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FLUCTUATIONS IN THE OCCUPATION TIME OF A SITE IN THE ASYMMETRIC SIMPLE EXCLUSION PROCESS

BY CÉDRIC BERNARDIN

Ecole Normale Supérieure de Cachan

We consider the simple asymmetric exclusion process with nonzero drift under the stationary Bernoulli product measure at density ρ. We prove that for dimension $d = 2$ the occupation time of the site 0 is diffusive as soon as $\rho \neq 1/2$. For dimension $d = 1$, if the density ρ is equal to $1/2$, we prove that the time t variance of the occupation time of the site 0 diverges in a certain sense at least as $t^{5/4}$.

1. Introduction. We are interested here in the variance of additive functionals of the simple exclusion process. The simple exclusion process $(\eta_t)_{t\geq 0}$ is a Markov process evolving on the configuration space $\Omega = \{0, 1\}^{\mathbb{Z}^d}$. If $\eta \in \Omega$ is a configuration, $\eta(x)$ is equal to 1 or 0 depending on whether the site x is occupied or not. First, let us fix a finite-range transition probability $p(\cdot)$ on $\mathbb{Z}^d$ whose associated mean is denoted by $m = \sum_x p(x)x$ and such that the symmetrization $s(\cdot) = \frac{1}{2}(p(\cdot) + p(-\cdot))$ is assumed to be irreducible. The process $(\eta_t)_{t\geq 0}$ is described by its generator $\mathcal{L}$, which acts on the local functions f as

$$(\mathcal{L}f)(\eta) = \sum_{x,y} p(y-x)\eta(x)\big(1-\eta(y)\big)\big(f(\eta^{x,y}) - f(\eta)\big) \tag{1.1}$$

and whose associated semigroup is denoted $(T_t)_{t\geq 0}$. Here and below $\eta^{x,y}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$, $\eta(y)$.

Recall that the process is conservative in the sense that no particles are created or destroyed. In fact, for each density $\rho \in [0, 1]$, the Bernoulli product measure ν_ρ over Ω obtained by placing a particle with probability ρ at each site x, independently from the other sites, is invariant for the exclusion process $(\eta_t)_{t\geq 0}$. Moreover, in the symmetric case, $p = s$, ν_ρ is also reversible. In the sequel, when we use expressions such as "if the density is ρ," it means that we will consider the process $(\eta_t)_{t\geq 0}$ starting from the probability measure ν_ρ.

We are interested here in the variance of the occupation time functional

$$\int_0^t \big(\eta_s(0) - \rho\big)\,ds \tag{1.2}$$

whose characterization is used to prove central limit theorems for additive functionals. In [3], an invariance principle for additive functionals under diffusive scaling was established by Kipnis and Varadhan in the symmetric case (for

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which the mean m of p is zero). In [2], Kipnis computes the variance of the occupation time using the duality properties of the symmetric simple exclusion. Later Varadhan [12] extended these results in the asymmetric case with mean $m=0$. More recently Sethuraman, Varadhan and Yau [11] prove invariance principles for the general asymmetric case for dimension $d \geq 3$, where transience estimates could be used. See [9] for a good review and extensions of all these results.

The remaining problem, then, concerns dimensions 1 and 2 for the asymmetric case with nonzero mean m. In [9], Sethuraman proves the invariance principle when f is a local function with finite limiting variance $\sigma^2(f)$, which is defined, if the limit exists, by

$$\sigma^2(f) = \lim_{t \to \infty} t^{-1} \sigma_t^2(f), \tag{1.3}$$

where $\sigma_t^2(f) = \mathbb{E}_\rho[\int_0^t f(\eta_s)\,ds]^2$ is the time t variance of the function f, and $\mathbb{E}_\rho$ is the expectation with respect to ν_ρ.

He shows that if $\sigma^2(\eta(0)-\rho) < +\infty$, $\sigma^2[(\eta(0)-\rho)(\eta(1)-\rho)] < +\infty$ for the dimension $d=1$ and if $\sigma^2(\eta(0)-\rho) < +\infty$ for the dimension $d=2$, then

$$\sigma^2(f) < +\infty \quad \Longleftrightarrow \quad \mathbb{E}_\rho(f) = 0. \tag{1.4}$$

From heuristic reasoning about the second-class particle, Sethuraman conjectures that if $m \neq 0$, then $\sigma^2(\eta(0)-\rho) < +\infty \Longleftrightarrow \rho \neq 1/2$ in dimension $d = 1, 2$ and that $\sigma^2[(\eta(0)-\rho)(\eta(1)-\rho)] < +\infty$ for all $0 \leq \rho \leq 1$ in dimension $d=1$ (cf. [9]).

In a recent paper Seppäläinen and Sethuraman [8] prove that, in dimension $d=1$, if $\rho \neq 1/2$, then $\sigma^2(\eta(0)-\rho) < +\infty$.

The aim of this paper is to establish some more positive answers to these conjectures. We prove the following two results:

1. For $d=2$ we prove that $\sigma^2(\eta(0)-\rho) < +\infty$ if $\rho \neq 1/2$ and $m \neq 0$ (this extends the result of Seppäläinen and Sethuraman to the two-dimensional setting).
2. For $d=1$, if the density ρ is equal to $1/2$ and $m \neq 0$, then $\sigma^2(\eta(0)-\rho) = +\infty$ and in fact that, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\sigma_t^2\big(\eta(0)-\rho\big) \geq C_\varepsilon t^{5/4-\varepsilon}. \tag{1.5}$$

The expected best exponent in (1.5) is not $5/4$ but $4/3$, and, in fact, it is conjectured that for $d=1$, for the total asymmetric exclusion process, the time correlations has the scaling form (cf. [6])

$$\begin{aligned}\mathbb{E}_\rho\big(\eta_t(j)\eta_0(0)\big) - \rho^2 &\sim \rho(1-\rho)\big(4\big(\rho(1-\rho)\big)^{1/3} t^{2/3}\big)^{-1} \\ &\qquad \times \tfrac{1}{8} g''\big((j-(1-2\rho)t)\big(4\big(\rho(1-\rho)\big)^{1/3} t^{2/3}\big)^{-1}\big),\end{aligned}$$

where the scaling function g is the second moment function of the solution of a first-order partial differential equation defined in [1].

Recently Yau [13] established that the diffusion coefficient in $d = 2$ diverges as $(\log t)^{2/3}$ in the leading order, and Landim, Quastel, Salmhofer and Yau [4] showed that the diffusion coefficient diverges at least as $t^{1/4}$ for $d = 1$ and as $(\log t)^{1/2}$ for $d = 2$.

The proof of the two results of this paper is based on the following argument.

Let the generator $\mathcal{L}$ be decomposed as $\mathcal{L} = \mathcal{S} + \mathcal{A}$, where $\mathcal{S} = (\mathcal{L} + \mathcal{L}^*)/2$ and $\mathcal{A} = (\mathcal{L} - \mathcal{L}^*)/2$ denote, respectively, the symmetric and antisymmetric parts of $\mathcal{L}$. Let f be an element of $\mathbb{L}^2(\nu_\rho)$. Since $\mathcal{L}$ is a nonpositive operator, for any $\lambda > 0$, we have the following two variational formulas (cf. Lemma 2.1):

$$
\begin{aligned}
&\langle f, (\lambda - \mathcal{L})^{-1} f\rangle \\
&\qquad = \sup_g \{2\langle f, g\rangle - \langle g, (\lambda - \mathcal{S})g\rangle - \langle \mathcal{A}g, (\lambda - \mathcal{S})^{-1}\mathcal{A}g\rangle\} && (1.6)\\
&\qquad = \inf_g \{2\langle (f + \mathcal{A}g), (\lambda - \mathcal{S})^{-1}(f + \mathcal{A}g)\rangle + \langle g, (\lambda - \mathcal{S})g\rangle\}, && (1.7)
\end{aligned}
$$

where the supremum and infimum are taken over all local functions g. These two variational formulas permit us to obtain lower and upper bounds of the Laplace transform of the time t variance of f by a good choice of test functions. To perform the calculations we use the duality expansion (as done in [5, 11] for the case $d \geq 3$). A key step, suggested to us by Yau (cf. [7] for the two-dimensional case), is then to approximate the terms in (1.6) involving $\mathcal{S}$ (the generator of the symmetric exclusion process) with similar terms involving the generator of the corresponding independent symmetric random walks. In this way the calculations can then be performed by standard Fourier calculus.

This article is organized as follows. In Section 2, we recall some definitions about the duality of the exclusion process and introduce Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$. In Section 3, we give results concerning approximations, in $\mathcal{H}_{-1}$, of the exclusion process by a system of free particles. This approximation is used in Section 4 to prove the super-diffusive behavior of the occupation time in dimension 1 with density $1/2$ and in Section 5 to prove the diffusivity of the occupation time in dimension 2 with a density different from $1/2$.

2. Recalling the duality of the simple exclusion process. We recall some definitions concerning duality and the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$. We fix a density ρ in $(0, 1)$ and note $\chi(\rho) = \rho(1 - \rho)$. Let $\mathcal{E}$ be the class of all finite subsets of $\mathbb{Z}^d$ and let $\mathcal{E}_n$ be the subsets of $\mathbb{Z}^d$ with n points. For each $A \in \mathcal{E}$, let Ψ_A be the local function

$$\Psi_A(\eta) = \prod_{x \in A} \frac{(\eta(x) - \rho)}{\sqrt{\chi(\rho)}}$$

and by convention $\Psi_\varnothing = 1$. It is easy to check that $\{\Psi_A; A \in \mathcal{E}\}$ is a Hilbertian basis of $\mathbb{L}^2(\nu_\rho)$. We will denote by $\mathcal{F}_n$ the subspace generated by $\{\Psi_A; A \in \mathcal{E}_n\}$. The functions of $\mathcal{F}_n$ are called functions of degree n.

All elements f of $\mathbb{L}^2(\nu_\rho)$ can be decomposed in the basis $\{\Psi_A; A \in \mathcal{E}\}$ and we write

$$f = \sum_{n\geq 0} \sum_{A\in\mathcal{E}_n} \mathfrak{f}(A)\Psi_A. \tag{2.1}$$

Note that the coefficients $\mathfrak{f}(A)$ depend on the density ρ but we will omit this fact in the notation since we will always work with a fixed density in the sequel.

Denote by the same symbol $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\rho$ the inner product on $\mathbb{L}^2(\nu_\rho)$ and the inner product with respect to the counting measure on $\mathbb{L}^2(\mathcal{E})$. Note that if f, g are two elements of $\mathbb{L}^2(\nu_\rho)$, then we have

$$\langle f, g\rangle = \langle \mathfrak{f}, \mathfrak{g}\rangle = \sum_{A\in\mathcal{E}} \mathfrak{f}(A)\mathfrak{g}(A).$$

We note the $\mathbb{L}^2$ norms by $\|\mathfrak{f}\|_0^2 = \|f\|_0^2 = \langle \mathfrak{f}, \mathfrak{f}\rangle$. Let $\mathfrak{G}_n$ be the subspace generated by the finite supported functions of degree n and let π_n be the projection in $\mathbb{L}^2(\mathcal{E})$ on $\mathfrak{G}_n$. Note that $\mathfrak{G}_1$ is just the set of local functions from $\mathbb{Z}^d$ into $\mathbb{R}$ with finite $\mathbb{L}^2$ norm.

For a subset A of $\mathbb{Z}^d$ and x, y in $\mathbb{Z}^d$ denote by $A_{x,y}$ the set defined by $A_{x,y} = A \setminus \{x\} \cup \{y\}$ if $x \in A$ and $y \notin A$, by $A_{x,y} = A \setminus \{y\} \cup \{x\}$ if $y \in A$ and $x \notin A$ and by $A_{x,y} = A$ otherwise. Denote by $s(\cdot)$ and $a(\cdot)$ the symmetric and antisymmetric parts of the transition probability $p(\cdot)$. In the basis $\{\Psi_A; A \in \mathcal{E}\}$, we have the following decomposition of the generator $\mathcal{L}$:

$$(\mathcal{L} f) = \sum_{A\in\mathcal{E}} (\mathfrak{L}\mathfrak{f})(A)\Psi_A,$$

$$(\mathcal{S} f) = \sum_{A\in\mathcal{E}} (\mathfrak{S}\mathfrak{f})(A)\Psi_A,$$

$$(\mathcal{A} f) = \sum_{A\in\mathcal{E}} (\mathfrak{A}\mathfrak{f})(A)\Psi_A,$$

where $\mathfrak{L} = \mathfrak{S} + \mathfrak{A}$ and $\mathfrak{S} = \mathfrak{L}^1$, $\mathfrak{A} = (1-2\rho)\mathfrak{L}^2 + 2\sqrt{\chi(\rho)}(\mathfrak{L}^+ - \mathfrak{L}^-)$, with

$$(\mathfrak{L}^1\mathfrak{f})(A) = (1/2) \sum_{x,y\in\mathbb{Z}^d} s(y-x)[\mathfrak{f}(A_{x,y}) - \mathfrak{f}(A)],$$

$$(\mathfrak{L}^2\mathfrak{f})(A) = \sum_{x\in A, y\notin A} a(y-x)[\mathfrak{f}(A_{x,y}) - \mathfrak{f}(A)],$$

$$(\mathfrak{L}^-\mathfrak{f})(A) = \sum_{x\notin A, y\notin A} a(y-x)\mathfrak{f}(A\cup\{x\}),$$

$$(\mathfrak{L}^+\mathfrak{f})(A) = \sum_{x\in A, y\in A} a(y-x)\mathfrak{f}(A-\{y\}).$$

Note that the operators $\mathfrak{L}^1$ and $\mathfrak{L}^2$ preserve the degree of functions but that $\mathfrak{L}^+$ (resp. $\mathfrak{L}^-$) increases (resp. decreases) the degree by 1. The operator $\mathfrak{A}$ has a unique decomposition of the form

$$\mathfrak{A} = \sum_{n\geq 0}(\mathfrak{A}_{n,n-1} + \mathfrak{A}_{n,n} + \mathfrak{A}_{n,n+1}), \tag{2.2}$$

where $\mathfrak{A}_{n,m}$ is the operator $\pi_m \mathfrak{A} \pi_n$ sending any function of degree n onto a function of degree m.

In the sequel, we will primarily consider functions of degree 1 for which we will use the notation $\mathfrak{f}(\cdot) = \mathfrak{f}(\{\cdot\})$. Hence we give the following expression of the operators $\mathfrak{A}_{11} = (1 - 2\rho)\mathfrak{B}_{11}$ and $\mathfrak{A}_{12} = 2\sqrt{\chi(\rho)}\mathfrak{B}_{12}$:

$$\begin{aligned}(\mathfrak{B}_{11}\mathfrak{f})(x) &= \sum_y a(y - x)[\mathfrak{f}(y) - \mathfrak{f}(x)],\\ (\mathfrak{B}_{12}\mathfrak{f})(\{x, y\}) &= a(y - x)[\mathfrak{f}(x) - \mathfrak{f}(y)].\end{aligned} \tag{2.3}$$

We start with the definitions of some resolvent norms and associated Hilbert spaces. Recall that $\mathcal{L}$ is an operator on $\mathbb{L}^2(\nu_\rho)$ whose symmetric part is $\mathcal{S}$ (itself the infinitesimal generator of the symmetric simple exclusion process with jump rate s) and antisymmetric part is $\mathcal{A}$. Denote by $\mathcal{D}(f) = -\langle f, \mathcal{L}f\rangle = -\langle f, \mathcal{S}f\rangle \geq 0$ the Dirichlet form of the process. The Dirichlet space $\mathcal{H}_1$ is defined by the completion with respect to the Dirichlet form $\mathcal{D}$, and the associated norm (resp. inner product) is denoted $\|\cdot\|_1 = \sqrt{\mathcal{D}}$ (resp. $\langle\cdot,\cdot\rangle_1$). By duality of the simple exclusion process, the Dirichlet space $\mathcal{H}_1$ corresponds to the space $\mathfrak{H}_1$, which is the completion of $\{\mathfrak{f} : \mathcal{E} \to \mathbb{R}$ with finite support s.t. $\langle \mathfrak{f}, -\mathfrak{L}\mathfrak{f}\rangle < \infty\}$. The corresponding norm (resp. inner product) will be denoted $\|\cdot\|_1$ (resp. $\langle\cdot,\cdot\rangle_1$).

Let $\mathcal{H}_{-1}$ (resp. $\mathfrak{H}_{-1}$) be the dual of $\mathcal{H}_1$ (resp. $\mathfrak{H}_1$) with respect to $\mathbb{L}^2(\nu_\rho)$ [resp. $\mathbb{L}^2(\mathcal{E})$]. This is the Hilbert space generated by the local functions (resp. finite supported functions) and the norm $\|\cdot\|_{-1}$ defined by

$$\begin{aligned}\|f\|_{-1}^2 &= \sup_g\{2\langle f, g\rangle - \|g\|_1^2\},\\ \|\mathfrak{f}\|_{-1}^2 &= \sup_{\mathfrak{g}}\{2\langle \mathfrak{f}, \mathfrak{g}\rangle - \|\mathfrak{g}\|_1^2\},\end{aligned} \tag{2.4}$$

where the supremum is carried over all local functions g (resp. finite supported functions $\mathfrak{g}$).

By duality of the simple exclusion process, we have of course $\|\mathfrak{f}\|_1^2 = \|f\|_1^2$ and $\|\mathfrak{f}\|_{-1}^2 = \|f\|_{-1}^2$. In the same way, for each $\lambda > 0$, we can define the Hilbert space $\mathcal{H}_{1,\lambda}$ (resp. $\mathfrak{H}_{1,\lambda}$) by completion with respect to the norm $\|f\|_{1,\lambda} = (\langle f, (\lambda - \mathcal{S})f\rangle)^{1/2}$ [resp. $\|\mathfrak{f}\|_{1,\lambda} = (\langle \mathfrak{f}, (\lambda - \mathfrak{S})\mathfrak{f}\rangle^{1/2}$]. Let $\mathcal{H}_{-1,\lambda}$ (resp. $\mathfrak{H}_{-1,\lambda}$) be the dual of $\mathcal{H}_{1,\lambda}$ (resp. $\mathfrak{H}_{1,\lambda}$) with respect to $\mathbb{L}^2(\nu_\rho)$ [resp. $\mathbb{L}^2(\mathcal{E})$]. The $\mathcal{H}_{-1,\lambda}$-norm (or $\mathfrak{H}_{-1,\lambda}$-norm) will be denoted $\|\cdot\|_{-1,\lambda}$.

LEMMA 2.1. *Let f be a local function and let $\lambda > 0$. Then we have the two variational formulas*

$$\langle f, (\lambda - \mathcal{L})^{-1} f\rangle = \sup_g \{2\langle f, g\rangle - \|g\|_{1,\lambda}^2 - \|\mathcal{A}g\|_{-1,\lambda}^2\}$$
$$= \inf_g \{\|f + \mathcal{A}g\|_{-1,\lambda}^2 + \|g\|_{1,\lambda}^2\}.$$

PROOF. Recall that, for $\lambda > 0$, $(\lambda - \mathcal{L})^{-1}$ is a bounded operator on $\mathbb{L}^2(\nu_\rho)$. Let $[(\lambda - \mathcal{L})^{-1}]_s$ denote the symmetric part of $(\lambda - \mathcal{L})^{-1}$. A simple computation shows that

$$\big[[(\lambda - \mathcal{L})^{-1}]_s\big]^{-1} = (\lambda - \mathcal{S}) - \mathcal{A}(\lambda - \mathcal{S})^{-1}\mathcal{A}.$$

Hence, we have

$$\langle f, (\lambda - \mathcal{L})^{-1} f\rangle = \langle f, [(\lambda - \mathcal{L})^{-1}]_s f\rangle$$
$$= \sup_g \big\{2\langle f, g\rangle - \big\langle g, \big[[(\lambda - \mathcal{L})^{-1}]_s\big]^{-1} g\big\rangle\big\}$$
$$= \sup_g \{2\langle f, g\rangle - \|g\|_{1,\lambda}^2 - \|\mathcal{A}g\|_{-1,\lambda}^2\}.$$

Using this formula with $\mathcal{S}$ in the place of $\mathcal{L}$, we have

$$\|\mathcal{A}g\|_{-1,\lambda}^2 = \sup_u \{2\langle \mathcal{A}g, u\rangle - \|u\|_{1,\lambda}^2\}.$$

Hence, we obtain

$$\langle f, (\lambda - \mathcal{L})^{-1} f\rangle = \sup_g \inf_u \{2\langle f, g\rangle - \|g\|_{1,\lambda}^2 - 2\langle \mathcal{A}g, u\rangle + \|u\|_{1,\lambda}^2\}$$
$$= \sup_g \inf_u \{2\langle f + \mathcal{A}u, g\rangle - \|g\|_{1,\lambda}^2 + \|u\|_{1,\lambda}^2\}$$
$$\le \inf_u \sup_g \{2\langle f + \mathcal{A}u, g\rangle - \|g\|_{1,\lambda}^2 + \|u\|_{1,\lambda}^2\}$$
$$= \inf_u \{\|f + \mathcal{A}u\|_{-1,\lambda}^2 + \|u\|_{1,\lambda}^2\}.$$

To prove that the inequality is in fact an equality, consider the functions u_λ and g_λ defined by

$$u_\lambda = (\lambda - \mathcal{S})^{-1}\mathcal{A}g_\lambda$$

with

$$\big[[(\lambda - \mathcal{L})^{-1}]_s\big]^{-1} g_\lambda = (\lambda - \mathcal{S})g_\lambda - \mathcal{A}(\lambda - \mathcal{S})^{-1}\mathcal{A}g_\lambda$$
$$= f.$$

Using the definitions of u_λ, g_λ, it is easy to show that

$$\langle f, (\lambda - \mathcal{L})^{-1} f\rangle = \langle f, g_\lambda\rangle = \|f + \mathcal{A}u_\lambda\|_{-1,\lambda}^2 + \|u_\lambda\|_{1,\lambda}^2$$

so that the inequality is an equality and we are done. $\square$

Hence, for $f \in \mathbb{L}^2(\nu_\rho)$, we have

$$\langle f, (\lambda - \mathcal{L})^{-1} f\rangle = \sup_g \{2\langle f, g\rangle - \|g\|_{1,\lambda}^2 - \|\mathcal{A}g\|_{-1,\lambda}^2\} \tag{2.5}$$

$$= \sup_{\mathfrak{g}} \{2\langle \mathfrak{f}, \mathfrak{g}\rangle - \|\mathfrak{g}\|_{1,\lambda}^2 - \|\mathfrak{A}\mathfrak{g}\|_{-1,\lambda}^2\} \tag{2.6}$$

$$= \inf_g \{2\|f + \mathcal{A}g\|_{-1,\lambda}^2 + \|g\|_{1,\lambda}^2\} \tag{2.7}$$

$$= \inf_{\mathfrak{g}} \{2\|\mathfrak{f} + \mathfrak{A}\mathfrak{g}\|_{-1,\lambda}^2 + \|\mathfrak{g}\|_{1,\lambda}^2\}, \tag{2.8}$$

where the supremum and infimum are taken over all local functions g (or all finite supported functions $\mathfrak{g}$).

3. Approximation by free particles. Although we are in dimension 1 or 2, since the approximation is valid for all dimensions, we give the proof for dimension $d \geq 1$ not necessarily less than 2. We denote by $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$. If $x = (x^{(1)}, \dots, x^{(n)}) \in (\mathbb{Z}^d)^n = \chi_n$ and $z \in \mathbb{Z}^d$, then the notation $x + ze_j$ stands for $(x^{(1)}, \dots, x^{(j)} + z, \dots, x^{(n)})$. $\mathcal{E}_n$ can be seen as a subclass of $\chi_n : \mathcal{E}_n = \{(x^{(1)}, \dots, x^{(n)}) \in \chi_n;\ \forall i \neq j, x^{(i)} \neq x^{(j)}\}$. Recall that the restriction of $\mathfrak{S} = \mathfrak{L}^1$ to $\mathfrak{S}_n$ is the infinitesimal generator of n particles in symmetric simple exclusion:

$$(\mathfrak{L}^1 \mathfrak{f})(A) = (1/2) \sum_{x,y \in \mathbb{Z}^d} s(y - x)[\mathfrak{f}(A_{x,y}) - \mathfrak{f}(A)]. \tag{3.1}$$

Here, $A \in \mathcal{E}_n$ and $\mathfrak{f} \in \mathfrak{S}_n$. The state space of this process $(\zeta_t)_{t\geq 0}$ is $\mathcal{E}_n$. We will denote by $\mathfrak{D}$ the Dirichlet form associated with $\mathfrak{L}^1$.

Consider now n free symmetric particles evolving on $\mathbb{Z}^d$. The state space of this process $(x_t)_{t\geq 0} = (x_t^{(1)}, \dots, x_t^{(n)})_{t\geq 0}$ is χ_n. The infinitesimal generator of n free particles with jump rates $s(\cdot)$ is given by

$$(\mathfrak{L}^{\text{free}} \mathfrak{f})(x) = \sum_{\substack{1 \leq j \leq n \\ z \in \mathbb{Z}^d}} s(z)[\mathfrak{f}(x + ze_j) - \mathfrak{f}(x)]. \tag{3.2}$$

Here $\mathfrak{f}$ is a finite supported function on χ_n. We have an another description of this process. Consider a Poisson process on $\mathbb{R}$ whose jump times are denoted by $T_1, \dots, T_k, \dots$. At time T_k, a particle is uniformly chosen between the n particles and this particle jumps of $z \in \mathbb{Z}^d$ with probability $s(z)$.

We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ ($\|\cdot\|_{0,\text{free}}$ being the associated norm) the inner product on $\mathbb{L}^2(\chi_n)$ defined by

$$\langle\langle\phi,\psi\rangle\rangle = \frac{1}{n!}\sum_{x\in\chi_n}\phi(x)\psi(x). \tag{3.3}$$

Let $\mathfrak{D}_{\text{free}}$ be the Dirichlet form associated with $\mathfrak{L}^{\text{free}}$. If ϕ is a local function on χ_n, then

$$\mathfrak{D}_{\text{free}}(\phi) = \frac{1}{n!}\sum_{\substack{x\in\chi_n,z\in\mathbb{Z}^d\\1\le j\le n}} s(z)[\phi(x+ze_j)-\phi(x)]^2. \tag{3.4}$$

THEOREM 3.1. *Let the dimension d and the degree $n\ge 1$ be fixed. There exist some positive constants $C_1\equiv C_1(n,d)$ and $C_2\equiv C_2(n,d)$ such that for each finite supported function $\mathfrak{f}$ on $\mathcal{E}_n$ there exists a symmetric function $\tilde{\mathfrak{f}}$ on χ_n with the following two properties*:

$$C_1\mathfrak{D}_{\text{free}}(\tilde{\mathfrak{f}}) \le \mathfrak{D}(\mathfrak{f}) \le C_2\mathfrak{D}_{\text{free}}(\tilde{\mathfrak{f}}) \tag{3.5}$$

and

$$C_1\|\tilde{\mathfrak{f}}\|_{0,\text{free}} \le \|\mathfrak{f}\|_0 \le C_2\|\tilde{\mathfrak{f}}\|_{0,\text{free}}. \tag{3.6}$$

PROOF. All the estimates depend on the dimension d but in the notation we will omit the dependence on d of the constants. Let T be the stopping time of the process $(x_t)_{t\ge 0}$,

$$T = \inf\{t\ge 0/\text{Card}\{x_t^{(1)},\dots,x_t^{(n)}\} = n\}.$$

We first prove there exists some positive integer $K\equiv K(d)$, depending only on the dimension d, such that, for each initial configuration of the process $x\in\chi_n$,

$$\mathbb{P}_x[T\le T_{Kn^2}] \ge \alpha_n > 0, \tag{3.7}$$

where α_n is a constant independent of x.

A configuration $x=(x^{(1)},\dots,x^{(n)})\in\chi_n$ will be called *good* if all the points of $\mathbb{Z}^d$, $x^{(1)},\dots,x^{(n)}$, are distinct. Otherwise it will be called *bad*. Denote by $\mathcal{G}_n$ the set of good configurations and by $\mathcal{B}_n$ the set of bad configurations. In the sequel we will use the following definition: if x and y are two configurations of χ_n, then the distance between x and y, $d(x,y)$, is defined as the infimum of the lengths of the trajectories $\mathcal{C}_k(x,y)=[x_0=x,\dots,x_k=y]$, between x and y, of the process $(x_t)_{t\ge 0}$. Note that, by irreducibility of $s(\cdot)$, d is well defined, by symmetry of $s(\cdot)$, d is symmetric and, of course, d is subadditive. We need to prove that, for some integer K and for any initial configuration, we are able to find a path of length at most Kn^2 so that the final configuration is good.

Consider all the lines parallel to the basis vector e_1 of $\mathbb{R}^d$ containing at least one particle of the configuration x. Denote by $L_1, \dots, L_k$ these lines and by n_α the number of particles on the line L_α so that $\sum_{\alpha=1}^k n_\alpha = n$.

Let L_α be a fixed line. We call $y^{(1)}, y^{(2)}, \dots, y^{(k_\alpha)}$ the different sites of the line where there is at least one particle and we assume $y_1^{(1)} < y_1^{(2)} < \cdots < y_1^{(k_\alpha)}$. We denote by m_j the number of particles on the site $y^{(j)}$ so that $\sum_{j=1}^{k_\alpha} m_j = n_\alpha$. Thanks to the irreducibility and translation invariance of the symmetrization, in K_1 steps, we can move a particle from site u to site $u + e_1$. Consequently, in $K_1 \sum_{i=1}^{k_\alpha - 1} m_i (m_{i+1} + \cdots + m_{k_\alpha})$ steps, for each $j \in \{2, \dots, k_\alpha\}$, we can move all the particles on the site $y^{(j)}$ to a site $w^{(j)} \in L_\alpha$ such that $w_1^{(j)} - w_1^{(j-1)} = m_{j-1}$. Now, for each j, in $K_1 m_j (m_j - 1)/2$ steps, we can move the particles which are on the site $w^{(j)}$ to the sites $w^{(j)}, w^{(j)} + e_1, \dots, w^{(j)} + (m_j - 1)e_1$, with one particle per site. After all these moves, the particles contained initially on the line L_α are always on the line L_α but with at most one particle per site: it is done in at most $K_1 \sum_{i=1}^{k_\alpha - 1} m_i (m_{i+1} + \cdots + m_{k_\alpha}) + K_1 \sum_{i=1}^{k_\alpha} m_i (m_i - 1)/2 \le \frac{3K_1}{2} n_\alpha^2$ steps. We do that for each line and we obtain a good configuration after at most $\frac{3K_1}{2} n^2$ steps.

It follows that $\mathbb{P}_x[T \le T_{Kn^2}] \ge \alpha_n > 0$, where α_n is a constant independent of x. By the Markov property, we obtain $\mathbb{P}_x(T \ge T_k) \le C_n \rho_n^k$, where $\rho_n \in (0, 1)$ and $C_n > 0$. In particular, $\mathbb{P}_x$-a.s., T is finite. Let us define $\tilde{\mathfrak{f}}$ on χ_n by

$$\tilde{\mathfrak{f}}(x) = \mathbb{E}_x[\mathfrak{f}(x_T)] \tag{3.8}$$

for $x \in \chi_n$.

Note that if $x \in \mathcal{E}_n$ $[i \ne j \Rightarrow x^{(i)} \ne x^{(j)}]$, then $\tilde{\mathfrak{f}}(x) = \mathfrak{f}(x)$. Consequently, using the Schwarz inequality and irreducibility of the symmetrization, it is easy to check the second inequality in (3.5).

The Dirichlet form $\mathfrak{D}_{\text{free}}(\tilde{\mathfrak{f}})$ can be decomposed into three terms:

$$\mathfrak{D}_{\text{free}}(\tilde{\mathfrak{f}}) = \frac{1}{n!}\{\mathcal{S}(\mathcal{G}_n, \mathcal{G}_n) + \mathcal{S}(\mathcal{B}_n, \mathcal{B}_n) + 2\mathcal{S}(\mathcal{G}_n, \mathcal{B}_n)\},$$

where

$$\mathcal{S}(\mathcal{G}_n, \mathcal{G}_n) = \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{x \in \mathcal{G}_n \cap (\mathcal{G}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x) - \tilde{\mathfrak{f}}(x + ze_j)\big)^2,$$

$$\mathcal{S}(\mathcal{G}_n, \mathcal{B}_n) = \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{x \in \mathcal{G}_n \cap (\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x) - \tilde{\mathfrak{f}}(x + ze_j)\big)^2,$$

$$\mathcal{S}(\mathcal{B}_n, \mathcal{B}_n) = \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{x \in \mathcal{B}_n \cap (\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x) - \tilde{\mathfrak{f}}(x + ze_j)\big)^2.$$

The first term is equal to

$$\mathcal{S}(\mathcal{G}_n, \mathcal{G}_n) = \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{x \in \mathcal{G}_n \cap (\mathcal{G}_n - ze_j)} \big(\mathfrak{f}(x) - \mathfrak{f}(x + ze_j)\big)^2$$

and so is bounded by $K_n \mathfrak{D}(\mathfrak{f})$, where K_n is a positive constant.

For the second term, we have

$$\begin{aligned}
\mathcal{S}(\mathcal{G}_n, \mathcal{B}_n) &= \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{x \in \mathcal{G}_n \cap (\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x) - \tilde{\mathfrak{f}}(x + ze_j)\big)^2 \\
&= \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{x \in \mathcal{G}_n \cap (\mathcal{B}_n - ze_j)} \big(\mathfrak{f}(x) - \tilde{\mathfrak{f}}(x + ze_j)\big)^2 \\
&= \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{x \in \mathcal{G}_n \cap (\mathcal{B}_n - ze_j)} \big[\mathbb{E}_{x+ze_j}\big(\mathfrak{f}(x_T) - \mathfrak{f}(x)\big)\big]^2 \\
&\le \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{x \in \mathcal{G}_n \cap (\mathcal{B}_n - ze_j)} \mathbb{E}_{x+ze_j}\big[\big(\mathfrak{f}(x_T) - \mathfrak{f}(x)\big)^2\big] \\
&= \sum_{k \ge 1} \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{\substack{x \in \mathcal{G}_n \cap (\mathcal{B}_n - ze_j) \\ y \in \mathcal{G}_n}} \mathbb{P}_{x+ze_j}[T = T_k, x_T = y]\big(\mathfrak{f}(y) - \mathfrak{f}(x)\big)^2 \\
&\le \sum_{k \ge 1} C_n \rho_n^k \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{\substack{x \in \mathcal{G}_n \cap (\mathcal{B}_n - ze_j) \\ y \in \mathcal{G}_n}} \big(\mathfrak{f}(y) - \mathfrak{f}(x)\big)^2 \mathbb{1}_{d(y, x+ze_j) \le k} \\
&\le \sum_{k \ge 1} C_n \rho_n^k \sum_{\substack{1 \le j \le n \\ z \in \mathbb{Z}^d}} s(z) \sum_{\substack{x \in \mathcal{G}_n \cap (\mathcal{B}_n - ze_j) \\ y \in \mathcal{G}_n}} \big(\mathfrak{f}(y) - \mathfrak{f}(x)\big)^2 \mathbb{1}_{d(y,x) \le k+1}
\end{aligned}$$

since we have seen that $\mathbb{P}_{x+ze_j}(T \ge T_k) \le C_n \rho_n^k$, where $\rho_n \in (0, 1)$ is independent of $x + ze_j$ and $x_{T_k} = y$ is possible only if $d(y, x + ze_j) \le k$.

Note now that if x and y are two good configurations, then one can find a path $\mathcal{C}_p(x, y) = [x_0 = x, \dots, x_p = y]$, where $p \le nd(x, y)$ and such that the path is contained in $\mathcal{G}_n$. Hence we get the following estimate:

$$\mathcal{S}(\mathcal{G}_n, \mathcal{B}_n) \le \sum_{k \ge 1} K_n \rho_n^k \sum_{\mathcal{C}_p = [x_0, \dots, x_p]} \big(\mathfrak{f}(x_p) - \mathfrak{f}(x_0)\big)^2 \mathbb{1}_{d(x_0, x_p) \le n(k+1)}, \tag{3.9}$$

where the summation is carried over all the paths $\mathcal{C}_p = [x_0, \dots, x_p]$ contained in $\mathcal{G}_n$,

$$\begin{aligned}
\mathcal{S}(\mathcal{G}_n, \mathcal{B}_n) &\le \sum_{k\ge 1} K_n \rho_n^k \sum_{\mathcal{C}_p=[x_0,\dots,x_p]} \big(\mathfrak{f}(x_p) - \mathfrak{f}(x_{p-1}) + \cdots \\
&\qquad\qquad + \mathfrak{f}(x_1) - \mathfrak{f}(x_0)\big)^2 \mathbb{1}_{p\le n(k+1)} \\
&\le \sum_{k\ge 1} K_n \rho_n^k \sum_{\mathcal{C}_p=[x_0,\dots,x_p]} \big(\mathfrak{f}(x_p) - \mathfrak{f}(x_{p-1}) + \cdots \\
&\qquad\qquad + \mathfrak{f}(x_1) - \mathfrak{f}(x_0)\big)^2 \mathbb{1}_{p\le n(k+1)} \\
&\le \sum_{k\ge 1} K_n \rho_n^k \sum_{\mathcal{C}_p=[x_0,\dots,x_p]} p \sum_{j=1}^{p} [\mathfrak{f}(x_j) - \mathfrak{f}(x_{j-1})]^2.
\end{aligned}$$

However, since s is finite-range, there exists a constant A_n such that, for all $j \in \{1, \dots, p-1\}$, there are at most $A_n p$ paths of length p containing the step $[x_{j-1}, x_j]$. Using the Schwarz inequality, it is now easy to check that, for some positive constant K_n, we have

$$\mathcal{S}(\mathcal{G}_n, \mathcal{B}_n) \le K_n \mathfrak{D}(\mathfrak{f}). \tag{3.10}$$

There remains the third term. Recall that, in the beginning of the proof, we proved that, for each configuration x, we are able to find a good configuration w_x such that $d(x, w_x) \le Kn^2$. When $s(z) > 0$ then $d(x, x + ze_j) = 1$ so that $d(x + ze_j, w_x) \le Kn^2 + 1 \le 2Kn^2$. We have

$$\begin{aligned}
\mathcal{S}(\mathcal{B}_n, \mathcal{B}_n) &= \sum_{\substack{1\le j\le n\\ z\in\mathbb{Z}^d}} s(z) \sum_{x\in\mathcal{B}_n\cap(\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x) - \tilde{\mathfrak{f}}(x+ze_j)\big)^2 \\
&= \sum_{\substack{1\le j\le n\\ z\in\mathbb{Z}^d}} s(z) \sum_{x\in\mathcal{B}_n\cap(\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x) - \tilde{\mathfrak{f}}(w_x) + \tilde{\mathfrak{f}}(w_x) - \tilde{\mathfrak{f}}(x+ze_j)\big)^2 \\
&\le 2 \sum_{\substack{1\le j\le n\\ z\in\mathbb{Z}^d}} s(z) \sum_{x\in\mathcal{B}_n\cap(\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x) - \tilde{\mathfrak{f}}(w_x)\big)^2 \\
&\quad + 2 \sum_{\substack{1\le j\le n\\ z\in\mathbb{Z}^d}} s(z) \sum_{x\in\mathcal{B}_n\cap(\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x+ze_j) - \tilde{\mathfrak{f}}(w_x)\big)^2 \\
&= 2 \sum_{\substack{1\le j\le n\\ z\in\mathbb{Z}^d}} s(z) \sum_{x\in\mathcal{B}_n\cap(\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x) - \mathfrak{f}(w_x)\big)^2 \\
&\quad + 2 \sum_{\substack{1\le j\le n\\ z\in\mathbb{Z}^d}} s(z) \sum_{x\in\mathcal{B}_n\cap(\mathcal{B}_n - ze_j)} \big(\tilde{\mathfrak{f}}(x+ze_j) - \mathfrak{f}(w_x)\big)^2.
\end{aligned}$$

Doing the same work as for the second term, it is easy to see that

$$\mathcal{S}(\mathcal{B}_n, \mathcal{B}_n) \le K_n \mathfrak{D}(\mathfrak{f}) \tag{3.11}$$

for a positive constant K_n.

We now have to prove the second set of inequalities concerning the $\mathbb{L}^2$ norm. The right-hand side inequality in (3.6) is evident. For the left-hand side inequality, we have

$$\|\tilde{\mathfrak{f}}\|_{0,\mathrm{free}}^2 = \frac{1}{n!} \sum_{x \in \chi_n} \big(\mathbb{E}_x[\mathfrak{f}(x_T)]\big)^2 = \frac{1}{n!} \sum_{x \in \mathcal{G}_n} (\mathfrak{f}(x))^2 + \frac{1}{n!} \sum_{x \in \mathcal{B}_n} \big(\mathbb{E}_x[\mathfrak{f}(x_T)]\big)^2.$$

The first sum is equal to $\|\mathfrak{f}\|_0^2$ and one can rewrite the second term as

$$\begin{aligned}
\frac{1}{n!} \sum_{x \in \mathcal{B}_n} \big(\mathbb{E}_x[\mathfrak{f}(x_T)]\big)^2 &= \frac{1}{n!} \sum_{k \ge 1} \sum_{\substack{x \in \mathcal{B}_n \\ y \in \mathcal{G}_n}} \mathfrak{f}(y)^2 \mathbb{P}_x\big[x_{T_k} = y; T = T_k\big] \\
&= \frac{1}{n!} \sum_{y \in \mathcal{G}_n} \mathfrak{f}(y)^2 \sum_{k \ge 1} \sum_{x \in \mathcal{B}_n} \mathbb{P}_x\big[x_{T_k} = y; T = T_k\big] \\
&\le \frac{1}{n!} \sum_{y \in \mathcal{G}_n} \mathfrak{f}(y)^2 \sum_{k \ge 1} \sum_{\substack{x\, s.t. \\ d(x,y) \le k}} \mathbb{P}_x[T = T_k] \\
&\le K_n \frac{1}{n!} \sum_{y \in \mathcal{G}_n} \mathfrak{f}(y)^2 \sum_{k \ge 1} k^d \rho_n^k \\
&\le K_n' \frac{1}{n!} \sum_{y \in \mathcal{G}_n} \mathfrak{f}(y)^2 \\
&= K_n' \|\mathfrak{f}\|_0^2. \qquad \square
\end{aligned}$$

Let us define the resolvent norms associated with the generator $\mathfrak{L}^{\mathrm{free}}$. The $\mathfrak{H}_1^{\mathrm{free}}$ and $\mathfrak{H}_{-1}^{\mathrm{free}}$ norms associated with the Dirichlet form $\mathfrak{D}_{\mathrm{free}}$ are defined by

$$\begin{aligned}
\|\phi\|_{1,\mathrm{free}}^2 &= \mathfrak{D}_{\mathrm{free}}(\phi), \\
\|\phi\|_{-1,\mathrm{free}}^2 &= \sup_{\psi} \{2\langle\langle \phi, \psi \rangle\rangle - \|\psi\|_{1,\mathrm{free}}^2\}
\end{aligned} \tag{3.12}$$

and the resolvent norm, $\|\cdot\|_{1,\mathrm{free},\lambda}$ and $\|\cdot\|_{-1,\mathrm{free},\lambda}$ are defined by

$$\begin{aligned}
\|\phi\|_{1,\mathrm{free},\lambda}^2 &= -\langle\langle \phi, (\mathfrak{L}^{\mathrm{free}} - \lambda)\phi \rangle\rangle = \mathfrak{D}_{\mathrm{free}}(\phi) + \lambda \langle\langle \phi, \phi \rangle\rangle, \\
\|\phi\|_{-1,\mathrm{free},\lambda}^2 &= \sup_{\psi} \{2\langle\langle \phi, \psi \rangle\rangle - \|\psi\|_{1,\mathrm{free},\lambda}^2\}.
\end{aligned} \tag{3.13}$$

Note here that we do not have necessary equivalence of the $\mathfrak{H}_{-1}^{\text{free}}$ and $\mathfrak{H}_{-1}$ norms because we do not have the equality $\langle\langle \tilde{\mathfrak{f}}, \tilde{\mathfrak{g}} \rangle\rangle = \langle \mathfrak{f}, \mathfrak{g} \rangle$. Nevertheless, let $\mathfrak{W}_n$ be the operator defined on $\mathbb{L}^2(\chi_n)$ by

$$\mathfrak{W}_n \mathfrak{f}(x) = \mathbb{1}_{x \in \mathcal{G}_n} \mathfrak{f}(x). \tag{3.14}$$

THEOREM 3.2. *For each n, there exists a constant $C = C(d, n)$ such that*

$$\|\mathfrak{f}\|_{-1,\lambda} \le C \|\mathfrak{W}_n \tilde{\mathfrak{f}}\|_{-1,\text{free},\lambda} \tag{3.15}$$

for all finite supported function over $\mathcal{E}_n$ and where $\tilde{\mathfrak{f}}$ is defined by (3.8).

PROOF. Indeed, we have

$$\begin{aligned}
\|\mathfrak{f}\|_{-1,\lambda}^2 &= \sup_{\mathfrak{g}} \{2\langle \mathfrak{f}, \mathfrak{g} \rangle - \|\mathfrak{g}\|_{1,\lambda}^2\} \\
&= \sup_{\mathfrak{g}} \{2\langle\langle \mathfrak{W}_n \tilde{\mathfrak{f}}, \tilde{\mathfrak{g}} \rangle\rangle - \|\mathfrak{g}\|_{1,\lambda}^2\} \\
&\le C \sup_{\mathfrak{g}} \{2\langle\langle \mathfrak{W}_n \tilde{\mathfrak{f}}, \tilde{\mathfrak{g}} \rangle\rangle - \|\tilde{\mathfrak{g}}\|_{1,\text{free},\lambda}^2\} \\
&\le C \|\mathfrak{W}_n \tilde{\mathfrak{f}}\|_{-1,\text{free},\lambda}^2.
\end{aligned}$$

The penultimate inequality follows from Theorem 3.1. $\square$

Hence it is possible to compare the $\mathfrak{H}_{-1,\lambda}$ norm of $\mathfrak{B}_{12}\mathfrak{f}$ with respect to the $\mathfrak{H}_{-1,\lambda}^{\text{free}}$ norm of $\mathfrak{T}_{12}\mathfrak{f}$, where $\mathfrak{T}_{12}$ is just the operator $\mathfrak{W}_2 \widetilde{\mathfrak{B}}_{12}$ and $\widetilde{\mathfrak{B}}_{12}$ is defined by

$$(\widetilde{\mathfrak{B}_{12}\mathfrak{f}})(x, y) = (\widetilde{\mathfrak{B}}_{12} \tilde{\mathfrak{f}})(x, y). \tag{3.16}$$

A simple computation shows that

$$(\mathfrak{T}_{12}\mathfrak{f})(x, y) = a(y - x)\big(\mathfrak{f}(x) - \mathfrak{f}(y)\big). \tag{3.17}$$

Hence, we have

$$\|\mathfrak{B}_{12}\mathfrak{f}\|_{-1,\lambda}^2 \le C \|\mathfrak{T}_{12}\tilde{\mathfrak{f}}\|_{-1,\text{free},\lambda}^2 = C \|\mathfrak{T}_{12}\mathfrak{f}\|_{-1,\text{free},\lambda}^2 \tag{3.18}$$

by Theorem 3.2 and since $\tilde{\mathfrak{f}} = \mathfrak{f}$ for the degree 1 functions $\mathfrak{f}$.

Now we give the expressions in terms of Fourier transform of the $\|\cdot\|_{1,\text{free},\lambda}$ and $\|\cdot\|_{-1,\text{free},\lambda}$ norms. If ψ is a function of $\mathbb{L}^2(\chi_n)$ and if $\hat{\psi}(s_1 \cdots s_n)$, $s_i \in [0, 1]^d$, is its Fourier transform, defined by

$$\hat{\psi}(s_1 \cdots s_n) = \frac{1}{\sqrt{n!}} \sum_{(x_1 \cdots x_n) \in \chi_n} e^{2i\pi(x_1 s_1 + \cdots + x_n s_n)} \psi(x_1 \cdots x_n)$$

and

$$\widehat{\mathfrak{L}^{\text{free}} \psi} = \widehat{\mathfrak{L}^{\text{free}}} \hat{\psi},$$

where

$$\widehat{\mathfrak{L}^{\text{free}}}\hat{\psi}(s_1\cdots s_n) = -\left[\sum_{j=1}^{n}\theta_d(s_j)\right]\hat{\psi}(s_1\cdots s_n)$$

with, for $u \in [0,1]^d$,

$$\theta_d(u) = 2\sum_{z\in\mathbb{Z}^d} s(z)\sin^2\big(\pi(u\cdot z)\big). \tag{3.19}$$

Using the Parseval equality, we get

$$\|\psi\|^2_{1,\text{free},\lambda} = \frac{1}{(2\pi)^{nd}}\int_{(s_1\cdots s_n)\in([0,1]^d)^n}\left(\lambda + \sum_{i=1}^{n}\theta_d(s_i)\right)|\hat{\psi}(s_1\cdots s_n)|^2\,ds_1\cdots ds_n$$

and

$$\|\psi\|^2_{-1,\text{free},\lambda} = \frac{1}{(2\pi)^{nd}}\int_{(s_1\cdots s_n)\in([0,1]^d)^n}\frac{|\hat{\psi}(s_1\cdots s_n)|^2}{\lambda+\sum_{i=1}^{n}\theta_d(s_i)}\,ds_1\cdots ds_n.$$

4. Superdiffusivity in dimension 1 for the density $\rho = 1/2$. We prove that, for dimension 1 and density $\rho = 1/2$, the occupation time of the site 0 in the general asymmetric simple exclusion process with nonzero mean has a superdiffusive behavior. Throughout this section, the density ρ is fixed and equal to $1/2$. Let $f_0(\eta) = (\eta(0)-\rho)$ and recall that the time t variance is defined by

$$\sigma_t^2(f_0) = \mathbb{E}_\rho\left[\int_0^t f_0(\eta(s))\,ds\right]^2.$$

THEOREM 4.1. *For all $\varepsilon > 0$, there exists a positive constant C_ε such that, for large t, we have*

$$\sigma_t^2(f_0) = \mathbb{E}_\rho\left[\int_0^t f_0(\eta(s))\,ds\right]^2 \geq C_\varepsilon t^{5/4-\varepsilon}.$$

To prove this theorem, we first obtain a lower bound for the Laplace transform of the function $\sigma_\cdot^2(f_0)$, which is contained in the following lemma.

LEMMA 4.2. *There exists a positive constant c such that, for all $\lambda > 0$, we have*

$$\langle f_0, (\lambda - \mathcal{L})^{-1} f_0\rangle \geq c\lambda^{-1/4}. \tag{4.1}$$

PROOF. By Theorem 2.1 in [10], we just have to prove this lemma for the total asymmetric exclusion process. In fact, the following estimates could be done for the general asymmetric exclusion process but the use of this theorem permits us to simplify the calculus. Hence, in the sequel, we assume total asymmetry to the right of the exclusion process.

Remark that $\mathfrak{f}_0 = \sqrt{\chi(\rho)}\delta_0$, where δ_0 is the characteristic function of the set $\{0\}$, is a degree-1 function.

We have

$$\langle \delta_0, (\lambda - \mathfrak{L})^{-1}\delta_0\rangle = \sup_{\mathfrak{g}}\{2\langle \delta_0, \mathfrak{g}\rangle - \|\mathfrak{g}\|_{1,\lambda}^2 - \|\mathfrak{A}\mathfrak{g}\|_{-1,\lambda}^2\}, \tag{4.2}$$

where the supremum is taken over all finite supported functions $\mathfrak{g}$. To obtain a lower bound, we can restrict this supremum over degree-1 functions. In this case, since the density is $\rho = 1/2$, there is a single term in the asymmetric part of the generator. If $\mathfrak{g}$ is of degree 1, we have $\mathfrak{A}\mathfrak{g} = \mathfrak{A}_{12}\mathfrak{g}$ and we recall that

$$(\mathfrak{A}_{12}\mathfrak{g})(\{x, y\}) = 0 \qquad \text{if } |y - x| > 1$$

and

$$(\mathfrak{A}_{12}\mathfrak{g})(\{x, x+1\}) = \tfrac{1}{2}\big(\mathfrak{g}(x) - \mathfrak{g}(x+1)\big).$$

Thanks to (3.18), we can replace the term $\|\mathfrak{A}_{12}\mathfrak{g}\|_{-1,\lambda}^2$ by $\beta\|\mathfrak{T}_{12}\mathfrak{g}\|_{-1,\text{free},\lambda}^2$ in (4.2) (β being a positive constant independent of λ), where $\mathfrak{T}_{12}$ is the operator defined by

$$(\mathfrak{T}_{12}\phi)(x, y) = 0 \qquad \text{if } |y - x| \neq 1,$$
$$(\mathfrak{T}_{12}\phi)(x, x+1) = \tfrac{1}{2}\big(\phi(x) - \phi(x+1)\big).$$

More exactly, with this definition, there exists some constant β such that

$$\langle \delta_0, (\lambda - \mathfrak{L})^{-1}\delta_0\rangle \geq \sup_{\phi\in\mathfrak{G}_1}\{2\langle\langle \delta_0, \phi\rangle\rangle - \|\phi\|_{1,\lambda,\text{free}}^2 - \beta\|\mathfrak{T}_{12}\phi\|_{-1,\text{free},\lambda}^2\}. \tag{4.3}$$

Using the Fourier transform, we have to estimate

$$\begin{aligned}\sup_{\phi}\bigg\{&\frac{1}{\pi}\int_0^1 \hat{\phi}(s)\,ds - \int_0^1 \big(\lambda + \theta_1(s)\big)|\hat{\phi}(s)|^2\,ds\\ &- \frac{\beta}{2}\int_{(s,t)\in[0,1]^2}\frac{(\sin(2\pi s) + \sin(2\pi t))^2}{\lambda + \theta_1(s) + \theta_1(t)}|\hat{\phi}(s+t)|^2\,ds\,dt\bigg\}.\end{aligned} \tag{4.4}$$

It is simple to check that

$$\begin{aligned}\int_{(s,t)\in[0,1]^2}&\frac{(\sin(2\pi s) + \sin(2\pi t))^2}{\lambda + \theta_1(s) + \theta_1(t)}|\hat{\phi}(s+t)|^2\,ds\,dt\\ &= 4\int_0^1 ds|\hat{\phi}(s)|^2\sin^2(\pi s)F_\lambda(s)\,ds,\end{aligned} \tag{4.5}$$

where

$$F_\lambda(s) = \int_0^1 \frac{\cos^2(\pi u)}{\lambda + 2\sin^2(\pi(u+s)/2) + 2\sin^2(\pi(u-s)/2)}\,du. \tag{4.6}$$

The supremum is taken over all the real local functions ϕ but it is easy to see that the supremum is the same if you consider the supremum over real functions in $\mathbb{L}^2(\mathbb{Z}) = \mathfrak{S}_1$. Note now that the expression in the variational formula (4.4), using (4.5), is just the integral of a quadratic expression in terms of $\hat{\phi}(s)$. Consequently, the last supremum (4.4) is equal to

$$\frac{1}{4\pi^2}\int_0^1 \frac{du}{\lambda + 2\sin^2(\pi u)(1+\beta F_\lambda(u))} \tag{4.7}$$

and the maximizer is given by

$$\hat{\phi}_\lambda(u) = \frac{1}{2\pi(\lambda + 2\sin^2(\pi u)(1+\beta F_\lambda(u)))}. \tag{4.8}$$

The maximizer is the Fourier transform of a real function since we have $\hat{\phi}_\lambda^*(1-s) = \hat{\phi}_\lambda(s)$ (here $\hat{\phi}_\lambda^*$ means the complex conjugate of the function $\hat{\phi}_\lambda$).

For $s \le 1/2$, using the fact that for $0 \le x \le 1$ we have

$$\sin\left(\frac{\pi x}{2}\right) \ge x \quad \text{and} \quad \cos\left(\frac{\pi s}{2}\right) \ge \frac{1}{\sqrt{2}},$$

we get the following inequalities:

$$\begin{aligned}
F_\lambda(s) &= \int_0^1 \frac{\cos^2(\pi v)}{\lambda + 4\sin^2(\pi s/2)\cos^2(\pi v/2) + 4\cos^2(\pi s/2)\sin^2(\pi v/2)}\,dv && (4.9)\\
&\le \int_0^1 \frac{dv}{\lambda + 4s^2\cos^2(\pi v/2) + 2v^2} && (4.10)\\
&\le \int_0^{1/2} \frac{dv}{\lambda + 2s^2 + 2v^2} + \int_{1/2}^1 \frac{dv}{2v^2} && (4.11)\\
&\le C\frac{1}{\sqrt{\lambda + s^2}}, && (4.12)
\end{aligned}$$

where C is a constant independent of s, λ.

Consequently, for some positive constant K, we have

$$\begin{aligned}
&\int_0^{1/2} \frac{du}{\lambda + 2\sin^2(\pi u)(1+\beta F_\lambda(u))}\\
&\qquad \ge \int_0^{1/2} \frac{du}{\lambda + 2\pi^2 u^2(1 + K/\sqrt{\lambda + u^2})}\\
&\qquad = \lambda^{-1/4} \int_0^{\lambda^{-3/4}/2} \frac{dz}{1 + 2\pi^2 z^2\sqrt{\lambda} + 2\pi^2 K z^2(1+\sqrt{\lambda}z^2)^{-1/2}}
\end{aligned}$$

with the change of variables $z = \lambda^{-3/4}u$.

Hence we obtain, for sufficiently small λ,

$$\int_0^1 \frac{du}{\lambda + 2\sin^2(\pi u)(1+\beta F_\lambda(u))} \geq \lambda^{-1/4} \int_0^1 \frac{dz}{1+2\pi^2(1+K)z^2}$$
$$\geq C\lambda^{-1/4},$$

where C is a positive constant independent of λ. This proves Lemma 4.2. $\square$

We now give a lemma which is just a refinement of Lemma 4.5 in [9]. Recall that we have defined $\sigma_t(f)$ by $\sigma_t^2(f) = \mathbb{E}_\rho[\int_0^t f(\eta(s))\,ds]^2$. We recall some notation of [9]. For each $t > 0$, T_t is the semigroup of the asymmetric simple exclusion process whose generator is given by $\mathcal{L}$. Let $U_f(t) = \int_0^t \langle T_s f, f\rangle\,ds$ and $V_f(t) = \int_0^t U_f(s)\,ds = \frac{1}{2}\sigma_t^2(f)$. Since f is a bounded function, there exists a constant C such that $V_f(t) \leq Ct^2$. Moreover, $\langle T_s f, f\rangle \geq 0$ by Lemma 2.2 of [9] so that U_f is nonnegative.

LEMMA 4.3. *Let $U(t)$ be a nonnegative function and let $V(t) = \int_0^t U(s)\,ds$. Suppose that there exists some constant $\theta > 0$ such that $\limsup_{t\to+\infty} t^{-\theta} V(t) < \infty$. Denote by $L(\lambda) = \int_0^\infty e^{-\lambda s} U(s)\,ds$ the Laplace transform of U and suppose that $\bar{\alpha}(U) = \sup\{\alpha \in [0,1]/ \liminf_{\lambda\to 0} \lambda^{\alpha+1} L(\lambda) > 0\}$ exists. Then for all $\varepsilon > 0$ we have*

$$\liminf_{t\to+\infty} t^{-(\bar{\alpha}+1-\varepsilon)} V(t) > 0.$$

PROOF. Let C_1 and C_ε be positive constants such that for large t and small λ we have

$$V(t) \leq C_1 t^\theta,$$
$$L(\lambda) \geq C_\varepsilon \lambda^{-(\bar{\alpha}+1-\varepsilon)}.$$

Note $\bar{\alpha}(U) = \bar{\alpha}$ and $\gamma = \bar{\alpha} - 2\varepsilon$ and suppose that $\liminf_{t\to+\infty} t^{-(1+\gamma)} V(t) = 0$. Then there exists some subsequence $\{t_n\}$, $t_n \uparrow \infty$, such that $V(t_n) \leq t_n^{\gamma+1}$. Since $L(\lambda) = \lambda \int_0^\infty e^{-\lambda s} V(s)\,ds$, we have the following inequalities:

$$C_\varepsilon \lambda^{-(\bar{\alpha}+2-\varepsilon)} \leq \int_0^{t_n} e^{-\lambda s} V(s)\,ds + \int_{t_n}^\infty e^{-\lambda s} V(s)\,ds \tag{4.13}$$

$$\leq t_n^{\gamma+2} + C_1 \int_{t_n}^\infty e^{-\lambda s} s^\theta\,ds \tag{4.14}$$

$$= t_n^{\gamma+2} + C_1 \lambda^{-\theta-1} \int_{\lambda t_n}^\infty e^s s^\theta\,ds. \tag{4.15}$$

We now take $\lambda = \lambda_n$, where $K_n = \lambda_n t_n = t_n^{1/\mu}$ with $\mu > 1$ so that $\lim_{n\to\infty} \lambda_n = 0$ and $\lim_{n\to\infty} K_n = \infty$. Note that $\int_x^\infty e^{-s} s^\theta\,ds \sim_{x\to\infty} e^{-x} x^\theta$. Multiplying by

$\lambda^{\bar{\alpha}+2-\varepsilon}$ and using this remark, we have

$$C_\varepsilon \leq t_n^{\gamma+\varepsilon-\bar{\alpha}} K_n^{\bar{\alpha}+2-\varepsilon} + C'_\varepsilon K_n^{1+\bar{\alpha}-\varepsilon} t_n^{\theta-\bar{\alpha}+\varepsilon-1} e^{-K_n} \tag{4.16}$$

$$\leq t_n^{\gamma+\varepsilon-\bar{\alpha}+(1/\mu)(\bar{\alpha}+2-\varepsilon)} + C'_\varepsilon t_n^{(1/\mu)(1+\bar{\alpha}-\varepsilon)+\theta-\bar{\alpha}+\varepsilon-1} e^{-t_n^{1/\mu}}, \tag{4.17}$$

where C'_ε is a positive constant. If μ is sufficiently large, then $\gamma + \varepsilon - \bar{\alpha} + (1/\mu)(\bar{\alpha} + 2 - \varepsilon)$ will be of the same sign as $\gamma + \varepsilon - \bar{\alpha} = -\varepsilon$, which is negative. For a such μ, the last estimate tends to 0 as n goes to infinity so that $C_\varepsilon = 0$, which yields a contradiction. $\square$

In our case, since we have proved $\bar{\alpha}(U_{f_0}) \geq 1/4$, we obtain that, for all $\varepsilon > 0$,

$$\liminf_{t\to\infty} t^{-(5/4-\varepsilon)} V_{f_0}(t) > 0$$

and Theorem 4.1 is proved.

5. Diffusivity in dimension $d = 2$ with density $\rho \neq 1/2$. In this section, we prove that, for dimension $d = 2$ and density $\rho \neq 1/2$, the limiting variance $\sigma^2(f_0) = \lim_{t\to\infty} t^{-1}\sigma_t^2(f_0)$ is finite. We recall that in [9] it is proved that $\sigma^2(f_0)$ is well defined and that $\sigma^2(f_0) = \lim_{\lambda\to 0}\langle f_0, (\lambda - \mathcal{L})^{-1} f_0\rangle$. Throughout this section, the density ρ is fixed and different from $1/2$.

THEOREM 5.1. *If the density is $\rho \neq 1/2$ and the dimension is $d = 2$, for the general asymmetric simple exclusion process with nonzero mean, the limiting variance $\sigma^2(f_0) = \lim_{t\to\infty} t^{-1}\sigma_t^2(f_0)$ is finite.*

Since $\sigma^2(f_0) = \lim_{\lambda\to 0}\langle f_0, (\lambda - \mathcal{L})^{-1} f_0\rangle$, we have to prove

$$\sup_{\lambda>0}\langle f_0, (\lambda - \mathcal{L})^{-1} f_0\rangle < \infty. \tag{5.1}$$

For simplicity of the calculus we will assume that the jumps of the exclusion process are just nearest neighbors. In fact, one can restrict the study to this case by Theorem 2.1 of [10]. We adopt the following notation: $a(e_1) = -a(-e_1) = a_1$, $a(e_2) = -a(-e_2) = a_2$, $s(e_1) = s(-e_1) = s_1$ and $s(e_2) = s(-e_2) = s_2$. Thanks to the irreducibility of $s(\cdot)$, $s_1 > 0$ and $s_2 > 0$ and since the mean of p is nonzero, $a_1^2 + a_2^2 \neq 0$. Recall that $\mathfrak{f}_0 = \sqrt{\chi(\rho)}\delta_0$ is a degree-1 function.

We have the following variational formula:

$$\langle \delta_0, (\lambda - \mathfrak{L})^{-1}\delta_0\rangle = \inf_{\mathfrak{g}}\left[2\|\delta_0 - \mathfrak{A}\mathfrak{g}\|_{-1,\lambda}^2 + \|\mathfrak{g}\|_{1,\lambda}^2\right],$$

where the infimum is taken over all finite supported functions.

If $\mathfrak{g}$ is of degree 1, then $\mathfrak{A}\mathfrak{g}$ has the form

$$\mathfrak{A}\mathfrak{g} = \mathfrak{A}_{11}\mathfrak{g} + \mathfrak{A}_{12}\mathfrak{g},$$

where

$$\begin{aligned}(\mathfrak{A}_{11}\mathfrak{g})(x) &= (1-2\rho)(\mathfrak{B}_{11}\mathfrak{g})(x)\\ &= (1-2\rho)\big[a_1\big(\mathfrak{g}(x+e_1)-\mathfrak{g}(x-e_1)\big)+a_2\big(\mathfrak{g}(x+e_2)-\mathfrak{g}(x-e_2)\big)\big]\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{A}_{12}\mathfrak{g})(\{x,x+e_1\}) &= 2\sqrt{\chi(\rho)}(\mathfrak{B}_{12}\mathfrak{g})(\{x,x+e_1\})\\ &= 2\sqrt{\chi(\rho)}\big[a_1\big(\mathfrak{g}(x)-\mathfrak{g}(x+e_1)\big)\big],\\ (\mathfrak{A}_{12}\mathfrak{g})(\{x,x+e_2\}) &= 2\sqrt{\chi(\rho)}(\mathfrak{B}_{12}\mathfrak{g})(\{x,x+e_2\})\\ &= 2\sqrt{\chi(\rho)}\big[a_2\big(\mathfrak{g}(x)-\mathfrak{g}(x+e_2)\big)\big],\\ (\mathfrak{A}_{12}\mathfrak{g})(\{x,y\}) &= 0 \qquad \text{if } |y-x|\neq 1.\end{aligned}$$

Using Theorem 3.2 we have the following inequalities:

$$\begin{aligned}&\inf_{\phi}\big[2\|\delta_0-\mathfrak{A}\phi\|_{-1,\lambda}^2+\|\phi\|_{1,\lambda}^2\big]\\ &\quad\le \inf_{\phi\in\mathfrak{G}_1}\big[2\|\delta_0-\mathfrak{A}\phi\|_{-1,\lambda}^2+\|\phi\|_{1,\lambda}^2\big]\\ &\quad= \inf_{\phi\in\mathfrak{G}_1}\big[2\|\delta_0-(1-2\rho)\mathfrak{B}_{11}\phi\|_{-1,\lambda}^2+\|\phi\|_{1,\lambda}^2+8\chi(\rho)\|\mathfrak{B}_{12}\phi\|_{-1,\lambda}^2\big]\\ &\quad\le C\inf_{\phi\in\mathfrak{G}_1}\big[\|\delta_0-(1-2\rho)\mathfrak{B}_{11}\phi\|_{-1,\text{free},\lambda}^2\\ &\qquad\qquad + \|\phi\|_{1,\text{free},\lambda}^2+8\chi(\rho)\|\mathfrak{T}_{12}\phi\|_{-1,\text{free},\lambda}^2\big],\end{aligned}$$

where C is a positive constant.

We have

$$\inf_{\phi\in\mathfrak{G}_1}\|\delta_0-\mathfrak{A}_{11}\phi\|_{-1,\text{free},\lambda}^2+\|\phi\|_{1,\text{free},\lambda}^2 \tag{5.2}$$

$$\begin{aligned}&= \frac{1}{4\pi^2}\inf_{\phi\in\mathfrak{G}_1}\left\{\int_{u\in[0,1]^2}\left\{\frac{|1+i\gamma(u)\hat{\phi}(u)|^2}{\lambda+\theta_2(u)}\right.\right.\\ &\qquad\left.\left.+\big(\lambda+\theta_2(u)\big)|\hat{\phi}(u)|^2\right\}du_1\,du_2\right\},\end{aligned}\tag{5.3}$$

where

$$\begin{aligned}\gamma(u) &= 2(1-2\rho)[a_1\sin(2\pi u_1)+a_2\sin(2\pi u_2)]\\ &= b_1\sin(2\pi u_1)+b_2\sin(2\pi u_2)\end{aligned}$$

and we recall that

$$\theta_2(u) = 4s_1\sin^2(\pi u_1)+4s_2\sin^2(\pi u_2).$$

The function ϕ_λ realizing the infimum of $\|\delta_0 - \mathfrak{A}_{11}\phi\|^2_{-1,\text{free},\lambda} + \|\phi\|^2_{1,\text{free},\lambda}$ is given by its Fourier transform:

$$\hat{\phi}_\lambda(u) = \frac{i\gamma(u)}{\gamma(u)^2 + (\lambda + \theta_2(u))^2}. \tag{5.4}$$

The minimizer is the Fourier transform of a real function since we have $\hat{\phi}^*_\lambda(1-s_1, 1-s_2) = \hat{\phi}_\lambda(s_1, s_2)$.

Moreover, this infimum is then equal to

$$\frac{1}{4\pi^2}\int_{u\in[0,1]^2} \frac{(\lambda + \theta_2(u))}{\gamma(u)^2 + (\lambda + \theta_2(u))^2}\, du, \tag{5.5}$$

and we have

$$\sup_{\lambda>0}\int_{u\in[0,1]^2} \frac{(\lambda + \theta_2(u))}{\gamma(u)^2 + (\lambda + \theta_2(u))^2}\, du < \infty. \tag{5.6}$$

Indeed, problems happen when $\theta_2(u) = 0$ [e.g., $u = (0,0), (0,1), (1,0), (1,1)$]. Let us examine the problem for $(0,0)$, the others being of the same nature. Near $(0,0)$ we may make the change of variables $x_1 = \sin(2\pi u_1)$, $x_2 = \sin(2\pi u_2)$, and it is easy to check that there exists some constant C (independent of λ) such that

$$\begin{aligned}\int_{u\in[0,1/4]^2} &\frac{(\lambda + \theta_2(u))}{\gamma(u)^2 + (\lambda + \theta_2(u))^2}\, du \\ &\le C\int_{x\in V} \frac{(\lambda + x_1^2 + x_2^2)}{(b_1x_1 + b_2x_2)^2 + (\lambda + x_1^2 + x_2^2)^2}\, dx,\end{aligned}$$

where V is a neighborhood of $(0,0)$. Now, using polar coordinates, the problem is equivalent to finding a uniform bound in λ for

$$\int_0^1 r\, dr \int_0^{\pi/2} \frac{\lambda + r^2}{(b_1^2 + b_2^2)r^2\sin^2(\phi) + (\lambda + r^2)^2}\, d\phi.$$

It is easy to check that this integral is uniformly bounded.

Hence we take this function ϕ_λ and, to prove (5.1), it remains to show that

$$\sup_{\lambda>0}\|\mathfrak{T}_{12}\phi_\lambda\|_{-1,\text{free},\lambda} < \infty.$$

We can compute the Fourier transform of $\mathfrak{T}_{12}\phi_\lambda$ and we have

$$(\widehat{\mathfrak{T}_{12}\phi_\lambda})(u,v) = -\frac{i}{1-2\rho}[\gamma(u) + \gamma(v)]\hat{\phi}_\lambda(u+v).$$

Consequently, we get

$$\|\mathfrak{T}_{12}\phi_\lambda\|^2_{-1,\text{free},\lambda} \tag{5.7}$$

$$= \frac{1}{4\pi^2(1-2\rho)^2}\int_{(u,v)\in([0,1]^2)^2} \frac{(\gamma(u) + \gamma(v))^2}{\lambda + \theta_2(u) + \theta_2(v)}|\hat{\phi}_\lambda(u+v)|^2\, du\, dv. \tag{5.8}$$

We have to check that the following integral I is finite:

$$I = \int_{(u,v)\in[0,1]^4} \frac{\gamma^2(u+v)}{(\gamma^2(u+v)+\theta_2^2(u+v))^2} \frac{(\gamma(u)+\gamma(v))^2}{\theta_2(u)+\theta_2(v)}\, du\, dv. \tag{5.9}$$

We begin with a lemma which will be useful in proving the convergence of the integral I.

LEMMA 5.2. *The function*

$$\begin{aligned} &J_{\alpha,\beta}(\varepsilon_1,\varepsilon_2) \\ &\quad = \big(\sin^2(\pi\varepsilon_1)+\sin^2(\pi\varepsilon_2)\big) \\ &\qquad \times \left[\big(\alpha\sin(2\pi\varepsilon_1)+\beta\sin(2\pi\varepsilon_2)\big) + \frac{(\sin^2(\pi\varepsilon_1)+\sin^2(\pi\varepsilon_2))^2}{(\alpha\sin(2\pi\varepsilon_1)+\beta\sin(2\pi\varepsilon_2))}\right]^{-2} \end{aligned}$$

is integrable in a neighborhood $V \subset \mathbb{R}\times\mathbb{R}$ of $(0,0)$ *for every value of the parameters α and β such that $\alpha^2+\beta^2 \neq 0$.*

PROOF. First, note that we can assume that $V \subset \mathbb{R}^+\times\mathbb{R}^+$ by a change of signs for α,β. Since we are near 0, we may make the change of variables $x_1 = \sin(2\pi\varepsilon_1)$, $x_2 = \sin(2\pi\varepsilon_2)$ and we have

$$\begin{aligned} &\int_V J_{\alpha,\beta}(\varepsilon_1,\varepsilon_2)\, d\varepsilon_1\, d\varepsilon_2 \\ &\qquad \le C\int_V \frac{(\varepsilon_1^2+\varepsilon_2^2)(\alpha\sin(2\pi\varepsilon_1)+\beta\sin(2\pi\varepsilon_2))^2}{[(\alpha\sin(2\pi\varepsilon_1)+\beta\sin(2\pi\varepsilon_2))^2+(\varepsilon_1^2+\varepsilon_2^2)^2]^2}\, d\varepsilon_1\, d\varepsilon_2 \\ &\qquad \le K\int_U \frac{(x_1^2+x_2^2)(\alpha x_1+\beta x_2)^2}{[(\alpha x_1+\beta x_2)^2+(x_1^2+x_2^2)^2]^2} \frac{dx_1}{\sqrt{1-x_1^2}} \frac{dx_2}{\sqrt{1-x_2^2}} \\ &\qquad \le K'\int_U \frac{(x_1^2+x_2^2)(\alpha x_1+\beta x_2)^2}{[(\alpha x_1+\beta x_2)^2+\nu(x_1^2+x_2^2)^2]^2}\, dx_1\, dx_2, \end{aligned}$$

where C, K, K' are positive constants and $U \subset \mathbb{R}^+\times\mathbb{R}^+$ is a neighborhood of $(0,0)$.

Assume that $U \subset [0,1]\times[0,1]$ and use polar coordinates to rewrite this last integral as follows:

$$\begin{aligned} &\int_U \frac{(x_1^2+x_2^2)(\alpha x_1+\beta x_2)^2}{[(\alpha x_1+\beta x_2)^2+(x_1^2+x_2^2)^2]^2}\, dx_1\, dx_2 \\ &\qquad \le \int_0^1 dr \int_0^{\pi/2} r\, d\phi \frac{(\alpha^2+\beta^2)r^4\sin^2(\phi+\phi_0)}{[(\alpha^2+\beta^2)r^2\sin^2(\phi+\phi_0)+r^4]^2} \\ &\qquad = (\alpha^2+\beta^2)\int_0^1 r\, dr \int_{\phi_0}^{\phi_0+\pi/2} d\phi\, \frac{\sin^2(\phi)}{[\sin^2(\phi)+r^2]^2}, \end{aligned}$$

where ϕ_0 is such that $\alpha/\sqrt{\alpha^2+\beta^2} = \sin(\phi_0)$ and $\beta/\sqrt{\alpha^2+\beta^2} = \cos(\phi_0)$. It is easy to check the convergence of this last integral and so the lemma is proved. □

LEMMA 5.3.
$$I = \int_{(u,v)\in[0,1]^4} \frac{\gamma^2(u+v)}{(\gamma^2(u+v)+\theta_2^2(u+v))^2} \frac{(\gamma(u)+\gamma(v))^2}{\theta_2(u)+\theta_2(v)}\, du\, dv < \infty.$$

PROOF. Let us denote by $F(u,v)$ the function appearing in the integral,

$$F(u,v) = \frac{\gamma^2(u+v)}{(\gamma^2(u+v)+\theta_2^2(u+v))^2} \frac{(\gamma(u)+\gamma(v))^2}{\theta_2(u)+\theta_2(v)}. \tag{5.10}$$

Since we are just interested in the convergence of this integral, by a trivial inequality, we can suppose that $\theta_2(u) = \sin^2(\pi u_1) + \sin^2(\pi u_2)$.

Note that the singularities of F correspond to the set

$$\mathcal{S} = \{(u,v) \in [0,1]^4 / u+v \in A \times A\},$$

where $A = \{0,1,2\}$.

In fact, many singularities have the same behavior and, by a change of variables, it is enough to study what happens for F in the neighborhood of $u+v=(0,0)$, $u+v=(0,1)$, $u+v=(1,1)$, $u+v=(0,2)$. We just give the proof for the singularity $(0,1)$.

We have hence u_1+v_1 near 0 and u_2+v_2 near 1.

We suppose that $(u,v) \in [0,1]^4$ is such that $u_1, v_1, \varepsilon_2 \in V$, $u_2+v_2 = 1+\varepsilon_2$. Here, V is a neighborhood in $\mathbb{R}$ of 0. Let us call $y_1 = u_1 - v_1$ (resp. $y_2 = u_2 - v_2$) and, since $u_1, v_1 \in V$, we will assume that $u_1 + v_1 = \varepsilon_1$, where $\varepsilon_1 \in V$ and $y_1 \in V$. We denote this domain by D.

It is easy to check that

$$\gamma(u+v) = b_1 \sin(2\pi\varepsilon_1) + b_2 \sin(2\pi\varepsilon_2)$$

and that

$$\begin{aligned}
&\left[\gamma(u+v) + \frac{\theta_2^2(u+v)}{\gamma(u+v)}\right]^2 \\
&\quad = \left[\big(b_1 \sin(2\pi\varepsilon_1) + b_2 \sin(2\pi\varepsilon_2)\big) + \frac{(\sin^2(\pi\varepsilon_1)+\sin^2(\pi\varepsilon_2))^2}{(b_1\sin(2\pi\varepsilon_1)+b_2\sin(2\pi\varepsilon_2))}\right]^2 \\
&\quad = \chi_{b_1,b_2}(\varepsilon_1,\varepsilon_2).
\end{aligned}$$

We also have

$$\begin{aligned}
\theta_2(u)+\theta_2(v) = 2\bigg[&\cos^2\left(\frac{\pi y_1}{2}\right)\sin^2\left(\frac{\pi\varepsilon_1}{2}\right) + \cos^2\left(\frac{\pi\varepsilon_1}{2}\right)\sin^2\left(\frac{\pi y_1}{2}\right) \\
&+ \cos^2\left(\frac{\pi y_2}{2}\right)\cos^2\left(\frac{\pi\varepsilon_2}{2}\right) + \sin^2\left(\frac{\pi\varepsilon_2}{2}\right)\sin^2\left(\frac{\pi y_2}{2}\right)\bigg] \\
\geq{}& \delta(\varepsilon_1,\varepsilon_2,v_1,v_2),
\end{aligned}$$

where δ is a continuous function which is positive if y_2 is not in a neighborhood of ± 1 and such that

$$\delta(\varepsilon_1, \varepsilon_2, y_1, y_2) \geq k\big(\varepsilon_1^2 + \varepsilon_2^2 + y_1^2 + (y_2 \mp 1)^2\big)$$

for y_2 in the neighborhood of ± 1 and where k is a positive constant.

Note therefore that the function $1/\delta$ is integrable on D.

We may rewrite the fourth term as follows:

$$\begin{aligned}\gamma(u) + \gamma(v) &= 2\big(b_1 \sin(\pi\varepsilon_1)\cos(\pi y_1) - b_2 \sin(\pi\varepsilon_2)\cos(\pi y_2)\big)\\ &= \big(b_1 \sin(2\pi\varepsilon_1) + b_2 \sin(2\pi\varepsilon_2)\big) + \rho(\varepsilon_1, \varepsilon_2, y_1, y_2),\end{aligned}$$

where ρ is a bounded function such that

$$|\rho(\varepsilon_1, \varepsilon_2, y_1, y_2)| \leq K(\varepsilon_1 + \varepsilon_2)\big(\varepsilon_1^2 + \varepsilon_2^2 + y_1^2 + (y_2 \mp 1)^2\big)$$

for y_2 in the neighborhood of ± 1 and where K is a positive constant.

Using this notation, we get the following inequalities:

$$\begin{aligned}&\int_{(u,v)\in D} F(u, v)\, ds\, dt\\ &\quad\leq C \int_{\substack{(\varepsilon_1,\varepsilon_2)\in V^2\\ y_1\in V, y_2\in[-1,1]}} \frac{((b_1 \sin(2\pi\varepsilon_1) + b_2 \sin(2\pi\varepsilon_2)) + \rho(\varepsilon_1, \varepsilon_2, y_1, y_2))^2}{\chi_{b_1,b_2}(\varepsilon_1, \varepsilon_2)\delta(\varepsilon_1, \varepsilon_2, y_1, y_2)}\, d\varepsilon\, dy\\ &\quad\leq 2C \int_{\substack{(\varepsilon_1,\varepsilon_2)\in V^2\\ y_1\in V, y_2\in[-1,1]}} \frac{(b_1 \sin(2\pi\varepsilon_1) + b_2 \sin(2\pi\varepsilon_2))^2}{\chi_{b_1,b_2}(\varepsilon_1, \varepsilon_2)\delta(\varepsilon_1, \varepsilon_2, y_1, y_2)}\, d\varepsilon\, dy\\ &\quad\qquad + 2C \int_{\substack{(\varepsilon_1,\varepsilon_2)\in V^2\\ y_1\in V, y_2\in[-1,1]}} \frac{\rho^2(\varepsilon_1, \varepsilon_2, y_1, y_2)}{\chi_{b_1,b_2}(\varepsilon_1, \varepsilon_2, y_1, y_2)\delta(\varepsilon_1, \varepsilon_2, y_1, y_2)}\, d\varepsilon\, dy\\ &\quad\leq M \int_{\substack{(\varepsilon_1,\varepsilon_2)\in V^2\\ y_1\in V, y_2\in[-1,1]}} \frac{1}{\delta(\varepsilon_1, \varepsilon_2, y_1, y_2)}\, d\varepsilon\, dy\\ &\quad\qquad + M \int_{\substack{(\varepsilon_1,\varepsilon_2)\in V^2\\ y_1\in V, y_2\in[-1,1]}} J_{b_1,b_2}(\varepsilon_1, \varepsilon_2)\, d\varepsilon\, dy\end{aligned}$$

since

$$\frac{\rho^2(\varepsilon_1, \varepsilon_2, y_1, y_2)}{\delta(\varepsilon_1, \varepsilon_2, y_1, y_2)} \leq \frac{K^2}{k}(\varepsilon_1^2 + \varepsilon_2^2)\big(\varepsilon_1^2 + \varepsilon_2^2 + y_1^2 + (y_2 \mp 1)^2\big)$$

for y_2 in a neighborhood of ± 1 and is bounded otherwise. However, the first integral in the last inequality is bounded because $1/\delta$ is integrable and the second integral is finite by Lemma 5.2.

Then we have proved that $\int_{(u,v)\in D} F(u,v)\,du\,dv < \infty$. The study of the other singularities could be done with comparable estimates. $\square$

This lemma completes the proof of Theorem 5.1. We recall the two following corollaries established in [9].

COROLLARY 5.4. *Assume that the density is* $\rho \neq 1/2$, *the dimension* d *is* 2 *and the mean of* p *is nonzero*. *Let* f *be a local function with mean* 0. *We have weak convergence in the uniform topology on* $C[0,\infty)$ *to Brownian motion*, *with respect to the initial measure* ν_ρ,

$$\lim_{\alpha\to\infty} \alpha^{-1/2} \int_0^{\alpha t} f(\eta_s)\,ds = B\big(\sigma^2(f)t\big).$$

PROOF. The proof is a trivial consequence of Lemmas 3.4 and 3.9 and Theorem 1.1 of [9]. $\square$

COROLLARY 5.5. *If the density* $\rho \neq 1/2$, *dimension* $d = 2$, *mean of* p *is nonzero*, *the position of the second-class particle* $R(t)$ *is* ν_ρ*-transient*.

PROOF. See [9], Section 6 for the proof.

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DÉPARTEMENT DE MATHÉMATIQUES
ECOLE NORMALE SUPÉRIEURE DE CACHAN
61 AVENUE DU PRÉSIDENT WILSON
94235 CACHAN CEDEX
FRANCE
E-MAIL: bernardi@cmla.ens-cachan.fr