Congruence Preservation and Recognizability

I. Guessarian
IRIF, CNRS & Université Denis Diderot-Paris 7

Joint work with

Patrick Cégielski & Serge Grigorieff
(Université Paris 12) (Université Paris 7)

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I. Guessarian

Outline

1. Original problem
2. Characterize Congruence Preservation Algebraically
3. Characterize Congruence Preservation via Lattice Closure
4. More on Algebras, Congruence preservation, Lattice closure
5. Case of $\mathbb{Z}$
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A Question by Jean-Eric Pin

\( \mathcal{L} \) lattice of finite subsets of \( \mathbb{N} \), (i.e. for \( L, L' \in \mathcal{L} \), \( L \cap L', L \cup L' \in \mathcal{L} \))

\( \mathcal{L} \) closed under decrement \( \implies \mathcal{L} \) closed under division

Closure under decrement: \( \forall L \in \mathcal{L}: (L - 1) = \{ n - 1 \mid n \in L, n - 1 \geq 0 \} \in \mathcal{L} \)

\( \{0, 3, 7\} - 1 = \{2, 6\} \)

Closure under division: \( \forall a \in \mathbb{N}, \forall L \in \mathcal{L}: L/a = \{ n \mid an \in L \} \in \mathcal{L} \)

\( \{0, 3, 7\}/3 = \{0, 1\} \)

Answer: YES ... and Much More...
Which functions $f : \mathbb{N} \to \mathbb{N}$ are such that

\[\forall \mathcal{L} \text{ lattice of finite subsets of } \mathbb{N} \]
\[\forall L \in \mathcal{L} \quad (L - 1) \in \mathcal{L} \implies \forall L \in \mathcal{L} \quad f^{-1}(L) \in \mathcal{L}\]

$(\star)$

**Theorem (CGG13)**

$f : \mathbb{N} \to \mathbb{N}$ satisfies $(\star) \iff f$ is congruence preserving

Idem for lattices of regular subsets of $\mathbb{N}$


**Definition**

$f$ congruence preserving $\iff$

for any congruence $\sim$ on $\mathbb{N}$: $x \sim y \implies f(x) \sim f(y)$
Congruences on $\langle \mathbb{N}, + \rangle$

$x \sim y$ iff; either $x = y$ or $\varphi(x) = \varphi(y)$ with $\varphi$

$$\varphi(x) = \begin{cases} x & \text{for } x \leq a \\ a + ( (x - a) \mod k ) & \text{for } x > a \end{cases} \quad a \geq 0, k \geq 1$$

frying pan monoid

$a \sim a + 8 \sim a + 16 \sim \cdots$ or $a + 3 \sim a + 11 \sim a + 19 \sim \cdots$
Morphisms, Congruences depend on signature

Morphism on $\langle \mathbb{N}, + \rangle \iff$ Morphism on $\langle \mathbb{N}, \times \rangle$

$\varphi(x) = 3x$  \hspace{1cm} $\psi(x) = \begin{cases} 1 & \text{iff } \exists n \ x = 2^n, \\ 0 & \text{otherwise.} \end{cases}$

$\langle \mathbb{N}, + \rangle$-congruence $\implies$ $\langle \mathbb{N}, \times \rangle$-congruence.

$\langle \mathbb{N}, + \rangle$-congruence $\not\iff$ $\langle \mathbb{N}, \times \rangle$-congruence.

$x \sim y$ iff $\psi(x) = \psi(y)$ is a $\langle \mathbb{N}, \times \rangle$-congruence and not a $\langle \mathbb{N}, + \rangle$-congruence: $2 \sim 4$ and $4 \sim 4$ but $(2 + 4) \not\sim (4 + 4)$
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Issue: capture congruence preservation

**Theorem (CGG13)**

\[ f : \mathbb{N} \to \mathbb{N} \text{ congruence preserving} \iff \]

1. \( \forall a, b \in \mathbb{N} \quad a - b \text{ divides } f(a) - f(b) \) or equivalently (justifying the denomination), \( \forall n \geq 1, \)
   \[ \forall a, b \in \mathbb{N} \quad (a \equiv b \mod n \implies f(a) \equiv f(b) \mod n) \]

2. and \( \forall a \in \mathbb{N} \quad f(a) \geq a \) or \( f \) constant

- Obvious example: Polynomials in \( \mathbb{N}[x] \)
- What else?
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Congruence preserving functions

\[ A = \langle A, \mathcal{O} \rangle \] algebra with operations \( \mathcal{O} \).

**Definition**

\( f : A^n \rightarrow A \) is congruence preserving iff, for any \( \mathcal{O} \)-congruence \( \sim \) on \( A \):

\[
\forall x_1, \ldots, x_n, y_1, \ldots, y_n \in A \\
\bigwedge_{i=1}^{i=n} x_i \sim y_i \implies f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n)
\]

Example: “Polynomial functions” \( = \) expressed by terms with constants in \( A \). What else?
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Representing functions \( \mathbb{N} \rightarrow \mathbb{Z} \)

We represent functions \( \mathbb{N} \rightarrow \mathbb{Z} \) by
series of polynomials in \( \mathbb{Q}[x] \) mapping \( \mathbb{N} \) into \( \mathbb{Z} \)

Binomial polynomial function \( \mathbb{N} \rightarrow \mathbb{N} \) in \( \mathbb{Q}[x] \)

\[
\binom{x}{0} = 1 \quad \binom{x}{n} = \frac{x(x-1) \cdots (x-n+1)}{n!}
\]

Functions \( \mathbb{N} \rightarrow \mathbb{Z} \) \( \equiv \) \( 1-1 \)

Infinite \( \mathbb{Z} \)-linear combinations of the binomial polynomials

NO CONVERGENCE PROBLEM: For every \( x \in \mathbb{N} \)
infinite sum \( \sum_{n \in \mathbb{N}} a_n \binom{x}{n} \) reduces to the finite sum \( \sum_{n \leq x} a_n \)
Characterize preservation of modular congruences

Theorem (CGG15)

If \( f : \mathbb{N} \rightarrow \mathbb{Z} \), then \((1) \iff (2)\)

\[
(1) \forall x, y \ x - y \text{ divides } f(x) - f(y)
\]

\[
(2) f(x) = a_0 + a_1 x + a_2 \frac{x(x - 1)}{2!} + a_3 \frac{x(x - 1)(x - 2)}{3!} + \cdots,
\]

where \( \ell \) divides \( a_n \) for all \( 2 \leq \ell \leq n \).

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**Characterize congruence preservation on** $\mathbb{N}$

$f : \mathbb{N} \rightarrow \mathbb{N}$ congruence preserving $\iff$

1. $\forall a, b \in \mathbb{N} \quad a - b$ divides $f(a) - f(b)$

2. and $\forall x \in \mathbb{N} \quad f(x) \geq x$ or $f$ constant.

**Theorem**

$f : \mathbb{N} \rightarrow \mathbb{N}$ Congruence preserving $\iff$ (1) and (2)

(1) $f(x) = a_0 + a_1 x + a_2 \frac{x(x - 1)}{2!} + a_3 \frac{x(x - 1)(x - 2)}{3!} + \cdots$, where $\ell$ divides $a_n$ for all $2 \leq \ell \leq n$

(2) $\forall x \in \mathbb{N} \quad f(x) \geq x$ or $f$ constant.
Characterize congruence preservation on $\mathbb{N}$

$f : \mathbb{N} \rightarrow \mathbb{N}$ congruence preserving $\iff$

1. $\forall a, b \in \mathbb{N}$, $a - b$ divides $f(a) - f(b)$
2. and $\forall x \in \mathbb{N}$, $f(x) \geq x$ or $f$ constant.

Theorem

$f : \mathbb{N} \rightarrow \mathbb{N}$ Congruence preserving $\iff$ (1) and (2)

(1) $f(x) = a_0 + a_1 x + a_2 \frac{x(x - 1)}{2!} + a_3 \frac{x(x - 1)(x - 2)}{3!} + \cdots$,

where $\ell$ divides $a_n$ for all $2 \leq \ell \leq n$

(2) $\forall x \in \mathbb{N}$, $f(x) \geq x$ or $f$ constant.
Corollary (CGG)

Non polynomial congruence preserving functions $\mathbb{N} \rightarrow \mathbb{N}$

$$f(x) = \lfloor e^{1/a} a^x x! \rfloor \quad \text{for} \ a \in \mathbb{N} \setminus \{0, 1\}$$

third kind Bessel function $g$

$$g(x) = \frac{\Gamma(1/2)}{2 \times 4^x \times x!} \int_1^\infty e^{-t/2} (t^2 - 1)^x dt$$

Idem for $f(x) = \lceil e^{1/a} a^x x! \rceil$

[CGG] Integral Difference Ratio Functions on Integers, LNCS 8808 (2014).

- What about other algebras ??
Algebraic characterization of congruence preservation

Abbrev: \( CP = \) congruence preserving

\[ f \text{ CP on } \langle \mathbb{N}, + \rangle \iff f \text{ infinite } \mathbb{Z}-\text{linear combination of binomial polynomials satisfying some conditions} \]

- On \( \mathbb{Z}, \mathbb{Z}_p \) with + and \( \times \): similar to \( \mathbb{N} \)
- On \( \langle \mathbb{N}, \times \rangle \): much simpler characterization

**Theorem**

\[ f : \mathbb{N} \rightarrow \mathbb{N}; \]

\[ f \text{ CP on } \langle \mathbb{N}, \times \rangle \]

\[ f(x) = f(1) \times x^k, \text{ with } k \in \mathbb{N} \]
Algebraic characterization of congruence preservation

Abbrev: CP = congruence preserving

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- On \(\langle \mathbb{N}, \times \rangle\): much simpler characterization

**Theorem**

\[ f : \mathbb{N} \longrightarrow \mathbb{N}; \]

\[ f \text{ CP on } \langle \mathbb{N}, \times \rangle \]

\[ f(x) = f(1) \times x^k, \text{ with } k \in \mathbb{N} \]
Congruence preservation on a non commutative algebras

\[ f \text{ CP on } \langle \mathbb{N}, \times \rangle \iff f(x) = f(1) \times x^k, \ k \in \mathbb{N}. \]

**Theorem**

*On the algebra of words with concatenation, \( S = \langle \Sigma^*, \cdot \rangle \)\*  
\[ f \text{ CP } \iff f : x \mapsto w_0xw_1xw_2 \cdots xw_k, \]  
\[ k \in \mathbb{N}, \ w_0, w_1, \ldots, w_k \in \Sigma^*. \]

Non trivial proof using restricted morphisms.

**affine complete algebras:** for all \( f, \) \( f \text{ CP } \iff f \text{ polynomial.} \)  
\( S, \langle \mathbb{N}, \times \rangle \) are affine complete.  
\( \langle \mathbb{N}, + \rangle \) is not affine complete
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Theorem (CGG14)

Algebra $\mathcal{N} = \langle \mathbb{N}, + \rangle$, $f : \mathbb{N} \rightarrow \mathbb{N}$, then \( (1) \iff (2) \)

\( (1) \) $f$ CP on $\mathcal{N}$ and, $\forall a \in \mathbb{N}$, $f(a) \geq a$,

\( (2) \) for every recognizable subset $L$ of $\mathcal{N}$ the smallest lattice of subsets of $\mathbb{N}$ containing $L$ and closed under $x \mapsto x - 1$ is also closed under $f^{-1}$.


- What about other algebras ??
Is congruence preservation characterized via lattice closure for any algebra?

Theorem ((1) ⇐⇒ (2) on $\mathcal{N} = \langle \mathbb{N}, + \rangle$)

(1) $f$ CP on $\mathcal{N}$ and, $\forall a \in \mathbb{N}, f(a) \geq a$,
(2) for every recognizable subset $L$ of $\mathcal{N}$ the smallest lattice of subsets of $\mathbb{N}$ containing $L$ and closed under $x \mapsto x - 1$ is also closed under $f^{-1}$.

Can be generalized to arbitrary algebras?

Theorem ( (1) ⇐⇒ (2) on $\mathcal{A} = \langle A, \mathcal{O} \rangle$)

(1) $f$ CP on $\mathcal{A}$ and, something else,
(2) for every recognizable subset $L$ of $\mathcal{A}$ the smallest lattice $\mathcal{L}_A(L)$ of subsets of $A$ containing $L$ and closed under some operations is also closed under $f^{-1}$.
Recognizability in algebra $\mathcal{A} = \langle A, \mathcal{O} \rangle$

**Definition**

$L$ is recognizable iff $L = \varphi^{-1}(F)$ with $\varphi : A \to M$ morphism, $M$ a finite algebra with same signature as $\mathcal{A}$, $F \subset M$.

**Examples**

- $\langle \mathbb{N}, + \rangle$-recognizable: finite sets, $1 + 3\mathbb{N}$, $\{2\} \cup \{\{5, 7\} + 8\mathbb{N}\}$, $F \cup \{F' + k\mathbb{N}\}$ (general form)

- $\langle \mathbb{Z}, + \rangle$-recognizable: $F + k\mathbb{Z}$ (general form)

- $\langle \Sigma^*, \cdot \rangle$-recognizable: regular sets.
Recognizability depends on signature

\[ \langle \mathbb{N}, + \rangle \text{-recognizable} \implies \langle \mathbb{N}, \times \rangle \text{-recognizable.} \]
\[ \langle \mathbb{N}, + \rangle \text{-recognizable} \nLeftarrow \langle \mathbb{N}, \times \rangle \text{-recognizable.} \]

\((1 + 3\mathbb{N})\) is \(\langle \mathbb{N}, +, \times \rangle\)-recognizable, but \(\{2^n \mid n \in \mathbb{N}\}\) is \(\langle \mathbb{N}, \times \rangle\)-recognizable and not \(\langle \mathbb{N}, + \rangle\)-recognizable.

- \(\langle \mathbb{N}, \times \rangle\)-recognizables:
  1. all \(\langle \mathbb{N}, + \rangle\)-recognizables \((L = F \cup \{F' + k\mathbb{N}\})\),
  2. all finite unions \(p_1^{L_1} \cdots p_n^{L_n}\), with \(p_1, \ldots, p_n\) primes in \(P\), \(L_1, \ldots, L_n\) \(\langle \mathbb{N}, + \rangle\)-recognizable.
  3. suitably completed
Generalization to algebra: $\mathcal{N}_x = \langle \mathbb{N}, \times \rangle$

Theorem ( (1) $\iff$ (2) on $\mathcal{N}_x = \langle \mathbb{N}, \times \rangle$)

(1) $f$ CP on $\mathcal{N}_x$ and, $\forall a \in \mathbb{N}$, $a$ divides $f(a)$,

(2) for every recognizable subset $L$ of $\mathcal{N}_x$ the smallest lattice $\mathcal{L}_{\mathcal{N}_x}(L)$ of subsets of $\mathbb{N}$ containing $L$ and closed under division is also closed under $f^{-1}$.

Division: $S \subset \mathbb{N}$, $a \in \mathbb{N}$, let $S/a = \{x/a | x \in S \text{ and } x/a \in \mathbb{N}\}$

\[
\begin{align*}
(9 + 5\mathbb{N})/5 &= \emptyset \\
(9 + 5\mathbb{N})/4 &= \{9, 14, 19, 24, 29, 34, 39, 44, \ldots \}/4 \\
&= \{6, 11, 16 \ldots \} = 6 + 5\mathbb{N}
\end{align*}
\]
Tentative Generalization to Algebra $\mathcal{A} = \langle A, \mathcal{O} \rangle$

$\text{gen}(a, A, \mathcal{O}) = \text{set generated by } a \text{ in } A$.

- $\text{gen}(a, \mathbb{N}, +) = \{a + n \mid n \in \mathbb{N}\}$
- $\text{gen}(a, \mathbb{N}, \times) = \{a \times n \mid n \in \mathbb{N}\}$
- Words with concatenation:
  $\text{gen}(a, \Sigma^*, \cdot) = \{w \cdot a \cdot w' \mid w, w' \in \Sigma^*\}$

**Theorem** (1) $\iff$ (2) ???

1. $f$ CP on $\mathcal{A}$ and, $\forall a \in A$, $f(a) \in \text{gen}(a, A, \mathcal{O})$
2. For every recognizable subset $L$ of $\mathcal{A}$ the smallest lattice $\mathcal{L}_A(L)$ of subsets of $A$ containing $L$ and closed under $o^{-1}$ for all $o \in \mathcal{O}$ is also closed under $f^{-1}$. 

Characterize Congruence Preservation via Lattice Closure
Affine complete algebras

Theorem

In algebra $\mathcal{A} = \langle A, \mathcal{O} \rangle$, if $f : A \rightarrow A$ defined by a polynomial and $L$ recognizable, then $f^{-1}(L)$ recognizable.

Algebra $\mathcal{A} = \langle A, \mathcal{O} \rangle$ is affine complete if:

for all $f$, $f$ CP $\iff$ $f$ polynomial.

PROBLEM: $f^{-1}(L)$ in lattice??

WHAT ELSE??
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To which structures does the lattice closure characterization of CP extend?

- Goal: generalize the algebraic characterization of congruence preservation via lattices to algebras beyond $\mathbb{N}$
- stable preorder preservation characterized via closure of lattices.
- congruence preservation characterized via closure of boolean algebras.
- except in some cases ....
Stable Preorder vs Congruence Preservation on \( \mathcal{A} = \langle A, \mathcal{O} \rangle \)

**Theorem**

1. \( f \) **SPP** on \( \mathcal{A} \) and, \( \forall a \in A, f(a) \in \text{gen}(a, A, \mathcal{O}) \)
2. for every subset \( L \) of \( \mathcal{A} \) the smallest **complete lattice** \( \mathcal{L}_A^\infty(L) \) of subsets of \( A \) containing \( L \) and closed under \( o^{-1} \) for all \( o \in \mathcal{O} \) is also closed under \( f^{-1} \).

**Theorem**

1. \( f \) **CP** on \( \mathcal{A} \) and, \( \forall a \in A, f(a) \in \text{gen}(a, A, \mathcal{O}) \)
2. for every subset \( L \) of \( \mathcal{A} \) the smallest **complete boolean algebra** \( \mathcal{B}_A^\infty(L) \) of subsets of \( A \) containing \( L \) and closed under \( o^{-1} \) for all \( o \in \mathcal{O} \) is also closed under \( f^{-1} \).
Sufficient conditions to characterize Congruence Preservation via Lattice closure

If \( f : A \rightarrow A \), and \( \mathcal{A} = \langle A, \mathcal{O} \rangle \)

- residually finite algebra
- containing a group operation,

then congruence preservation is characterized via lattices

\[ f \text{ CP and } \forall a \in A, \ f(a) \in \text{gen}(a, A, \mathcal{O}) \]

for every recognizable \( L \), the smallest lattice \( \mathcal{L}_{\mathcal{A}}(L) \) of subsets of \( A \) containing \( L \) and closed under \( o^{-1} \) for all \( o \in \mathcal{O} \) is also closed under \( f^{-1} \).
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$\mathcal{Z} = \langle \mathbb{Z}, + \rangle$

**Theorem**

$\mathcal{Z} = \langle \mathbb{Z}, + \rangle, \ f : \mathbb{Z} \rightarrow \mathbb{Z} \iff \begin{array}{c} 1 \iff 2 \iff 3 \iff 4 \end{array}$

1. $f$ is CP on $\mathcal{Z}$.
2. $|x - y|$ divides $|f(x) - f(y)|$ for all $x, y \in \mathbb{Z}$.
3. For every recognizable subset $L$ of $\mathcal{Z}$, the lattice $\mathcal{L}_{\mathcal{Z}}(L)$ is closed under $f^{-1}$.
4. $f(x) = \sum_{n \in \mathbb{N}} a_n P_n(x)$ where, for all $2 \leq \ell \leq n$, $\ell$ divides $a_n$.

Similar theorem for $\mathcal{Z} = \langle \mathbb{Z}, +, \times \rangle$ and $\mathcal{Z}_p = \langle \mathbb{Z}_p, +, \times \rangle$
moral of the story: CP functions correspond to functions definable in terms of the algebra operations... in general ...

affine complete non commutative algebras different from the free monoid?