi^*\left(\left|\langle \pi_t, G\rangle - \langle \pi_0, G\rangle - \int_0^t \langle \pi_s, \Delta G\rangle \mathrm{d}s\right| > \epsilon\right)
\end{aligned}
$$

since the above set is open.

Moreover, by the Doob and Chebichev inequalities, the estimate (3.4) yields

$$\mathbb{P}_{\mu^N}\left[\left|M^N(t)\right| > \varepsilon\right] \le 4\varepsilon^{-2}\mathbb{E}_{\mu^N}\left[(M^N(T))^2\right] \le \frac{C(G,T)}{\varepsilon^2 N}$$

and by the Chebichev inequality, we have

$$\mathbb{P}_{\mu^N}\left[\left|\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}(\nabla_N G)((x+1)/N)(p_x(T)-p_x(0))\right| > \varepsilon\right] \le \frac{C(G,T)}{\epsilon^2 N}.$$

Letting $N_k$ go to infinity, we get that all limit points $\mathbb{Q}^*_{\Pi}$ are concentrated on trajectories $\pi_t$ satisfying

$$\langle \pi_t, G\rangle = \langle \pi_0, G\rangle + \int_0^t \langle \pi_s, \Delta G\rangle \mathrm{d}s. \quad \Box \tag{3.5}$$

We have now to prove that the limit trajectories are absolutely continuous with respect to the Lebesgue measure. This is an application of the entropy inequality and we just give a sketch for the proof. We refer the reader to the Lemma 1.6, p. 73, of [8] for more details.

**Lemma 3.4.** *All limit points $\mathbb{Q}^*_{\Pi}$ of $(\mathbb{Q}^N_{\Pi})_{N\ge 1}$ are concentrated on absolutely continuous measures (with respect to the Lebesgue measure) $\pi(\mathrm{d}u) = \pi(u)\mathrm{d}u$ such that $\pi \in \mathbb{L}^2(\mathbb{T}, \mathrm{d}u)$:*

$$\mathbb{Q}^*_{\Pi}\left\{\pi;\ \pi(\mathrm{d}u) = \pi(u)\mathrm{d}u,\ \int_{\mathbb{T}} \pi^2(u)\mathrm{d}u < +\infty\right\} = 1.$$

**Proof.** Let us denote the probability measure $\mu_*(\Pi^N)^{-1}$ on $C([0,T], \mathcal{M}_a)$ by $\mathbb{P}^*_{\Pi}$. Consider a bounded continuous function $J : \mathcal{M}_a \to \mathbb{R}^+$. By the entropy inequality,

$$\mathbb{E}_{\mathbb{Q}^N_{\Pi}}[J(\pi)] \le K_0 + \frac{1}{N}\log \mathbb{E}_{\mathbb{P}^*_{\Pi}}\left[\mathrm{e}^{NJ(\pi)}\right].$$

By the Laplace–Varadhan theorem, the second term converges as $N$ goes to infinity to

$$\sup_{\pi\in\mathcal{M}_a}[J(\pi) - I_0(\pi)]$$

where $I_0$ represents the large deviation rate function for the random measure $\pi$ under $\mathbb{P}^*_{\Pi}$. We compute now this rate function. The Log-Laplace transform of $r_0$ under $\mu_*$ is:

$$L_*(\theta) = \log(\mu_*(\mathrm{e}^{\theta r_0})) = \theta^2/2$$

and the Legendre transform $h$ of $L_*$ is given by

$$h(\alpha) = \sup_\theta\{\theta\alpha - L_*(\theta)\} = \alpha^2/2.$$

Hence, the large deviation rate function $I_0 : \mathcal{M}_a \to [0,\infty)$ is equal to

$$I_0(\pi) = \begin{cases} \frac{1}{2}\int_0^1 \pi^2(u)\mathrm{d}u & \text{if } \pi(\mathrm{d}u) = \pi(u)\mathrm{d}u \\ \infty & \text{otherwise.}\end{cases}$$

It is easy to check that $I_0$ is an increasing limit of bounded and continuous functionals $(J_k)_k$ and by the monotone convergence theorem, we obtain

$$\mathbb{E}_{\mathbb{Q}^*_\Pi}[I_0(\pi)] \le K_0$$

and we are done. □

In order to prove an uniqueness result of weak solutions for the heat equation, we need to consider time dependent functions $G$. The details can be found in [8].

## 4. Hydrodynamic limit for the energy field

We turn now to the energy field $\{\mathcal{E}_x = (p_x^2 + r_x^2)/2;\ x \in \mathbb{T}_N\}$. We recall that $\Phi_t^N$ is the empirical measure associated to the time-dependent energy field $\mathcal{E}^N(t)$ and that $\mathbb{Q}^N_\Phi$ is the law of $(\Phi_t^N)_{t\ge 0}$. Hence, $\mathbb{Q}^N_\Phi$ is a probability measure on $C([0,T],\mathcal{M}_a^+)$. The hydrodynamic equation of the energy is obtained in three steps. In the first step, we prove that the sequence $(\mathbb{Q}^N_\Phi)_N$ is relatively compact. In the second step, we prove a replacement lemma and in the third step, we give a $\mathbb{L}^2$ estimate showing that the martingale terms are negligible as $N$ goes to infinity.

A simple computation shows that

$$\mathcal{L}_N \mathcal{E}_x = (p_{x+1} r_x - p_x r_{x-1}) + \frac{1}{2}\Delta(p_x^2). \tag{4.1}$$

It appears that the current of energy is not a gradient so that only one discrete integration by parts is possible. Let $G : \mathbb{T} \to \mathbb{R}$ be a twice differentiable continuous function. We have

$$\langle \Phi_t^N, G\rangle = \langle \Phi_0^N, G\rangle - \sum_x (\nabla_N G)(x/N) \int_0^t p_x(s) r_{x-1}(s)\mathrm{d}s \tag{4.2}$$

$$+ \frac{1}{2N}\sum_x (\Delta_N G)(x/N)\int_0^t p_x^2(s)\mathrm{d}s + M^N(t) \tag{4.3}$$

where $M^N$ is a martingale. We are interested in the second term of the right hand side of (4.2). Let us define the local function $h$ by

$$h(\omega) = -\frac{p_0}{2}(r_0 + r_{-1}) - \frac{p_0^2}{4}. \tag{4.4}$$

A simple computation shows that

$$r_{x-1}p_x = \mathcal{L}_N(\tau_x h) + \frac{1}{4}\Delta(p_x^2) + \frac{1}{2}\nabla(p_{x+1}p_x + r_x^2). \tag{4.5}$$

Here $\tau_x h$ is $h$ translated by $x$:

$$(\tau_x h)(\omega) = h(\tau_x\omega) = h(\omega(x+\cdot)).$$

Notice that the relation (4.5) is in fact an exact *fluctuation–dissipation* equation (cf; [14]) in the sense that the current of the energy (given by $r_{x-1}p_x + \frac{1}{2}\nabla(p_x^2)$) is expressed as the sum of a discrete gradient and a dissipative term $\mathcal{L}_N(\tau_x h)$.

The second term in the right hand side of (4.2) can hence be rewritten as

$$
\begin{aligned}
-\sum_{x}(\nabla_N G)(x/N)\int_0^t (p_x(s)r_{x-1}(s))\mathrm{d}s = {} & \frac{1}{2N}\sum_{x\in\mathbb{T}_N}(\Delta_N G)(x/N) \\
& \times \int_0^t \{p_x(s)p_{x+1}(s) + r_x^2(s)\}\mathrm{d}s \\
& - \frac{1}{4N}\sum_{x\in\mathbb{T}_N}(\Delta_N G)(x/N)\int_0^t \nabla(p_x^2(s))\mathrm{d}s \\
& - \sum_{x\in\mathbb{T}_N}(\nabla_N G)(x/N)\int_0^t \mathcal{L}_N(\tau_x h)(\omega_s)\mathrm{d}s.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\langle \Phi_t^N, G\rangle = {} & \langle \Phi_0^N, G\rangle + \int_0^t \langle \Phi_s^N, \Delta_N G\rangle \mathrm{d}s \\
& + \frac{1}{2N}\sum_{x\in\mathbb{T}_N}(\Delta_N G)(x/N)\int_0^t \left\{p_x(s)p_{x+1}(s) - \frac{1}{2}(\nabla_x p^2)(s)\right\}\mathrm{d}s \\
& - \sum_{x\in\mathbb{T}_N}(\nabla_N G)(x/N)\int_0^t \mathcal{L}_N(\tau_x h)(\omega_s)\mathrm{d}s + M^N(t).
\end{aligned}
$$

By standard stochastic calculus,

$$
\begin{aligned}
Z^N(t) = {} & \frac{1}{N^2}\sum_{x\in\mathbb{T}_N}(\nabla_N G)(x/N)\tau_x h(\omega(t)) - \frac{1}{N^2}\sum_{x\in\mathbb{T}_N}(\nabla_N G)(x/N)\tau_x h(\omega(0)) \\
& - \int_0^t \left[\sum_{x\in\mathbb{T}_N}(\nabla_N G)(x/N)\mathcal{L}_N(\tau_x h(\omega(s)))\right]\mathrm{d}s
\end{aligned}
$$

is a martingale and we get hence

$$
\begin{aligned}
\langle \Phi_t^N, G\rangle = {} & \langle \Phi_0^N, G\rangle + \int_0^t \langle \Phi_s^N, \Delta_N G\rangle \mathrm{d}s \\
& + \frac{1}{2N}\sum_{x\in\mathbb{T}_N}(\Delta_N G)(x/N)\int_0^t \left\{p_x(s)p_{x+1}(s) - \frac{1}{2}(\nabla_x p^2)(s)\right\}\mathrm{d}s \\
& - \frac{1}{N^2}\sum_{x\in\mathbb{T}_N}(\nabla_N G)(x/N)\tau_x h(\omega(t)) + \frac{1}{N^2}\sum_{x\in\mathbb{T}_N}(\nabla_N G)(x/N)\tau_x h(\omega(0)) \\
& + Z^N(t) + M^N(t).
\end{aligned}
$$

**Lemma 4.1.** *The sequence $(\mathbb{Q}_\Phi^N)_{N\geq 1}$ is relatively compact.*

**Proof.** The proof is similar to the proof of Lemma 3.1. We have to prove that the sequence of laws of the process $(\langle \Phi_t^N, G\rangle)_{t\geq 0}$ is relatively compact for each $G \in C^2(\mathbb{T})$. We use the criterion of relative compactness given in Lemma 3.2. Let us denote by $X^N(t)$ the quantity

$$X^N(t) = \int_0^t \langle \Phi_s^N, \Delta_N G \rangle \mathrm{d}s + \frac{1}{2N} \sum_{x \in \mathbb{T}_N} (\Delta_N G)(x/N) \int_0^t \left\{ p_x(s) p_{x+1}(s) - \frac{1}{2} (\nabla p_x^2)(s) \right\} \mathrm{d}s$$

and by $Y^N(t)$ the following

$$Y^N(t) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} (\nabla_N G)(x/N) \tau_x h(\omega(t)) - \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} (\nabla_N G)(x/N) \tau_x h(\omega(0)).$$

By Lemma 2.1, it is easy to show there exists some constant $C$ independent of $N$ such that for $0 \le s \le t \le T$

$$\mathbb{E}_{\mu^N} \left[ |X^N(t) - X^N(s)| \right] \le C$$

and

$$\mathbb{E}_{\mu^N} \left[ |Y^N(t) - Y^N(s)| \right] \le C N^{-1}.$$

Since $X^N(0) = Y^N(0) = 0$, we have:

- $\forall t \in [0, T], \forall \varepsilon > 0, \exists A = A(t, \varepsilon) > 0,\ \sup_N \mathbb{E}_{\mu^N}[|X^N(t)| \ge A] \le \varepsilon$
- $\limsup_{\gamma \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu^N}[w(X^N, \gamma) > \varepsilon] = 0$

and

- $\forall t \in [0, T], \forall \varepsilon > 0, \exists A = A(t, \varepsilon) > 0,\ \sup_N \mathbb{E}_{\mu^N}[|Y^N(t)| \ge A] \le \varepsilon$
- $\limsup_{\gamma \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu^N}[w(Y^N, \gamma) > \varepsilon] = 0.$

A simple computation of the quadratic variation of the martingale term

$$W^N(t) = Z^N(t) + M^N(t) \tag{4.6}$$

shows that

$$\mathbb{E}_{\mu^N} \left[ (W^N(t) - W^N(s))^2 \right] \le \frac{C}{N^2} \int_s^t \sum_{x \in \mathbb{T}_N} \mathbb{E}_{\mu^N} \left[ \mathcal{E}_x^2(\omega_v) \right] \mathrm{d}v. \tag{4.7}$$

Since the total energy $N^{-1} \sum_{x \in \mathbb{T}_N} \mathcal{E}_x(v)$ is conserved by the dynamics, the Schwarz inequality and the initial condition assure that

$$\mathbb{E}_{\mu^N} \left[ (W^N(t) - W^N(s))^2 \right] \le C(t-s) \mu^N \left[ \left( N^{-1} \sum_{x \in \mathbb{T}_N} \mathcal{E}_x \right)^2 \right] \le C'(t-s). \tag{4.8}$$

Since the martingale $W^N(t)$ is equal to zero for $t = 0$, we have:

- $\forall t \in [0, T], \forall \varepsilon > 0, \exists A = A(t, \varepsilon) > 0,\ \sup_N \mathbb{E}_{\mu^N}[|W^N(t)| \ge A] \le \varepsilon$
- $\limsup_{\gamma \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu^N}[w(W^N, \gamma) > \varepsilon] = 0.$

These three estimates on $X^N, Y^N, W^N$ yield the relative compactness of $(\mathbb{Q}_\Phi^N)_{N \ge 1}$. $\square$

**Lemma 4.2.** *Let $\mathbb{Q}^*_\Phi$ be a limit point of the sequence $(\mathbb{Q}^N_\Phi)_{N\geq 1}$. Then $\mathbb{Q}^*_\Phi$ is concentrated on trajectories $\phi_t \in C([0,T], \mathcal{M}_a^+)$ satisfying*

$$\langle \phi_t, G\rangle = \langle \phi_0, G\rangle + \int_0^t \langle \phi_s, \Delta G\rangle \mathrm{d}s. \tag{4.9}$$

**Proof.** As in the proof of Lemma 3.3, we replace the discrete Laplacian by the continuous Laplacian. Moreover, by the definition of $X^N$, $Y^N$ and $W^N$, we have

$$\langle \Phi_t^N, G\rangle = \langle \Phi_0^N, G\rangle + X^N(t) - Y^N(t) + W^N(t). \tag{4.10}$$

We have seen in the proof of the preceding lemma that

$$\mathbb{E}_{\mu^N}\left[|Y^N(t)|\right] \leq CN^{-1}$$

and hence $Y^N(t)$ vanishes in the limit.

We have to prove that

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}\left[\left|\frac{1}{2N}\sum_{x\in\mathbb{T}_N}(\Delta_N G)(x/N)\int_0^t \left\{p_x(s)p_{x+1}(s) - \frac{1}{2}(\nabla p_x^2)(s)\right\}\mathrm{d}s\right| \geq \delta\right] = 0 \tag{4.11}$$

Lemma 5.5 shows that the term

$$\frac{1}{2N}\sum_{x\in\mathbb{T}_N}(\Delta_N G)(x/N)\int_0^t \{p_x(s)p_{x+1}(s)\}\,\mathrm{d}s$$

is negligible as $N$ goes to infinity.

The second term is easier to control. Indeed, if $G$ is sufficiently smooth, a discrete integration by parts can be performed and we have the bound:

$$\mathbb{E}_{\mu^N}\left[\left|\frac{1}{2N}\sum_{x\in\mathbb{T}_N}(\Delta_N G)(x/N)\int_0^t (\nabla p_x^2)(s)\mathrm{d}s\right|\right]$$
$$\leq C(G)\mathbb{E}_{\mu^N}\left[\frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{T}_N} p_x^2(s)\mathrm{d}s\right] \tag{4.12}$$

where $C(G)$ is a constant depending on $G$. By Lemma 2.1, the right hand side of the inequality goes to 0. If $G$ is only $C^2$, we have to approximate $G$ by a smooth function and use Lemma 2.1. We have then proved Eq. (4.11).

Moreover, it is proved at the end of Section 5 that

$$\lim_{N\to\infty}\mathbb{E}_{\mu^N}\left[|W^N(t)|^2\right] = 0.$$

The arguments given in Lemma 3.3 can be applied in the same way in order to complete the proof. □

**Remark 4.3.** The limit (4.11) is known in the hydrodynamic limit literature as a replacement lemma (cf [8]). It is proved here for a very particular function ($g(p) = p_0p_1 - 1/2\nabla p_0^2$) taking advantage of an integration by parts property for $g$. For more general $g$, the standard proofs of the replacement lemma ([8], p. 84–92) can not be applied directly because of the lack of good control of large velocities. This error appeared in the first version of the paper and we thank the referee for pointing out this mistake.

**Lemma 4.4.** *All limit points $\mathbb{Q}^*_\Phi$ of $(\mathbb{Q}^N_\Phi)_{N\geq 1}$ are concentrated on absolutely continuous measures (with respect to the Lebesgue measure) $\phi(\mathrm{d}u) = \phi(u)\mathrm{d}u \in \mathcal{M}_a^+$ such that $\phi \in \mathbb{L}^1(\mathbb{T}, \mathrm{d}u)$:*

$$\mathbb{Q}^*_\Phi \left\{ \phi;\ \phi(\mathrm{d}u) = \phi(u)\mathrm{d}u,\ \int_{\mathbb{T}} \phi(u)\mathrm{d}u < +\infty \right\} = 1.$$

**Proof.** Same proof as for Lemma 3.4. □

We have now all the elements needed to achieve the proof of the hydrodynamic limit for the energy field (cf. [8]).

## 5. An $\mathbb{L}^2$ estimate

In this section, we prove $\mathbb{L}^2$ estimates in order to show that the martingales which appear in the derivation of the hydrodynamic limit are vanishing as $N$ goes to infinity. This estimate is used to prove Lemma 5.5.

**Lemma 5.1.** *Assume that there exists some constant $E_0 > 0$ such that the initial states $(\mu^N)_{N\geq 1}$ satisfy*

$$\limsup_{N\to\infty} \mu^N \left[ \frac{1}{N} \sum_{x\in\mathbb{T}_N} \mathcal{E}_x^2 \right] \leq E_0 \tag{5.1}$$

*then at any time $t$,*

$$\lim_{N\to\infty} \mathbb{E}_{\mu^N} \left[ \frac{1}{N^2} \int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s \right] = 0. \tag{5.2}$$

**Proof.** A simple computation shows that we can not obtain this fact by considering the entropy inequality with respect to Gaussian measures because if $\nu$ is a standard Gaussian measure then $\nu(e^{\gamma x^4}) = +\infty$ for every choice of $\gamma$. This simple remark means that we have to work with a microcanonical description.

The proof is done in three steps:

(1) We use the entropy inequality in the microcanonical ensemble and the Feynman–Kac formula to estimate

$$\mathbb{E}_{\mu^N} \left[ \frac{1}{N^2} \int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s \right]$$

by

$$C(\gamma)\nu_N + 1/\gamma.$$

Here, $\nu_N$ is the largest eigenvalue for the symmetric operator $NS_N + N^{-2}\sum_x p_x^4$ in $\mathbb{L}^2(\lambda^{N,E})$ where $\lambda^{N,E}$ is the microcanonical ensemble of length $N$ with energy $E = O(N)$. The parameter $\gamma > 0$ is arbitrary and will go to infinity after all the other parameters.

(2) We divide the system in $N/K$ boxes of length $K$. The largest eigenvalue of the operator $S_N + N^{-1}\sum_x p_x^4$ can be bounded from above by the largest eigenvalue $\nu_{N,K}$ of the operator $NS_K + \sum_x p_x^4$ with respect to the microcanonical ensemble $\lambda^{K,1}$. This step is accomplished in Lemma 5.2.

(3) Letting $N$ go to infinity and using a compactness argument involving the Dirichlet form, we bound from above $\limsup_{N\to\infty} \nu_{N,K}$ by $C/K$ where $C > 0$. We let $K \to \infty$ and then $\gamma \to \infty$ to conclude.

We need to introduce some microcanonical notations. Let $\Lambda$ be a finite subset of $\mathbb{Z}$ with cardinality $|\Lambda|$. We denote $\Theta_\Lambda^R$ the hyper-sphere with dimension $|\Lambda|-1$ and radius $\sqrt{R}$ embedded in $\mathbb{R}^\Lambda$:

$$\Theta_\Lambda^R = \left\{ (\theta_x)_{x\in\Lambda};\ \sum_{x\in\Lambda} \theta_x^2 = R \right\}.$$

The uniform measure on $\Theta_\Lambda^R$ is denoted $\lambda^{\Lambda,R}$. In the special case $\Lambda = \{0, 1, \ldots, N-1\}$, we denote $\Theta_\Lambda^R$ by $\Theta_N^R$ and $\lambda^{\Lambda,R}$ by $\lambda^{N,R}$. Recall that the total energy is conserved by the dynamics. Hence, if the process $(\omega^N(t))_{t\geq 0}$ starts at time 0 on the hypersphere

$$\Theta_{2N}^{2Ne} = \left\{ (\omega_x)_{x\in\mathbb{T}_N} \in \mathbb{R}^{2N}\ ;\ \sum_{x\in\mathbb{T}_N} \mathcal{E}_x = \frac{1}{2}\sum_{x\in\mathbb{T}_N} (p_x^2 + r_x^2) = Ne \right\}$$

then at any time $t$, it remains on this hyper-sphere. $\lambda^{2N,2Ne}$ is the microcanonical invariant measure for the dynamics on $\Theta_{2N}^{2Ne}$.

Call $\rho^N(\mathrm{d}e)$ the law of the energy in the initial state $\mu^N$ and $\nu^{2N,2Ne}$ the conditional law of $\mu^N$ given $N^{-1}\sum_{x\in\mathbb{T}_N}\mathcal{E}_x = e$, so that

$$\mu^N(f) = \int_0^\infty \nu^{2N,2Ne}(f)\mathrm{d}\rho^N(e)$$

for any integrable function $f : \Omega^N \to \mathbb{R}$. At any time $t$, since the total energy is conserved, we have

$$S_t^N\mu^N = \int_0^\infty \mathrm{d}\rho^N(e)(S_t^N\nu^{2N,2Ne}).$$

Recall that $S_t^N$ is the semi-group associated to the dynamics $(\omega^N(t))_{t\geq 0}$ on $\mathbb{T}_N$ accelerated by $N^2$. If $\mu_N$ is equal to the standard Gaussian equilibrium measure $\mu_*^N$, the conditional law of $\mu_*^N$ given $N^{-1}\sum_{x\in\mathbb{T}_N}\mathcal{E}_x = e$ is $\lambda^{2N,2Ne}$. The law of the energy is then denoted by $\zeta^N(\mathrm{d}e)$:

$$\mu_*^N(f) = \int_0^\infty \lambda^{2N,2Ne}(f)\mathrm{d}\zeta^N(e).$$

By definition of the entropy, we have

$$H(\mu^N|\mu_*^N) = \int_0^\infty \mathrm{d}\rho^N(e)H(\nu^{2N,2Ne}|\lambda^{2N,2Ne}) + H(\rho^N|\zeta^N). \tag{5.3}$$

Inequality (1.6) gives hence the existence of a constant $C_0$ such that

$$\int \mathrm{d}\rho^N(e) H(\nu^{2N,2Ne}|\lambda^{2N,2Ne}) \le C_0 N.$$

We have

$$\mathbb{E}_{\mu^N}\left(\frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s\right) = \int_0^\infty \mathrm{d}\rho^N(e)\mathbb{E}_{\nu^{2N,2Ne}}\left[\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\int_0^t p_x^4(s)\mathrm{d}s\right].$$

Let $\gamma$ be a positive real. By the entropy inequality on $\Theta_{2N}^{2Ne}$, we have

$$\begin{aligned}&\mathbb{E}_{\nu^{2N,2Ne}}\left[\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\int_0^t p_x^4(s)\mathrm{d}s\right]\\ &\quad\le \frac{H(\nu^{2N,2Ne}|\lambda^{2N,2Ne})}{\gamma N} + \frac{1}{\gamma N}\log \mathbb{E}_{\lambda^{2N,2Ne}}\left(\exp\left\{\frac{\gamma}{N}\sum_{x\in\mathbb{T}_N}\int_0^t p_x^4(s)\mathrm{d}s\right\}\right).\end{aligned} \tag{5.4}$$

Hence, by integration with respect to $\rho^N$, we get

$$\begin{aligned}&\mathbb{E}_{\mu^N}\left(\frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s\right)\\ &\quad\le \frac{C_0}{\gamma} + \int_0^\infty \mathrm{d}\rho^N(e)\left[\frac{1}{\gamma N}\log \mathbb{E}_{\lambda^{2N,2Ne}}\left(\exp\left\{\frac{\gamma}{N}\sum_{x\in\mathbb{T}_N}\int_0^t p_x^4(s)\mathrm{d}s\right\}\right)\right].\end{aligned} \tag{5.5}$$

Let $\phi$ be a smooth density with respect to $\lambda^{2N,2Ne}$ on the sphere $\Theta_{2N}^{2Ne}$. The Dirichlet form $D_{N,2Ne}(\phi)$ of $\phi$ is defined by

$$D_{N,2Ne}(\phi) = \frac{1}{2}\sum_{x\in\mathbb{T}_N}\int (p_{x+1}\partial_{p_x}\sqrt{\phi} - p_x\partial_{p_{x+1}}\sqrt{\phi})^2\mathrm{d}\lambda^{2N,2Ne}(\omega). \tag{5.6}$$

By the Feynman–Kac formula (A1.7.2 of [8]) and the variational formula for the largest eigenvalue of an operator in a Hilbert space (A3.1.1 of [8]), the second quantity in the right hand side of (5.4) is bounded by

$$\frac{t}{\gamma N}\sup_\phi\left\{\frac{\gamma}{N}\sum_{x\in\mathbb{T}_N}\int p_x^4\phi\mathrm{d}\lambda^{2N,2Ne} - N^2 D_{N,2Ne}(\phi)\right\} \tag{5.7}$$

where the supremum is carried over the set of smooth densities $\phi(\omega)$ with respect to $\lambda^{2N,2Ne}$. Since the Dirichlet form concerns only the velocity configuration, this variational formula is essentially a variational formula for functions depending only on the velocities. To make this argument rigorous, we have to use conditional expectation. We denote by $p = (p_x)_{x\in\mathbb{T}_N}$ the velocities configuration of the total configuration $\omega = (\omega_x)_{x\in\mathbb{T}_N} = (p_x, r_x)_{x\in\mathbb{T}_N}$ and by $\mathcal{U} = (2N)^{-1}\sum_{x\in\mathbb{T}_N} r_x^2$ the interaction energy. The law of $\mathcal{U}$ under $\lambda^{2N,2Ne}$ is denoted by $\pi^{N,2Ne}(u)$. The Dirichlet form $D_{N,2Ne}$ is convex. If $\lambda^{2N,2Ne}(\phi|p,\mathcal{U})$ is the conditional expectation with respect to the velocity configuration $p$ and the interaction energy $\mathcal{U}$, then by the Jensen inequality

$$D_{N,2Ne}(\lambda^{2N,2Ne}(\phi|p,\mathcal{U})) \leq D_{N,2Ne}(\phi).$$

Let us then denote by $F(p,u)$ the function $F(p,u) = \lambda^{2N,2Ne}(\phi|p,\mathcal{U}=u)$. It is a standard fact that given $\{\mathcal{U}=u\}$, the law of $p$ is the uniform measure $\lambda^{N,2N(e-u)}$. Let us define

$$\Phi(u) = \int \mathrm{d}\lambda^{N,2N(e-u)}(p)F(p,u).$$

Since $\phi$ is a density with respect to $\lambda^{2N,2Ne}$, we have

$$\int \mathrm{d}\pi^{N,2Ne}(u)\Phi(u) = 1$$

so that $F(p,u)/\Phi(u)$ is a density with respect to the uniform measure $\lambda^{N,2N(e-u)}$.

We introduce

$$A_N^e = \frac{1}{\gamma N}\sup_{\phi}\left\{\frac{\gamma}{N}\sum_{x\in\mathbb{T}_N}\int p_x^4\phi\mathrm{d}\lambda^{N,2Ne} - N^2\bar{D}_{N,2Ne}(\phi)\right\} \tag{5.8}$$

where the supremum is now carried over $\phi(p)$ depending only on $(p_x; x\in\mathbb{T}_N)$ such that $\int\phi(p)\mathrm{d}\lambda^{N,2Ne}(p) = 1$. Here, $\lambda^{N,2Ne}$ is the uniform measure on the sphere of constant kinetic energy

$$\Theta_N^{2Ne} = \left\{(p_x)_{x\in\mathbb{T}_N}\ ;\ \frac{1}{N}\sum_x p_x^2 = 2e\right\}$$

and the Dirichlet form $\bar{D}_{N,2Ne}(\phi)$ is

$$\bar{D}_{N,2Ne}(\phi) = \frac{1}{2}\sum_{x\in\mathbb{T}_N}\int(p_{x+1}\partial_{p_x}\sqrt{\phi} - p_x\partial_{p_{x+1}}\sqrt{\phi})^2\mathrm{d}\lambda^{N,2Ne}(p). \tag{5.9}$$

With these notations and since the function $\sum_{x\in\mathbb{T}_N}p_x^4$ is a function only of the velocities, we can bound the quantity of interest (5.7) by

$$t\sup_{\Phi}\int\mathrm{d}\pi^{N,2Ne}(u)\Phi(u)A_N^{e-u} \tag{5.10}$$

where the supremum is taken over the functions $\Phi$ such that $\int\mathrm{d}\pi^{N,2Ne}(u)\Phi(u) = 1$. □

**Lemma 5.2.** *Fix an integer $K > 0$. For sufficiently large $N$, we have*

$$A_N^e \leq \frac{w_{N,\gamma}(K)}{\gamma}e^2 \tag{5.11}$$

*where $w_{N,\gamma}(K)$ is defined in (5.14).*

**Proof.** We divide the torus $\mathbb{T}_N$ of length $N$ into $N/K$ blocks of length $K$:

$$\Lambda_r = \{rK, rK+1, \ldots, (r+1)K-1\}, \quad r = 0, \ldots, N/K-1.$$

Notice that if $\phi$ is a density with respect to $\lambda^{N,Ne}$, then

$$\begin{aligned}\bar{D}_{N,Ne}(\phi) &\geq \sum_{r=0}^{N/K-1}\left\{\sum_{\substack{x,y\in\Lambda_r\\|x-y|=1}}\int(p_y\partial_{p_x}\sqrt{\phi}-p_x\partial_{p_y}\sqrt{\phi})^2\mathrm{d}\lambda^{N,2Ne}\right\}\\ &\geq \sum_{r=0}^{N/K-1}\left\{\sum_{\substack{x,y\in\Lambda_r\\|x-y|=1}}\int(p_y\partial_{p_x}\sqrt{\phi_r}-p_x\partial_{p_y}\sqrt{\phi_r})^2\mathrm{d}\lambda^{N,2Ne}\right\}\\ &= \sum_{r=0}^{N/K-1}\left\{\sum_{\substack{x,y\in\Lambda_r\\|x-y|=1}}\int(p_y\partial_{p_x}\sqrt{\phi_r}-p_x\partial_{p_y}\sqrt{\phi_r})^2\mathrm{d}\lambda_r^{N,2Ne}\right\}\end{aligned}$$

where $\phi_r$ is the conditional expectation of $\phi$ with respect to $\{p_x;\, x\in\Lambda_r\}$ and $\lambda_r^{N,2Ne}$ the marginal of $(p_x)_{x\in\Lambda_r}$. The penultimate inequality follows from convexity of the Dirichlet form. By the definition of the conditional expectation, we get for the first term in $A_N^e$,

$$\frac{1}{N}\int\sum_{x\in\mathbb{T}_N}p_x^4\phi\mathrm{d}\lambda^{N,2Ne}=\frac{1}{N}\sum_{r=0}^{N/K-1}\int\sum_{x\in\Lambda_r}p_x^4\phi_r\mathrm{d}\lambda_r^{N,2Ne}.$$

We may bound $A_N^e$ from above by:

$$A_N^e\leq\frac{1}{\gamma N}\sum_{r=0}^{N/K-1}\sup_{\phi}\left\{\frac{\gamma}{N}\int\sum_{x\in\Lambda_r}p_x^4\phi_r\mathrm{d}\lambda_r^{N,2Ne}-N^2\bar{D}_{N,2Ne}^{\Lambda_r}(\phi_r)\right\}\tag{5.12}$$

with

$$\bar{D}_{N,2Ne}^{\Lambda_r}(\phi_r)=\sum_{\substack{x,y\in\Lambda_r\\|x-y|=1}}\int(p_y\partial_{p_x}\sqrt{\phi_r}-p_x\partial_{p_y}\sqrt{\phi_r})^2\mathrm{d}\lambda_r^{N,2Ne}.$$

By the change of variables $q_x=N^{-1/2}p_x$, we can rewrite the right hand side of (5.12) in the following form

$$\sup_{\phi}\left\{\sum_{r=0}^{N/K-1}\left[\int\gamma\sum_{x\in\Lambda_r}q_x^4\phi_r\mathrm{d}\lambda_r^{N,2e}-N\bar{D}_{N,2e}^{\Lambda_r}(\phi_r)\right]\right\}$$

where now the supremum is taken over all densities $\phi$ with respect to the uniform measure $\lambda^{N,2e}$.

Let $m(d\alpha_0,\ldots,d\alpha_{N/K-1})$ be the law of the kinetic energies $(\sum_{x\in\Lambda_0}q_x^2/2,\ldots,\sum_{x\in\Lambda_{N/K-1}}q_x^2/2)$ in boxes $\Lambda_0,\ldots,\Lambda_{N/K-1}$. This measure is a probability measure on the simplex $\Sigma^{N/K,e}=\{(\alpha_0,\ldots,\alpha_{N/K-1})\in\mathbb{R}^+;\ \alpha_0+\cdots+\alpha_{N/K-1}=e\}$. We have

$$\begin{aligned}&\sum_{r=0}^{N/K-1}\int\left[\gamma\sum_{x\in\Lambda_r}q_x^4\phi_r\mathrm{d}\lambda_r^{N,2e}-N\bar{D}_{N,2e}^{\Lambda_r}(\phi_r)\right]=\int_{\Sigma^{N/K,e}}\mathrm{d}m(\alpha_0,\ldots,\alpha_{N/K-1})\\ &\quad\times\sum_{r=0}^{N/K-1}\lambda_r^{N,2e}\left(\gamma\sum_{x\in\lambda_r}q_x^4\phi_r-N\sum_{\substack{x,y\in\Lambda_r\\|x-y|=1}}(q_y\partial_{q_x}\sqrt{\phi_r}-q_x\partial_{q_y}\sqrt{\phi_r})^2\,\Big|\,\sum_{z\in\Lambda_r}q_z^2=2\alpha_r\right).\end{aligned}$$

It is well known that the conditional law of $\lambda_r^{N,2e}$ given $\{\sum_{z\in\Lambda_r} q_z^2 = 2\alpha_r\}$ is the uniform probability measure $\lambda^{\Lambda_r,2\alpha_r}$. Hence, we have obtained the following upper bound for $A_N^e$:

$$A_N^e \le \frac{1}{\gamma}\sup_{\phi}\left\{\int_{\Sigma^{N/K,e}} \mathrm{d}m(\alpha_0,\ldots,\alpha_{N/K-1})\sum_{r=0}^{N/K-1}\lambda^{\Lambda_r,2\alpha_r}(a_{N,\gamma}(\phi_r))\right\}$$

where

$$a_{N,\gamma}(\phi_r) = \left(\gamma\sum_{x\in\Lambda_r} q_x^4\phi_r - N\sum_{\substack{x,y\in\Lambda_r\\|x-y|=1}}(q_y\partial_{q_x}\sqrt{\phi_r} - q_x\partial_{q_y}\sqrt{\phi_r})^2\right).$$

The supremum of this expression is taken under the set of functions $\phi(q)$ which are densities with respect to $\lambda^{N,2e}$. But $a_{N,\gamma}(\phi_r)$ depends only on $\phi_r = \lambda^{N,2e}(\phi|(q_x)_{x\in\Lambda_r})$. Moreover, we have that

$$\forall r\in\{0,\ldots,N/K-1\}\int_{\Sigma^{N/K,e}}\mathrm{d}m(\alpha_0,\ldots,\alpha_{N/K-1})\lambda^{\Lambda_r,2\alpha_r}(\phi_r) = 1.$$

By the change of variables $u_x = (2\alpha_r)^{-1}q_x$ in $a_{N,\gamma}(\phi_r)$ and $\psi_{r,\alpha_r}((u_x)_{x\in\Lambda_r}) = \phi_r(((2\alpha_r)^{-1}q_x)_{x\in\Lambda_r})$, we have

$$\lambda^{\Lambda_r,2\alpha_r}[a_{N,\gamma}(\phi_r)] = 4\alpha_r^2\lambda^{\Lambda_r,1}[a_{N,\gamma}(\psi_{r,\alpha_r})].$$

We call $c_r(\alpha_r) = \lambda^{\Lambda_r,1}(\psi_{r,\alpha_r})$ and we have

$$\begin{cases}\forall r\in\{0,\ldots,N/K-1\},\ \displaystyle\int_{\Sigma^{N/K,e}}\mathrm{d}m(\alpha_0,\ldots,\alpha_{N/K-1})c_r(\alpha_r) = 1\\ \displaystyle\int_{\Sigma^{N,e}}\mathrm{d}m(\alpha_0,\ldots,\alpha_{N/K-1})\sum_{r=0}^{N/K-1}\alpha_r^2 c_r(\alpha_r)\\ \displaystyle\quad = \frac{1}{4}\lambda^{N,e}\left(\phi\sum_{r=0}^{N/K-1}\left[\sum_{x\in\Lambda_r}q_x^2\right]^2\right)\le e^2.\end{cases} \tag{5.13}$$

Notice that if $\alpha_r$ is fixed then

$$\lambda^{\Lambda_r,1}[a_{N,\gamma}(\psi_{r,\alpha_r})] = c_r(\alpha_r)\lambda^{\Lambda_r,1}\left[a_{N,\gamma}\left(\frac{\psi_{r,\alpha_r}}{c_r(\alpha_r)}\right)\right]$$

and $\frac{\psi_{r,\alpha_r}}{c_r(\alpha_r)}$ is a density with respect to $\lambda^{\Lambda_r,2\alpha_r}$.

Let us now introduce the quantity

$$\begin{aligned} w_{N,\gamma}(K) &= \sup_{\psi}\left\{\lambda^{\Lambda_0,1}(a_{N,\gamma}(\psi))\right\}\\ &= \sup_{\psi}\lambda^{\Lambda_0,1}\left\{\gamma\sum_{x\in\Lambda_0}q_x^4\psi - N\sum_{\substack{x,y\in\Lambda_0\\|x-y|=1}}(q_y\partial_{q_x}\sqrt{\psi}-q_x\partial_{q_y}\sqrt{\psi})^2\right\}\end{aligned} \tag{5.14}$$

where the supremum is taken over all densities $\psi$ on $\{(q_x)_{x\in\Lambda_0}; \sum_{x\in\Lambda_0}q_x^2 = 1\}$ with respect to $\lambda^{\Lambda_0,1}$. Of course, this expression does not depend on the choice $\Lambda_0 = \{0,\ldots,K-1\}$ and is the same if we replace $\Lambda_0$ by $\Lambda_r$.

It is easy to show, using (5.13), that $A_N^e$ can hence be bounded from above by

$$4 \sup_{c_0,\ldots,c_{N/K-1}} \int_{\Sigma^{N/K,e}} \mathrm{d}m(\alpha_0,\ldots,\alpha_{N/K-1}) \sum_{r=0}^{N/K-1} \alpha_r^2 c_r w_N(K) \tag{5.15}$$

where the supremum is taken over positives functions $c_0(\alpha_0),\ldots,c_{N/K-1}(\alpha_{N/K-1})$ such that

$$\forall r \in \{0,\ldots,N/K-1\}, \quad \int_{\Sigma^{N/K,e}} \mathrm{d}m(\alpha_0,\ldots,\alpha_{N/K-1}) c_r = 1$$

and

$$\int_{\Sigma^{N/K,e}} \mathrm{d}m(\alpha_0,\ldots,\alpha_{N/K-1}) \sum_{r=0}^{N/K-1} \alpha_r^2 c_r \leq e^2/4$$

$w_{N,\gamma}(K)$ depends only on $\gamma$, $N$, $K$ and not on $\alpha_r$. Hence, the preceding expression is bounded by $e^2 w_{N,\gamma}(K)$ and we have (5.11). □

Recall (5.10). We have

$$\begin{aligned} &\mathbb{E}_{\mu^N}\left(\frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s\right) \\ &\quad \leq \frac{C_0}{\gamma} + t\frac{w_{N,\gamma}(K)}{\gamma}\int_0^\infty \mathrm{d}\rho^N(e) \sup_{\Phi(u)} \int_0^e \Phi(u)(e-u)^2 \mathrm{d}\pi^{N,2Ne}(u) \end{aligned} \tag{5.16}$$

where the sup is carried over densities $\Phi(u)$ with respect to $\pi^{N,2Ne}$. It follows that

$$\mathbb{E}_{\mu^N}\left(\frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s\right) \leq \frac{C_0}{\gamma} + 2t\frac{w_{N,\gamma}(K)}{\gamma}\int_0^\infty \mathrm{d}\rho^N(e)e^2. \tag{5.17}$$

Notice that by the Schwarz inequality,

$$\int_0^\infty e^2 \mathrm{d}\rho^N(e) = \mu^N\left[\left(N^{-1}\sum_{x\in\mathbb{T}_N}\mathcal{E}_x\right)^2\right] \leq \mu^N\left[N^{-1}\sum_{x\in\mathbb{T}_N}\mathcal{E}_x^2\right]$$

and recall that by hypothesis,

$$\limsup_{N\to\infty} \mu^N\left[N^{-1}\sum_{x\in\mathbb{T}_N}\mathcal{E}_x^2\right] \leq E_0.$$

Hence, the limsup of the right hand side of (5.17) is bounded by $[C_0/\gamma + 2E_0 t \limsup_{N\to+\infty} \frac{w_{N,\gamma}(K)}{\gamma}]$. We prove now the following lemma.

**Lemma 5.3.**

$$\limsup_{N\to\infty} w_{N,\gamma}(K) \leq 3\gamma/K. \tag{5.18}$$

**Proof.** Recall the definition (5.14) and introduce the set

$$\mathcal{F}(C) = \left\{ \psi : \mathbb{R}_0^{\Lambda} \to \mathbb{R}^+; \int \psi \mathrm{d}\lambda^{\Lambda_0,1} = 1, \int \mathrm{d}\lambda^{\Lambda_0^1} \sum_{\substack{x,y\in\Lambda_0 \\ |x-y|=1}} (q_y \partial_{q_x} \sqrt{\psi} - q_x \partial_{q_y} \sqrt{\psi})^2 \leq C \right\}.$$

Notice that on the unit sphere $\{(q_x)_{x\in\Lambda_0};\ \sum_{x\in\Lambda_0} q_x^2 = 1\}$, we have the trivial inequality:

$$\sum_{x\in\Lambda_0} q_x^4 \leq \sum_{x\in\Lambda_0} q_x^2 = 1.$$

Hence, if $\psi$ is a density on the unit sphere not belonging to $\mathcal{F}(\gamma/N)$,

$$\int \mathrm{d}\lambda^{\Lambda_0,1} \left( \gamma \sum_{x\in\Lambda_0} q_x^4 \psi - N \sum_{\substack{x,y\in\Lambda_0 \\ |x-y|=1}} (q_y \partial_{q_x} \sqrt{\psi} - q_x \partial_{q_y} \sqrt{\psi})^2 \right) \leq 0.$$

To prove (5.18), we can hence restrict the supremum on the set $\mathcal{F}(\gamma/N)$ and evaluate

$$\gamma \sup_{\psi\in\mathcal{F}(\gamma/N)} \lambda^{\Lambda_0,1} \left( \sum_{x\in\Lambda_0} q_x^4 \psi \right)$$

$\mathcal{F}(\gamma/N)$ is a compact set for the weak topology and the Dirichlet form is lower semicontinuous. It follows that:

$$\limsup_{N\to\infty} w_{N,\gamma}(K) \leq \gamma \sup_{\psi\in\mathcal{F}(0)} \lambda^{\Lambda_0,1} \left( \sum_{x\in\Lambda_0} q_x^4 \psi \right). \tag{5.19}$$

We will check at the end of the proof that every element of $\mathcal{F}(0)$ is smooth. Then it is easy to show that a smooth density $\psi$ with Dirichlet form $\sum_{\substack{x,y\in\Lambda_0 \\ |x-y|=1}} \lambda^{\Lambda_0,1} (q_y \partial_{q_x} \sqrt{\psi} - q_x \partial_{q_y} \sqrt{\psi})^2$ equal to 0 is the function $\psi = \mathbf{1}$.

Hence, we get

$$\gamma \sup_{\psi\in\mathcal{F}(0)} \lambda^{\Lambda_0,1} \left( \sum_{x\in\Lambda_0} q_x^4 \psi \right) \leq \gamma \lambda^{\Lambda_0,1} \left( \sum_{x\in\Lambda_0} q_x^4 \right) = \gamma K \lambda^{\Lambda_0,1}(q_0^4).$$

Now the equivalence of ensembles says that $\lambda^{\Lambda_0,1}(q_0^4) \sim_{K\to\infty} 3/K^2$ and we obtain (5.18).

We prove now the smoothness property of elements of $\mathcal{F}(0)$. It follows from the hypoellipticity of the operator $\sum_{x,x+1\in\Lambda_0} (q_{x+1}\partial_{q_x} - q_x \partial_{q_{x+1}})^2$. Indeed, let us introduce the operator

$$X_{x,y} = (q_y \partial_{q_x} - q_y \partial_{q_x}), \quad x, y \in \Lambda_0$$

and remark that

$$[X_{x,y}, X_{y,z}] = X_{x,y} X_{y,z} - X_{y,z} X_{x,y} = X_{x,z}$$

for every $x, y, z \in \Lambda_0$. It follows that the Lie algebra generated by the vector fields

$$\left\{ X_{x,x+1};\ x \in \Lambda_0, x+1 \in \Lambda_0 \right\}$$

is of maximal dimension on the unit sphere $\{(q_x)_{x\in\Lambda_0};\ \sum_{x\in\Lambda_0} q_x^2 = 1\}$. We apply now the classical Hörmander's theorem ([7]) to obtain the desired smoothness of elements of $\mathcal{F}(0)$ (see [4] for details).

By lemma 5.3, we get

$$\limsup_{N\to\infty} \mathbb{E}_{\mu^N}\left(\frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s\right) \le C_0\gamma^{-1} + 6tE_0K^{-1}. \tag{5.20}$$

Letting $K$ go to infinity and then $\gamma$ to infinity, we obtain the result. □

**Corollary 5.4.** *Assume that there exists some constant $E_0 > 0$ such that the initial states $(\mu^N)_{N\ge 1}$ satisfy*

$$\limsup_{N\to\infty} \mu^N\left[\frac{1}{N}\sum_{x\in\mathbb{T}_N} \mathcal{E}_x^2\right] \le E_0. \tag{5.21}$$

*Let $u, v$ be positive reals such that $u + v = 4$ and $u > 0$. $k \in \mathbb{T}_N$ is assumed fixed. Then for any macroscopic time $t$,*

$$\lim_{N\to\infty} \mathbb{E}_{\mu^N}\left[\frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{T}_N} (p_x^u r_{x+k}^v)(s)\mathrm{d}s\right] = 0. \tag{5.22}$$

**Proof.** Essentially, it follows from Hölder's inequality. Recall that the total energy $\mathcal{E}^N$ is time invariant. Hence, for each $t$, elementary inequalities follow

$$\begin{aligned}
\mathbb{E}_{\mu^N}\left[N^{-2}\int_0^t \sum_{x\in\mathbb{T}_N} \mathcal{E}_x^2(s)\mathrm{d}s\right] &\le \int_0^t \mathrm{d}s\,\mathbb{E}_{\mu^N}\left[\left(\frac{1}{N}\sum_{x \text{ in } \mathbb{T}_N} \mathcal{E}_x(s)\right)^2\right] \\
&= t\mu^N\left[\left(\frac{1}{N}\sum_{x \text{ in}\mathbb{T}_N} \mathcal{E}_x\right)^2\right] \\
&\le t\mu_N\left[\frac{1}{N}\sum_{x\,\in\,\mathbb{T}_N} \mathcal{E}_x^2\right].
\end{aligned}$$

We get

$$\limsup_{N\to\infty} \mathbb{E}_{\mu^N}\left[N^{-2}\int_0^t \sum_{x\in\mathbb{T}_N} \mathcal{E}_x^2(s)\mathrm{d}s\right] \le tE_0. \tag{5.23}$$

Use Hölder's inequality to bound

$$\mathbb{E}_{\mu^N}\left[\frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{T}_N} (p_x^u r_{x+k}^v)(s)\mathrm{d}s\right]$$

by

$$\left(\mathbb{E}_{\mu^N}\left[\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\int_0^t p_x^4(s)\mathrm{d}s\right]\right)^{u/4}\left(\mathbb{E}_{\mu^N}\left[\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\int_0^t r_x^4(s)\mathrm{d}s\right]\right)^{v/4}.$$

By (5.23), the last term is less than

$$E_0^{v/4}\left(\mathbb{E}_{\mu^N}\left[\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\int_0^t p_x^4(s)\mathrm{d}s\right]\right)^{u/4}.$$

By the preceding lemma, we are done. □

We have now all elements to prove that the quadratic variation of the martingal $W^N$ defined in (4.6) is negligible as $N$ goes to infinity. The quadratic variation on the martingale $W^N$ is equal to

$$\frac{1}{2}\sum_{y\in\mathbb{T}_N}\mathbb{E}_{\mu^N}\left[\int_0^t \mathrm{d}s\left\{X_{y,y+1}\mathcal{F}_N\right\}^2(\omega(s))\right] \tag{5.24}$$

where

$$\mathcal{F}_N(\omega)=\sum_{x\in\mathbb{T}_N}G(x/N)\mathcal{E}_x(\omega)+\frac{1}{N}\sum_{x\in\mathbb{T}_N}(\nabla_N G)(x/N)\tau_x h(\omega).$$

It is easy to show that (5.24) can be bounded by a finite sum of terms of the form

$$\mathbb{E}_{\mu^N}\left[\frac{1}{N^2}\int_0^t\sum_{x\in\mathbb{T}_N}(p_x^u r_{x+k}^v)(s)\mathrm{d}s\right],\quad u>0,u+v=4. \tag{5.25}$$

It follows by the preceding corollary that

$$\lim_{N\to\infty}\mathbb{E}_{\mu^N}[(W^N(t))^2]=0.$$

**Lemma 5.5.** *Assume that there exists some constant $E_0>0$ such that the initial states $(\mu^N)_{N\geq 1}$ satisfy*

$$\limsup_{N\to\infty}\mu^N\left[\frac{1}{N}\sum_{x\in\mathbb{T}_N}\mathcal{E}_x^2\right]\leq E_0. \tag{5.26}$$

*Let $H:\mathbb{T}\to\mathbb{R}$ be a continuous function. Then at any time $t$ and for any $\delta>0$, we have*

$$\lim_{N\to\infty}\mathbb{P}_{\mu^N}\left[\left|\frac{1}{N}\sum_{x\in\mathbb{T}_N}H(x/N)\int_0^t p_x(s)p_{x+1}(s)\mathrm{d}s\right|>\delta\right]=0. \tag{5.27}$$

**Proof.** With the notations introduced in Lemma 5.1, we have

$$\mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\int_0^t\sum_{x\in\mathbb{T}_N}H(x/N)p_x p_{x+1}(s)\mathrm{d}s\right|\right)\leq\frac{C_0}{\gamma}+\int_0^\infty \mathrm{d}\rho^N(e)$$

$$\times \left[ \frac{1}{\gamma N} \log \mathbb{E}_{\lambda^{2N,2Ne}} \left( \exp \left\{ \gamma \left| \sum_{x \in \mathbb{T}_N} \int_0^t H(x/N) p_x(s) p_{x+1}(s) \mathrm{d}s \right| \right\} \right) \right]. \tag{5.28}$$

Since $e^{|x|} \le e^x + e^{-x}$ and since

$$\limsup_{N\to\infty} N^{-1} \log(a_N + b_N) \le \limsup_{N\to\infty} N^{-1} \log a_N + \limsup_{N\to\infty} N^{-1} \log b_N$$

it is enough to prove

$$\lim_{\gamma\to\infty} \lim_{N\to\infty} \int_0^\infty \mathrm{d}\rho^N(e) \left[ \frac{1}{\gamma N} \log \mathbb{E}_{\lambda^{2N,2Ne}} \right.$$
$$\left. \times \left( \exp \left\{ \gamma \sum_{x \in \mathbb{T}_N} \int_0^t H(x/N) p_x(s) p_{x+1}(s) \mathrm{d}s \right\} \right) \right] = 0. \tag{5.29}$$

By the Feynman–Kac formula (A1.7.2 of [8]) and the variational formula for the largest eigenvalue of an operator in a Hilbert space (A3.1.1 of [8]), the integrand in (5.29) is bounded by

$$\frac{t}{\gamma N} \sup_{\phi} \left\{ \gamma \sum_{x \in \mathbb{T}_N} H(x/N) \int p_x p_{x+1} \phi \mathrm{d}\lambda^{2N,2Ne} - N^2 D_{N,2Ne}(\phi) \right\} \tag{5.30}$$

where the supremum is carried over the set of smooth densities $\phi(\omega)$ with respect to $\lambda^{2N,2Ne}$.

Notice now that $X_{x,x+1}^2(p_x p_{x+1}) = -4 p_x p_{x+1}$. Since the operator $X_{x,x+1}$ is antisymmetric in $\mathbb{L}^2(\lambda^{2N,2Ne})$, an integration by parts can be performed. For any $A > 0$, by the Schwartz inequality, we get

$$\begin{aligned}
&\gamma \sum_{x \in \mathbb{T}_N} H(x/N) \int p_x p_{x+1} \phi \mathrm{d}\lambda^{2N,2Ne} \\
&\quad \le \frac{\|H\|_\infty A \gamma}{4} \sum_{x \in \mathbb{T}_N} \int (X_{x,x+1} p_x p_{x+1})^2 \phi \mathrm{d}\lambda^{2N,2Ne} \\
&\qquad + \frac{\gamma \|H\|_\infty}{4A} \int \sum_{x \in \mathbb{T}_N} (X_{x,x+1}(\sqrt{\phi}))^2 \mathrm{d}\lambda^{2N,2Ne} \\
&\quad = \frac{\|H\|_\infty A \gamma}{4} \sum_{x \in \mathbb{T}_N} \int (X_{x,x+1} p_x p_{x+1})^2 \phi \mathrm{d}\lambda^{2N,2Ne} + \frac{\gamma \|H\|_\infty}{2A} D_{N,2Ne}(\phi).
\end{aligned} \tag{5.31}$$

Since $(X_{x,x+1}(p_x p_{x+1}))^2 = (p_{x+1}^2 - p_x^2)^2 \le 2(p_{x+1}^4 + p_x^4)$, with the choice $A = \gamma \|H\|_\infty N^{-1}$, we have $N^2 - \gamma \|H\|_\infty (2A)^{-1} \ge N^2/2$ and we have then to prove that for any $\gamma > 0$

$$\lim_{N\to\infty} \frac{t}{2\gamma} \int \mathrm{d}\rho^N(e) \sup_{\phi} \left\{ \frac{\|H\|_\infty^2}{N^2} \sum_{x \in \mathbb{T}_N} p_x^4 \phi \mathrm{d}\lambda^{2N,2Ne} - N D_{N,2Ne}(\phi) \right\} = 0. \tag{5.32}$$

This is proved in Lemma 5.1. □

## 6. Discussion

We proved in the preceding section that if

$$\limsup_{N\to\infty} \mu^N \left[ \frac{1}{N} \sum_{x\in\mathbb{T}_N} \mathcal{E}_x^2 \right] \leq E_0 \tag{6.1}$$

then, at any time $t$,

$$\lim_{N\to\infty} \mathbb{E}_{\mu^N} \left[ \frac{1}{N^2} \int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s \right] = 0. \tag{6.2}$$

In fact, we expect better and we conjecture that if (6.1) is satisfied then

$$\limsup_{N\to\infty} \mathbb{E}_{\mu^N} \left[ \frac{1}{N} \int_0^t \sum_{x\in\mathbb{T}_N} p_x^4(s)\mathrm{d}s \right] \leq E_0' \tag{6.3}$$

where $E_0'$ is a constant independent of $N$.

In order to motivate this conjecture, consider only the symmetric part $\mathcal{S}_N$ of the generator $\mathcal{L}_N$. $N^2\mathcal{S}_N$ is the generator of a particle system denoted by $(p^N(t))_{t\geq 0}$. Hence, $p^N(t) = (p_x^N(t))_{x\in\mathbb{T}_N}$ satisfy the following stochastic differential equations:

$$\mathrm{d}p_x^N(t) = Np_{x-1}^N \mathrm{d}W_{x-1,x}(t) - Np_{x+1}^N \mathrm{d}W_{x,x+1}(t) - N^2 p_x^N \mathrm{d}t \tag{6.4}$$

where $x \in \mathbb{T}_N$ and $\{W_{x,x+1}\}_{x\in\mathbb{T}_N}$ are independent standard Brownian motions. The term $N^2$ (resp. $N$) in $N^2\mathcal{S}_N$ (resp. (6.4)) corresponds to the acceleration of time by $N^2$ (diffusive scale). This reversible system is of gradient type and the derivation of the hydrodynamic limit is for the main part standard by following the entropy method of [6]. Nevertheless, as for the process $\omega^N$, our proof of the hydrodynamic limit for $p^N$ works only if (6.2) is valid. Of course, the results of Section 6 are directly applicable for the process $p^N$ so that we can prove (6.2) for the process $p^N$. But a $H_{-1}$ norm argument (cf. [8], chapter 5) shows that (6.3) is true for the process $p^N$ starting from a good initial state $\mu^N$. This gives reasons to suspect that (6.3) is also true for the process $\omega^N$.

Notice that the method used here to derive the hydrodynamic limit for $\omega^N$ is the entropy method of [6], [14]. It is of some interest to notice that if we try to obtain the hydrodynamical behavior using the relative entropy method of Yau [15], we have to establish (6.3) (even for the particular simple model $p^N$!). Hence, even if we are able to obtain the hydrodynamic limit for $p^N$ by the *entropy method*, we are not able to obtain it using the *relative entropy method*.

To finish, let us say that essentially the model is non linear. Using the "adapted" relative entropy method of [12], the non linear hydrodynamic limit for the slight modified model $(\omega_\gamma^N(t))_{t\geq 0}$ described in the introduction should be obtained under the assumption (6.3).

## Acknowledgements

It is a pleasure to thank S. Olla for giving me the model and for useful discussions. I am also grateful to J. Fritz, C. Landim and the referee for very pertinent comments, and to S.R.S. Varadhan for his help in the proof of Lemma 5.1.

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