Series expansions of the deflection angle in the scattering problem for power-law potentials

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Abstract

We present a rigorous study of the classical scattering for any two-body inter-particle potential of the form $v(r) = g/r^\gamma$, with $\gamma > 0$, for repulsive ($g > 0$) and attractive ($g < 0$) interactions. We first derive an explicit series expansion of the deflection angle in the impact factor b. Then, we study carefully the modifications of the results when a regularization (softening) is introduced in the potential at small scales. We check and illustrate all the results with the exact integration of the equations of motion.

1 Introduction

Scattering of particles are present in many physical processes in a broad area of Physics, as atomic (e.g. [?]), plasma (e.g. [?]), astrophysics (e.g. [?]), active matter (e.g. [?]), etc. On this subject, a seminal paper was published by Ernest Rutherford in 1911 [?], in which he studied the deflection of α and β particles by an atom. He calculated analytically the angle of deflection of the incident particles with the nucleus. His calculations, compared to experimental data (see [?] for references), permitted to conclude that the atom is basically "empty" with a charge concentrated in the center, surrounded by the electron cloud, which lead to the "planetary" model of the atom.

These two-body collisions play also a central role in collisional processes in Coulomb plasmas (see e.g. [?]), in self-gravitating systems (as pointed out by Chandrasekhar in a seminal paper [?]), and, in general, in systems of particles with power law interactions [?, ?]. In order to write kinetic equations which describe the evolution of such systems, it is necessary to solve the two-body problem, i.e. to compute the final velocities after a scattering event. For example, let us consider for simplicity the Boltzmann equation which describes the evolution of the one-point distribution function $f(\mathbf{r}, \mathbf{v}; t)$ of a system of particles interacting with the inter-particle potential

$$v(r) = \frac{g}{r^\gamma}. \tag{1}$$

For simplicity, we will write the Boltzmann equation for a spatially homogeneous and isotropic three dimensional system, in which collective effects are neglected. In this case, it has the simple form

$$\frac{\partial \varphi}{\partial t}(\mathbf{v}_1; t) = 2\pi \int d\mathbf{v}_2 \int_0^\infty db\, b\, u\, G(\mathbf{v}_2', \mathbf{v}_1', \mathbf{v}_2, \mathbf{v}_1; t), \tag{2}$$

where $\varphi(\mathbf{v}; t)$ is the velocity probability function and

$$G(\mathbf{v}_2', \mathbf{v}_1', \mathbf{v}_2, \mathbf{v}_1; t) = \varphi(\mathbf{v}_2'; t)\varphi(\mathbf{v}_1'; t) - \varphi(\mathbf{v}_2; t)\varphi(\mathbf{v}_1; t), \tag{3}$$

$\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of the particles before the collision, $\mathbf{v}_1'$ and $\mathbf{v}_2'$ the velocities of the particles after the collision, b the impact factor and u the modulus of the relative velocity, i.e.

$$u = \|\mathbf{v}_2 - \mathbf{v}_1\|. \tag{4}$$

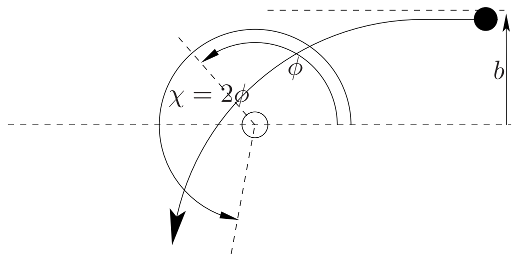

Figure 1: Collision in the center of mass frame. The black dot represents the fictitious (reduced) particle, and the white dot the center of mass of the particles, which is at rest.

Despite the apparent simplicity of Eq. (**??**), the main difficulty consists in computing the final velocities $\mathbf{v}_i'$ as a function of the initial ones $\mathbf{v}_i$ ($i = 1, 2$). In the center of mass frame (see Appendix **??** for details), the angle ϕ (see Fig. **??**) can be calculated as a function of the impact factor b using the formula

$$\phi(b) = \int_{r_{min}}^{\infty} \frac{(b/r^2)dr}{\sqrt{1 - (b/r)^2 \mp (b_0/r)^\gamma}}, \tag{5}$$

where r_{min} is the largest root of the denominator. The "minus" sign in the denominator corresponds to a repulsive interaction while the "plus" sign to an attractive one and b_0 is the characteristic scale:

$$b_0 = \left(\frac{2|g|}{mu^2}\right)^{1/\gamma}. \tag{6}$$

In the case of Coulomb and gravitational interaction ($\gamma = 1$), there exists an analytical expression of the deflection angle, the Rutherford formula [**?**]: for the repulsive case

$$\phi(b/b_0) = \arctan\left(\frac{2b}{b_0}\right); \tag{7}$$

and for the attractive one

$$\phi(b/b_0) = \pi - \arctan\left(\frac{2b}{b_0}\right). \tag{8}$$

In the case of different interactions, the process is well known only on the qualitative level or in particular cases (see e.g. [**?**, **?**, **?**, **?**, **?**, **?**, **?**]). In the cases in which explicit solutions are not possible to compute, it is natural to perform an asymptotic expansion in the adimensional variable b/b_0. Inspecting for example the gravitational case (**??**), we see that it is possible to write the solution in the form of two asymptotic series: one in powers of b/b_0, valid for $b/b_0 \leqslant 1/2$, and another one in powers of b_0/b, valid for $b/b_0 \geqslant 1/2$. It is then natural to ask the following questions

1. Is it possible to write, for $\gamma \neq 1$, an expression in the form of a power series of Eq. (**??**) for small b/b_0 and for large b/b_0?
2. If the answer of the previous question is positive, what are the exponent(s) of the power series?
3. In the case in which the answer to the previous question is positive, do the coefficients of these power series have simple analytical expressions?
4. If so, do the convergence radius of the two power series for small and large b/b_0 match?

It is not trivial to answer the questions listed above. A naive expansion in power series of b/b_0 of Eq. (**??**), gives in many cases divergent integrals, which indicates that the series are not in powers of b/b_0. Moreover, special care should be given for attractive interactions and $\gamma > 2$, where the centrifugal barrier could not be sufficient to prevent particles to crash.

In the present paper we will give the answers to the previous questions. We will derive the full asymptotic power series, one valid for small b/b_0, and another one for large b/b_0, which extend the result [?] valid for $\gamma > 2$, and we will show that their convergence radius match. Morevover, we will study how the trajectories change when introducing a regularization at small scales in the potential.

2 Summary of the results

2.1 Pure power-law interactions

In this paper we have derived the full asymptotic series solution of Eq. (**??**) (in the cases in which it well-defined) for both attractive and repulsive potentials. We first denote

$$\beta = (\gamma/2)^{1/\gamma}|1-2/\gamma|^{\frac{2-\gamma}{2\gamma}}. \tag{9}$$

Theorem 1 *We assume repulsive interactions, that is the minus sign in Eq.* (**??**) *with an arbitrary* $\gamma > 0$.
(*i*) *For* $b > \beta b_0$, *we have*

$$\phi(b/b_0) = \sqrt{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n\Gamma((n\gamma+1)/2)}{2n!\Gamma(1+n(\gamma/2-1))}(b_0/b)^{\gamma n}. \tag{10}$$

(*ii*) *For* $b < \beta b_0$, *we have*

$$\phi(b/b_0) = \sum_{n=0}^{+\infty}\alpha_n(b/b_0)^{2n+1}, \tag{11}$$

where, for $n \in \mathbb{N}$,

$$\alpha_n = \alpha_n(\gamma) = \frac{(-1)^n\sqrt{\pi}}{(2n+1)n!}\frac{\Gamma\left(1+(2n+1)/\gamma\right)}{\Gamma\left(1/2-n+(2n+1)/\gamma\right)}.$$

Theorem 2 *We assume attractive interactions, that is the plus sign in Eq.* (**??**).
(*i*) *For* $b > \beta b_0$ *and* $\gamma > 0$ *arbitrary, we have*

$$\phi(b/b_0) = \sqrt{\pi}\sum_{n=0}^{+\infty}\frac{\Gamma((n\gamma+1)/2)}{2n!\Gamma(1+n(\gamma/2-1))}(b_0/b)^{\gamma n}. \tag{12}$$

(*ii*) *For* $b < \beta b_0$, $\gamma < 2$ *and if*

$$\gamma \notin \left\{2\frac{2k+1}{2\ell+1}, k, \ell \in \mathbb{N}, k < \ell\right\},$$

we have

$$\phi(b/b_0) = \sum_{n=0}^{+\infty}a_n(b/b_0)^{\frac{2\gamma}{2-\gamma}n} + \sum_{q=0}^{+\infty}c_q(b/b_0)^{2q+1}, \tag{13}$$

where, for $q \in \mathbb{N}$,

$$\begin{aligned} c_q &= (-1)^q\binom{-1/2}{q} \times \frac{\Gamma((2q+1)/\gamma+1)\Gamma((q+1/2)(1-2/\gamma))}{(2q+1)\Gamma(q+1/2)} \\ &= \frac{\Gamma((2q+1)/\gamma+1)\Gamma((q+1/2)(1-2/\gamma))}{\sqrt{\pi}(2q+1)q!}, \end{aligned} \tag{14}$$

$$a_0 = \frac{\pi}{2-\gamma}, \tag{15}$$

and, for $n \geqslant 1$,

$$a_n = -\frac{\sqrt{\pi}\Gamma\left(\frac{\gamma n}{\gamma-2}+\frac{1}{2}\right)}{2n\Gamma\left(\frac{2n}{\gamma-2}\right)n!}. \tag{16}$$

(iii) If $\gamma = \gamma_{k,\ell} = 2\frac{2k+1}{2\ell+1} \in]0,2[$ for some $k,\ \ell \in \mathbb{N}$ with $k < \ell$, then

$$\begin{aligned}\phi(b/b_0) = &\sum_{\substack{n\in\mathbb{N}\ s.t.\\ 1+n\frac{2k+1}{k-\ell}\notin -2\mathbb{N}}} a_n(\gamma_{k,\ell})(b/b_0)^{\frac{2\gamma_{k,\ell}}{2-\gamma_{k,\ell}}n} + \sum_{\substack{q\in\mathbb{N}\ s.t.\\ (2q+1)\frac{k-\ell}{2k+1}\notin -\mathbb{N}}} c_q(\gamma_{k,\ell})(b/b_0)^{2q+1}\\ &+ \sum_{\substack{n,q\in\mathbb{N}\ s.t.\\ (2q+1)(\ell-k)=n(2k+1)}} \frac{\sqrt{\pi}(-1)^q(b/b_0)^{2q+1}}{2n\Gamma(-n-q-1/2)n!q!}\Bigg(2\ln(b/b_0)\\ &\quad + \frac{\Gamma'}{\Gamma}(-n-q-1/2)\frac{2}{\gamma_{k,\ell}} + \frac{2}{\gamma_{k,\ell}}\gamma_0 - H_q - \frac{2-\gamma_{k,\ell}}{\gamma_{k,\ell}}(H_n+1/n)\Bigg).\end{aligned} \tag{17}$$

Here, $H_N = \sum_{p=1}^{N} 1/p$ is the harmonic sum of order N and γ_0 is Euler's constant.
(iv) For $\gamma \geqslant 2$, particles crash in a finite time if $b \leqslant \beta b_0$. If $b > \beta b_0$, we have

$$\phi\left(\frac{b}{b_0}\right) = \frac{\pi}{2\sqrt{1-b_0^2/b^2}} \qquad \text{if } \gamma = 2 \text{ and } b > \beta b_0 = b_0, \tag{18a}$$

$$\phi\left(\frac{b}{b_0}\right) \approx -\frac{\ln(1-\beta b_0/b)}{2\sqrt{\gamma-2}} \qquad \text{if } \gamma > 2 \text{ and } b \approx \beta b_0. \tag{18b}$$

Remark 1 *Statement (i) of Theorems **??** and **??** are due to [?] for $\gamma > 2$. The formulas extend to arbitrary γ positive.*

Remark 2 *In Eq. (**??**) (i), the coefficient is the same as in Eq. (**??**) (i), up to the $(-1)^n$ factor.*

Remark 3 *In the statement of Theorem **??** when $\gamma < 2$, we emphasize that in the generic (ii) case $\gamma \notin \{2\frac{2k+1}{2\ell+1}, k,\ell \in \mathbb{N}, k < \ell\}$, which we call the unexceptional cases, then $\phi(b/b_0)$ is the sum of two power series with different exponents, one does not depend on γ, the other one does. In particular, for a given γ, if we want a first or second order expansion of ϕ for $b/b_0 \ll 1$, we need to order the exponents in Eqs. (**??**) and (**??**). For instance, noticing that $2\gamma/(2-\gamma) < 1$ as soon as $\gamma < 2/3$, for hard collisions ($b/b_0 \ll 1$), we obtain:*

$$\phi = \frac{\pi}{2-\gamma} + \begin{cases} \dfrac{\Gamma(1+1/\gamma)\Gamma(1/2-1/\gamma)}{\sqrt{\pi}} b/b_0 + o(b/b_0) & \text{if } 2/3 < \gamma < 2,\\ \dfrac{3}{4}(b/b_0)\ln(b_0/b) + o((b/b_0)\ln(b_0/b)) & \text{if } \gamma = 2/3,\\ -\dfrac{\sqrt{\pi}\Gamma(1/2-\gamma/(2-\gamma))}{\Gamma(2/(2-\gamma))}(b/b_0)^{2\gamma/(2-\gamma)} + o((b/b_0)^{2\gamma/(2-\gamma)}) & \text{if } 0 < \gamma < 2/3. \end{cases} \tag{19}$$

*In the exceptional cases (iii), then logarithmic corrections appear (see Eq. (**??**)).*

2.2 Hard collisions with regularized interactions

We have calculated the modification of the above results, at first order and for hard collisions, when a regularization is applied at small scales in the potential (which is a standard procedure, e.g. in molecular dynamics simulations). In this case the angle ϕ is given by the formula

$$\phi_\epsilon(b,b_0) = \frac{b}{r_{min}}\int_0^1 \frac{dx}{\sqrt{1-(\frac{bx}{r_{min}})^2 \pm \frac{b_0^\gamma}{\epsilon^\gamma}\mathcal{V}(\frac{r_{min}}{\epsilon x})}}. \tag{20}$$

In the conclusion section we will give an example of the use of these results. In order to be able to make explicit calculations, we will consider two regularizations commonly used in the astrophysical literature (see e.g. [?, ?]), the *Plummer potential*

$$v^{\mathrm{Pl}}(r,\epsilon) = \frac{g}{(r^2+\epsilon^2)^{\gamma/2}} \tag{21}$$

and the compact softening

$$v^{\mathrm{co}}(r,\epsilon) = \begin{cases} \dfrac{g}{r^\gamma} & \text{if } r \geqslant \epsilon \\ \dfrac{g}{\epsilon^\gamma}\mathrm{v}\,(r/\epsilon) & \text{if } 0 \leqslant r \leqslant \epsilon, \end{cases} \tag{22}$$

where v is a function on $[0,1]$ such that $\mathrm{v}(1)=1$. For these regularized potentials, we do not expect series expansions with analytically simple coefficients. We have however been able to compute the following second order expansions (the explicit coefficients are given in Section **??**):

Theorem 3 *We consider repulsive interactions, that is the minus sign in Eq.* (**??**).
(*i*) *For the Plummer softening, when $\epsilon < b_0$ are fixed, we have, for small b/b_0,*

$$\phi_\epsilon(b,b_0) = B^{\mathrm{Pl}}_{\epsilon/b_0}(\gamma)(b/b_0) + \mathcal{O}((b/b_0)^3),$$

where the coefficient $B^{\mathrm{Pl}}_{\epsilon/b_0}(\gamma)$ is given in Eq. (**??**) *in subsection* **??**.
(*ii*) *For the compact softening, when $\epsilon < b_0$ are fixed and for small b/b_0, the deflection angle ϕ_ϵ is not affected by the softening, hence we have the same asymptotic behavior as in Eq.* (**??**)*, namely*

$$\phi_\epsilon(b,b_0) = \alpha_1(\gamma)(b/b_0) + \mathcal{O}((b/b_0)^3).$$

(*iii*) *For the Plummer (resp. compact) softening, when $\epsilon > b_0$ (resp. $\epsilon > b_0(\max \mathrm{v})^{1/\gamma}$) and b/ϵ small, we have*

$$\phi_\epsilon(b,b_0) = \frac{\pi}{2} - \tilde{B}_{\epsilon/b_0}(\gamma)b/\epsilon + o(b/\epsilon),$$

where the coefficient $\tilde{B}_{\epsilon/b_0}(\gamma)$ is given in Eq. (**??**) *in Subsection* **??** *(resp. Eq.* (**??**) *in Subsection* **??**).

Theorem 4 *We consider attractive interactions, that is the plus sign in Eq.* (**??**) *and either the Plummer or the compact softening. Let us fix $\epsilon > 0$ and $b_0 > 0$ arbitrary. Then, for b/ϵ small, we have*

$$\phi_\epsilon(b,b_0) = \frac{\pi}{2} + C_{\epsilon/b_0}(\gamma)b/\epsilon + o(b/\epsilon),$$

where $C_{\epsilon/b_0}(\gamma)$ is given in Eq. (**??**) *in Subsection* **??**.

Remark 4 *Let us point out that the statement of Theorem* **??** *holds true independently whether ϵ/b_0 is small or not.*

Proposition 1 *We consider the case $\gamma > 2$ with either the Plummer or the compact softening. Then, there exists a threshold $\epsilon_*(b_0,\gamma)$, depending only on b_0 and γ, such that*

- *if $\epsilon \leqslant \epsilon_*(b_0,\gamma)$, then the angle ϕ_ϵ diverges to $+\infty$ for some critical impact factor b;*
- *if $\epsilon > \epsilon_*(b_0,\gamma)$, then ϕ_ϵ is a smooth function of b/b_0 for $b/b_0 \geqslant 0$.*

For the Plummer softening, we have

$$\epsilon_*^{Pl}(b_0,\gamma) = b_0\left(\frac{\gamma-2}{\gamma+2}\right)^{\frac{1}{2}+\frac{1}{\gamma}}, \tag{23}$$

and the expression of $\epsilon_*(b_0,\gamma)$ in the case of compact softening (see Eq. (**??**)) *is slightly more involved but still proportional to b_0.*

2.3 Numerical checking and discussion

First of all, we show in Fig. **??** the truncated series expansions Eqs. (**??**), (**??**), (**??**) and (**??**) for the pure power-law case, for several values of γ: $\gamma = 1/2$ and $\gamma = 7/4$, for repulsive and attractive interactions and $\gamma = 2/3$ and $\gamma = 6/7$ for attractive interactions, the "exceptional case", see Eq. (**??**). We plot the numerical solution ("exact") obtained by numerical integration of Eq. (**??**) and the truncated series with different number of terms, showing the convergence towards the "exact" solution. In the inset we show the relative error of the numerical solution and the series with the largest number of terms (nmax) plotted in the main figure. We observe, as expected, that the maximum difference between the two solutions is at the convergence radius of the series, denoted by a vertical dotted line. We observe that the truncation of the respective series to the tenth term provides an excellent approximation for both $b/b_0 \in [0, \beta[$ and $b/b_0 \in]\beta, +\infty[$.

It is interesting to study the different kind of trajectories inspecting the first terms of the asymptotic series. In the case of repulsive potentials, the maximum value that the angle ϕ can take is $\pi/2$, which corresponds to particles coming back in their original direction. In the case of attractive potentials, different cases arise depending on the value of γ:

- For $0 < \gamma < 2$, the leading order value of ϕ for $b/b_0 \ll 1$ is $\pi/(2-\gamma)$. A number n_{loops} of loops may appear in the trajectory, that can be calculated by using the formula:

$$n_{loops} = \text{floor}\left(\frac{1}{2-\gamma}\right). \tag{24}$$

  A typical trajectory for γ close to 2 is illustrated in Fig. **??**, with $\gamma = 1.95$ and $b/b_0 = 0.6\beta$, for which $n_{\text{loops}} = 12$.

- For $\gamma > 2$, we have formation of pairs for impact factors smaller than a critical one. For impact factors exactly at the critical one there is the phenomena of *orbiting*, in which the particles are trapped into a circular orbit. For larger impact factors the collision is well behaved. We illustrate this behavior in Fig. **??**.

The numerical checking for regularized potentials can be found in Section **??**.

3 Series expansions for pure power-law potentials

For the general case $\gamma \neq 1$, we do not expect to be able to derive an explicit expression through elementary functions for the angle ϕ as a function of b/b_0, as we did for $\gamma = 1$ in Eqs. (**??**) and (**??**). However, it is possible to express the integral (**??**) as a sum of a series. It is important to note that the angle ϕ is a function of the ratio of b and b_0 (see Eq. (**??**) in Appendix **??**). We will seek therefore for power series of $(b/b_0)^\sigma$ for some suitable σ, not necessarily integer. As a first step, we perform the substitution $r = r_{min}/x$, $0 < x \leqslant 1$, in Eq. (**??**), yielding

$$\phi(b/b_0) = \frac{b}{r_{min}} \int_0^1 \frac{dx}{\sqrt{1 - (bx/r_{min})^2 \mp (b_0 x/r_{min})^\gamma}}. \tag{25}$$

We recall that the "minus" sign in the denominator corresponds to a repulsive interaction while the "plus" sign to an attractive one. We will see that it is necessary to use two power series: one valid for the weak scattering regime ($b \gg b_0$) and another one for the strong scattering regime ($b \ll b_0$). We will see that the radius of convergence of both series match, and therefore the solution is fully described by the two power series.

The main difficulty in finding the power series is that a naive Taylor expansion in b/b_0 in Eq. (**??**) does not work. We shall proceed by first identifying an appropriate small parameter, which we call generically δ, and make a first Taylor expansion in δ; then, we expand δ in terms of b/b_0 and substitute in the first expansion. In the following we detail the procedure for each case.

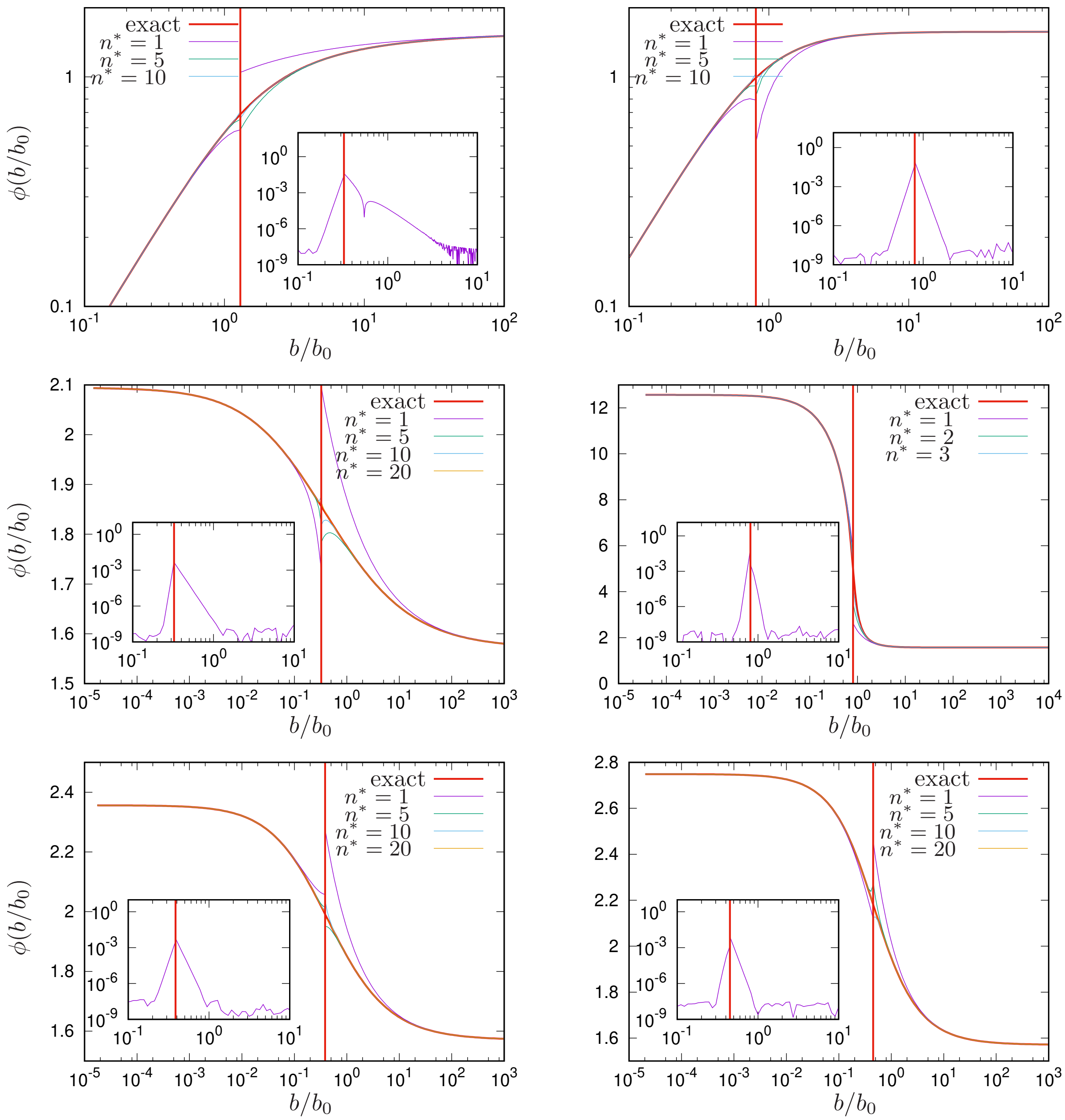

Figure 2: Top: repulsive interaction, with $\gamma = 1/2$ (left) and $\gamma = 7/4$ (right). Middle: attractive interaction with the same values of γ. Bottom: two "exceptional" attractive cases (see Eq. (**??**)), with $\gamma = 2/3$ (left) and $\gamma = 6/7$ (right). The integer n^* corresponds to the number of terms summed in the first series, the number of terms summed in the second series is chosen such that the final exponents are as close as possible. Inset: relative error for maximal n^*.

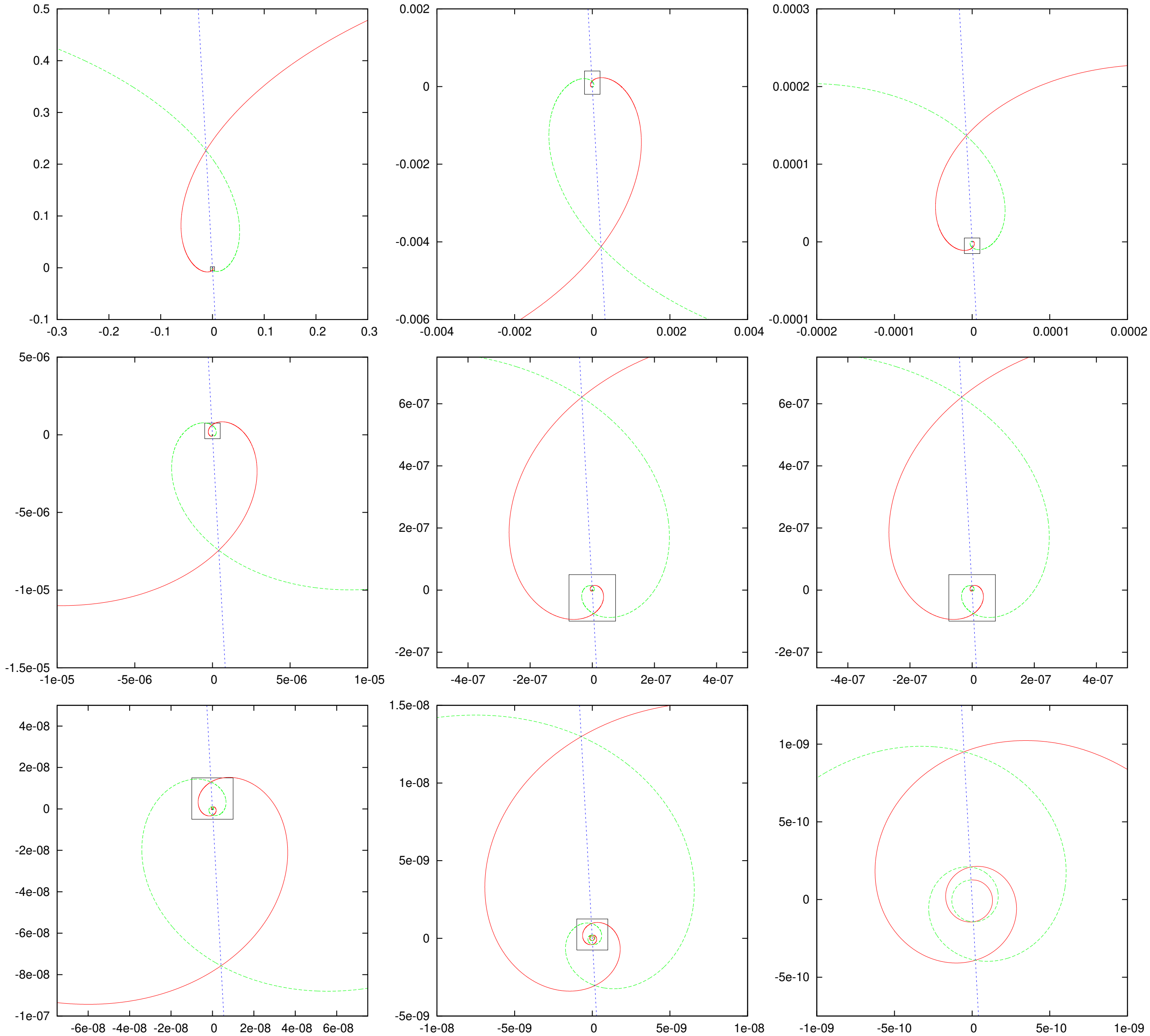

Figure 3: Near-collision in the center of mass frame for $\gamma = 1.95$ and $b/b_0 = 0.6\beta$. The dotted line is the axis of symmetry of the trajectory. The square in each plot represents the frame of the next plot (which have to be read from left to right and top to down). The first half part of the trajectory — from $x = +\infty$ to the axis of symmetry — is plotted in red, the other half of the trajectory in green. The points of intersection of the trajectory lie on the axis of symmetry.

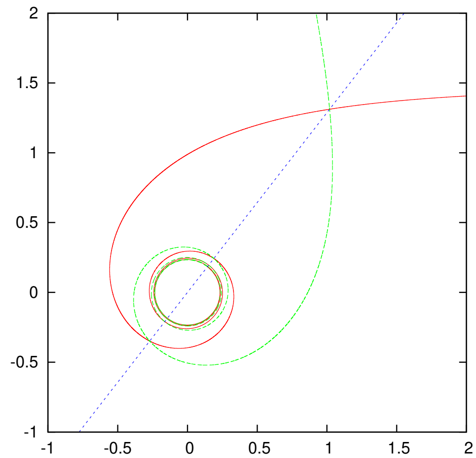

Figure 4: A trajectory in the center of mass frame for attractive interaction $\gamma = 2.05$ and $b/b_0 = \beta + 1.4 \times 10^{-6}$ (only a portion of the trajectory is plotted). The first half part of the trajectory — from $x = +\infty$ to the axis of symmetry — is plotted in red, the other half of the trajectory in green. The points of intersection of the trajectory lie on the axis of symmetry.

3.1 The regime of *soft* collisions for attractive and repulsive interactions (proof of Theorem ?? (i) and ?? (i))

The regime of *soft* collisions corresponds to the case in which the scale b_0 is small compared to the impact factor b. In this regime the trajectories of the particles are weakly perturbed. In this Subsection, γ is any positive number. In this case the appropriate small parameter is

$$\delta = (b_0/r_{min})^\gamma = \mp[(b/r_{min})^2 - 1].$$

From Eq. (??) we obtain

$$\frac{r_{min}}{b}\phi(b/b_0) = \int_0^1 \frac{dx}{\sqrt{1 - x^2 \mp \delta(x^\gamma - x^2)}}.$$

We proceed in two steps: we first prove that ϕ is a power series in $(b_0/b)^\gamma$ for b sufficiently large, and then identify the coefficients in the expansion.

We want an expansion of the above integral using that δ is a small parameter. It is then natural to write it under the form

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1 \mp \delta\frac{x^\gamma - x^2}{1-x^2}}}$$

and to expand the second square root in power series. This is possible since the expression $(x^\gamma - x^2)/(1-x^2)$ is bounded on $[0,1]$ (for $\gamma > 0$) and this implies that $(r_{min}/b)\phi(b/b_0)$ is actually a power series in δ. Moreover, since

$$\frac{b}{r_{min}} = \sqrt{1 \pm (b_0/r_{min})^\gamma} = \sqrt{1 \pm (b_0/b)^\gamma (b/r_{min})^\gamma},$$

it is easy to show that b/r_{min}, thus also $\delta = \pm((b/r_{min})^2 - 1)$, is itself a power series of the variable $(b_0/b)^\gamma$ (with positive radius). By substitution and Cauchy product, ϕ is a power series in $(b_0/b)^\gamma$ for b sufficiently large, that is there exists some coefficients $\kappa_n(\gamma)$, $n \in \mathbb{N}$, such that, for b large enough,

$$\phi = \sum_{n=0}^{+\infty} \kappa_n(\gamma)(b_0/b)^{\gamma n}.$$

In addition, from the above computation, we know that each coefficient $\kappa_n(\gamma)$ is a finite sum of the type

$$\sum_{k=0}^{n} C(n,k) \int_0^1 \left(\frac{x^\gamma - x^2}{1-x^2}\right)^k \frac{dx}{\sqrt{1-x^2}},$$

the integrals coming from the expansion of the integral $(r_{min}/b)\phi$ in powers of δ, and the coefficients $C(n,k)$ of the Cauchy products and the substitution. In particular, each coefficient $\kappa_n(\gamma)$ is an analytic function of γ in $(0,+\infty)$ (and even in the half-space $\{\mathrm{Re} > 0\}$).

We now identify the coefficients $\kappa_n(\gamma)$ by considering the two expansions valid for $\gamma > 2$ and b large,

$$\phi = \sqrt{\pi} \sum_{n=0}^{+\infty} \frac{\Gamma((n\gamma+1)/2)}{2n!\Gamma(1+n(\gamma/2-1))} (\mp(b_0/b)^\gamma)^n = \sum_{n=0}^{+\infty} \kappa_n(\gamma)(b_0/b)^{\gamma n},$$

where the first equality, valid for $\gamma > 2$, comes from [?]. By uniqueness of the power series expansions, we deduce that if $\gamma > 2$, then for all $n \in \mathbb{N}$,

$$\kappa_n(\gamma) = (\mp 1)^n \sqrt{\pi} \frac{\Gamma((n\gamma+1)/2)}{2n!\Gamma(1+n(\gamma/2-1))}. \tag{26}$$

Since κ_n is an analytic function in $(0,+\infty)$ and both $\gamma \mapsto \Gamma((n\gamma+1)/2)$ and $\gamma \mapsto 1/\Gamma(1+n(\gamma/2-1))$ are analytic in $(0,+\infty)$, we deduce from the principle of permanence for analytic functions that Eq. (**??**) holds true for any $\gamma > 0$.

We may now compute the radius of convergence. If $\gamma > 2$, this has been carried out in [?] using the generalized Stirling formula $\Gamma(s+1) \approx (s/\mathrm{e})^s\sqrt{2\pi s}$ when $s \to +\infty$, showing the convergence of the series for $b_0/b < 1/\beta$. The generalization to $\gamma \leqslant 2$ follows from the same type of computations, combined with Euler's reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$. This proves Eq. (**??**) and Eq. (**??**).

3.2 The regime of *hard* collisions for repulsive interactions (proof of Theorem ?? *(ii)*)

This corresponds to the minus sign in Eq. (**??**). In this Subsection again, γ is any positive number. It is then easy to check that $r_{min} \approx b_0$ for small b/b_0. In this case the appropriate small parameter δ is

$$\delta = (b/r_{min})^2 \sim (b/b_0)^2 \ll 1.$$

Substituting $(b_0/r_{min})^\gamma = 1-\delta$ in Eq. (**??**), we obtain the expression

$$\phi(b/b_0) = \sqrt{\delta} \int_0^1 \frac{dx}{\sqrt{1-x^\gamma+\delta(x^\gamma-x^2)}},$$

Since the quantity $(x^\gamma - x^2)/(1-x^\gamma)$ is bounded on $[0,1]$, the above integral is here again a power series in δ:

$$\int_0^1 \frac{dx}{\sqrt{1-x^\gamma+\delta(x^\gamma-x^2)}} = \sum_{p=0}^{+\infty} \binom{-1/2}{p} \delta^p I_p, \tag{27}$$

where the integrals

$$I_p = \int_0^1 \frac{(x^\gamma-x^2)^p}{(1-x^\gamma)^{p+1/2}}\, dx,$$

$p \in \mathbb{N}$, may be expressed, after using the substitution $x^\gamma = \cos^2(\vartheta)$, with the help of the Γ function:

$$I_0 = \frac{\sqrt{\pi}\Gamma(1+1/\gamma)}{\Gamma(1/2+1/\gamma)}, \qquad I_1 = \frac{2\Gamma(1+3/\gamma)}{3\Gamma(-1/2+3/\gamma)} - \frac{2\Gamma(1+1/\gamma)}{\gamma\Gamma(1/2+1/\gamma)}, \qquad \text{etc.}$$

Furthermore, by definition of r_{min}, we have

$$\sqrt{\delta} = \frac{b}{b_0}(1-\delta)^{1/\gamma}. \tag{28}$$

This implicit relation provides $\sqrt{\delta}$ as the sum of a power series in b/b_0, i.e.,

$$\sqrt{\delta} = \sum_{n=0}^{\infty} \lambda_n (b/b_0)^{2n+1}. \tag{29}$$

The coefficients λ_n can be calculated inserting Eq. (**??**) in Eq. (**??**). We claim therefore that there exists some coefficients α_n ($n \in \mathbb{N}$), depending only on n and γ, such that

$$\phi(b/b_0) = \sum_{n=0}^{+\infty} \alpha_n (b/b_0)^{2n+1}, \tag{30}$$

with

$$\alpha_n = \frac{(-1)^n \sqrt{\pi}}{(2n+1)n!} \frac{\Gamma\left(1+(2n+1)/\gamma\right)}{\Gamma\left(1/2-n+(2n+1)/\gamma\right)}.$$

The form of α_n can be verified by calculating the coefficients λ_n and inserting Eq. (**??**) in Eq. (**??**). The computation of the convergence radius of this series follows from straightforward computations involving, as in [**?**], the generalized Stirling formula and (for $\gamma > 2$) Euler's reflection formula. This proves Eq. (**??**).

3.3 The regime of *hard* collisions for attractive interactions (proof of Theorem ?? (*ii*), (*iii*) and (*iv*))

We focus now on the plus sign in Eq. (**??**) in the regime $b \ll b_0$. As we shall see, the situation is drastically different since the qualitative behavior strongly depends on γ. In this section, we wish to give, for $b \ll b_0$, a series expansion of ϕ analoguous to (**??**). For this regime, we shall consider the small parameter $\delta = (r_{min}/b)^2 \ll 1$ and substitute $(b_0/r_{min})^\gamma = \delta^{-1} - 1$ in Eq. (**??**) to obtain the expression

$$\phi(b/b_0) = \int_0^1 \frac{dx}{\sqrt{x^\gamma - x^2 + \delta(1-x^\gamma)}}, \tag{31}$$

which tends, as $\delta \to 0$, to $\int_0^1 (x^\gamma - x^2)^{-1/2} dx$, which is finite only for $0 < \gamma < 2$. This already leads us to study separately the cases $\gamma < 2$ and $\gamma \geqslant 2$ separately.

3.3.1 The case $0 < \gamma < 2$, γ unexceptional (*ii*)

We use the change of variables $y = x\delta^{-1/\gamma}$ in Eq. (**??**), which is adapted to our problem, to deduce

$$\phi(b/b_0) = \delta^{\frac{2-\gamma}{2\gamma}} \int_0^{\delta^{-1/\gamma}} \frac{dy}{\sqrt{y^\gamma - \delta^{2/\gamma-1} y^2 + 1 - \delta y^\gamma}}.$$

The idea is now to expand the integrand in power series in $\delta^{2/\gamma-1}$, arguing as in sections **??** and **??**. We obtain

$$\delta^{-\frac{2-\gamma}{2\gamma}} \phi(b/b_0) = \int_0^{\delta^{-1/\gamma}} \frac{dy}{\sqrt{y^\gamma + 1 - \delta y^\gamma}} (1 - \delta^{2/\gamma-1} H_1(y))^{-1/2},$$

with $H_1(y) = y^2/(y^\gamma + 1 - \delta y^\gamma)$. It is elementary to prove that $\delta^{2/\gamma-1} H_1(y)$ is increasing in y from 0 to 1, hence we may Taylor expand

$$\delta^{-\frac{2-\gamma}{2\gamma}} \phi(b/b_0) = \sum_{n=0}^{+\infty} (-\delta^{2/\gamma-1})^n \binom{-1/2}{n} \int_0^{\delta^{-1/\gamma}} \frac{y^{2n} dy}{(y^\gamma + 1 - \delta y^\gamma)^{n+1/2}},$$

where $\binom{-1/2}{n} = (\prod_{j=0}^{n-1}(-1/2+j))/n!$. The integral may be expressed through the hypergeometric function (see [**?**], chapter 15) ${}_2F_1 = F$:

$$\begin{aligned}\int_0^{\delta^{-1/\gamma}} \frac{y^{2n}dy}{(y^\gamma+1-\delta y^\gamma)^{n+1/2}} &= \frac{\delta^{-\frac{2n+1}{\gamma}}}{2n+1} F\left(n+1/2, \frac{2n+1}{\gamma}, \frac{2n+1}{\gamma}+1, 1-1/\delta\right),\\ &= \frac{\delta^{-\frac{2n+1}{\gamma}}}{2n+1}\delta^{n+1/2}\frac{2}{2-\gamma}F(n+1/2,1,1-(n+1/2)(2-\gamma)/\gamma,\delta)\\ &\quad+\frac{\delta^{-\frac{2n+1}{\gamma}}}{2n+1}\delta^{\frac{2n+1}{\gamma}}\frac{\Gamma((2n+1)/\gamma+1)\Gamma((n+1/2)(1-2/\gamma))}{\Gamma(n+1/2)}\\ &\quad\times F\left(\frac{2n+1}{\gamma}, 1+(n+1/2)(2-\gamma)/\gamma, 1+(n+1/2)(2-\gamma)/\gamma, \delta\right)\\ &= \frac{\delta^{-\frac{2n+1}{\gamma}}}{2n+1}\delta^{n+1/2}\frac{2}{2-\gamma}F(n+1/2,1,(n+1/2)(1-2/\gamma)+1,\delta)\\ &\quad+\frac{\Gamma((2n+1)/\gamma+1)\Gamma((n+1/2)(1-2/\gamma))}{(2n+1)\Gamma(n+1/2)}(1-\delta)^{-\frac{2n+1}{\gamma}},\end{aligned}$$

by using the functional relation 15.3.8 in [**?**] and the fact that $F(a,b,b,z) = (1-z)^{-a}$. These formulas hold when $\gamma/2$ is not of the form $(2k+1)/(2\ell+1)$ for some k, $\ell \in \mathbb{N}$ with $k < \ell$, since then $1-(n+1/2)(2-\gamma)/\gamma$ is never a nonpositive integer. This is precisely the unexceptional γ's. Reporting these expressions, we infer

$$\phi(b/b_0) = \phi_I(b/b_0) + \phi_{II}(b/b_0),$$

where

$$\phi_I(b/b_0) = \frac{2}{2-\gamma}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\binom{-1/2}{n}F(n+1/2,1,(n+1/2)(1-2/\gamma)+1,\delta)$$

and

$$\phi_{II}(b/b_0) = \delta^{\frac{2-\gamma}{2\gamma}}\sum_{n=0}^{+\infty}(-1)^n(\delta^{2/\gamma-1})^n\binom{-1/2}{n}(1-\delta)^{-\frac{2n+1}{\gamma}} \times \frac{\Gamma((2n+1)/\gamma+1)\Gamma((n+1/2)(1-2/\gamma))}{(2n+1)\Gamma(n+1/2)}.$$

In the series ϕ_{II}, we observe that, by definition of δ, we have $b/b_0 = \delta^{\frac{1}{\gamma}-\frac{1}{2}}(1-\delta)^{-1/\gamma}$, thus

$$\delta^{\frac{2-\gamma}{2\gamma}}(\delta^{2/\gamma-1})^n(1-\delta)^{-\frac{2n+1}{\gamma}} = (b/b_0)^{2n+1}.$$

By using Stirling's formula and the complement formula, we easily obtain

$$(-1)^n\binom{-1/2}{n}\frac{\Gamma((2n+1)/\gamma+1)\Gamma((n+1/2)(1-2/\gamma))}{(2n+1)\Gamma(n+1/2)} \approx -\frac{\beta^{-2n-1}}{\gamma n\sqrt{2-\gamma}\sin(\pi(n+1/2)(2-\gamma)/\gamma)},$$

where $\beta = (\gamma/2)^{1/\gamma}(2/\gamma-1)^{\frac{2-\gamma}{2\gamma}}$ (see Eq. (**??**)). Therefore,

$$\phi_{II}(b/b_0) = \sum_{n=0}^{+\infty}(-1)^n(b/b_0)^{2n+1}\binom{-1/2}{n} \times \frac{\Gamma((2n+1)/\gamma+1)\Gamma((n+1/2)(1-2/\gamma))}{(2n+1)\Gamma(n+1/2)},$$

which is a power series in b/b_0 of radius β.

Let us now turn to ϕ_I. The series is very slowly converging, since $\binom{-1/2}{n} \approx (-1)^{n+1}\sqrt{\pi/n}$. In particular, we know that $\phi(0^+) = a_0(\gamma) = \pi/(2-\gamma)$ and indeed

$$\frac{\pi}{2-\gamma} = \phi_I(0^+) + 0 = \frac{2}{2-\gamma}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\binom{-1/2}{n} = \frac{2}{2-\gamma}\arcsin(1),$$

but the remainder $\sum_{n=N+1}^{+\infty} \frac{(-1)^n}{2n+1}\binom{-1/2}{n}$ is of order $1/\sqrt{N}$. If we truncate the series ϕ_I, we then have a quite large error even on the zeroth order term. As a consequence, we shall try to give a power series expansion of ϕ_I in suitable powers of b/b_0. We claim that there exist numbers (depending on γ only) a_n, $n \in \mathbb{N}$, such that

$$\phi_I(b/b_0) = \sum_{n=0}^{+\infty} a_n (b/b_0)^{\frac{2\gamma}{2-\gamma}n}. \tag{32}$$

By expanding the hypergeometric function in power series, it is clear that $\phi_I(b/b_0)$ is a power series in δ with positive radius, namely

$$\phi_I(b/b_0) = \frac{2}{2-\gamma}\sum_{p=0}^{+\infty} A_p \delta^p \qquad \text{with} \quad A_p = \sum_{n=0}^{+\infty} d_{n,p},$$

where

$$d_{n,p} = \frac{(-1)^n}{2n+1}\binom{-1/2}{n}\frac{\Gamma(n+1/2+p)\Gamma((n+1/2)(1-2/\gamma)+1)}{\Gamma(n+1/2)\Gamma((n+1/2)(1-2/\gamma)+1+p)}.$$

Moreover, δ is related to b/b_0 through the formula $b/b_0 = \delta^{\frac{1}{\gamma}-\frac{1}{2}}(1-\delta)^{-1/\gamma}$, or

$$(b/b_0)^{\frac{2\gamma}{2-\gamma}} = \delta(1-\delta)^{-\frac{2}{2-\gamma}} = \delta + \frac{2}{2-\gamma}\delta^2 + \ldots = \sum_{k=0}^{+\infty}(-1)^k\binom{-\frac{2}{2-\gamma}}{k}\delta^{k+1}$$

hence, inverting this relation, we see that δ is a power series (with coefficients depending on γ only) of $(b/b_0)^{\frac{2\gamma}{2-\gamma}}$ with positive radius. The result Eq. (**??**) then follows by substitution.

We wish now to obtain an explicit expression for the coefficients a_n. The strategy is to equate the coefficients in the expansions in powers of δ in

$$\phi_I(b/b_0) = \sum_{n=0}^{+\infty}\left(\frac{2}{2-\gamma}\sum_{p=0}^{+\infty} d_{n,p}\delta^p\right) \qquad \text{and} \qquad \phi_I(b/b_0) = \sum_{p=0}^{+\infty} a_p\left(\sum_{k=0}^{+\infty}(-1)^k\binom{-\frac{2}{2-\gamma}}{k}\delta^k\right)^p,$$

and then solve the linear, upper triangular, system relating the a_p and A_p (through the coefficients $d_{n,p}$), the sum over n being performed at the end of the calculation. We have then obtained the formulas Eqs. (**??**), (**??**) and (**??**), thus proving Eq. (**??**).

3.3.2 The case $0 < \gamma < 2$, γ exceptional *(iii)*

It remains to study the case where $\gamma/2$ is of the form $\gamma_{k,\ell}/2 = \frac{2k+1}{2\ell+1}$, which we shall call exceptional. We then fix two integers k, ℓ with $0 \leqslant k < \ell$. The idea will be to pass to the limit in the formula given in Eq. (**??**) when γ unexceptional tends to $\gamma_{k,\ell}$. Notice that we may write Eq. (**??**), for γ unexceptional, under the form

$$\begin{aligned}\phi(b/b_0) = &\sum_{\substack{n\in\mathbb{N}\text{ s.t.}\\ 1+n\frac{2k+1}{k-\ell}\notin -2\mathbb{N}}} a_n(b/b_0)^{\frac{2\gamma}{2-\gamma}n} + \sum_{\substack{q\in\mathbb{N}\text{ s.t.}\\ (2q+1)\frac{k-\ell}{2k+1}\notin -\mathbb{N}}} c_q(b/b_0)^{2q+1} \\ &+ \sum_{\substack{n,q\in\mathbb{N}\text{ s.t.}\\ (2q+1)(\ell-k)=n(2k+1)}}\left(a_n(b/b_0)^{\frac{2\gamma}{2-\gamma}n} + c_q(b/b_0)^{2q+1}\right).\end{aligned}$$

Passing to the limit as $\gamma \to \gamma_{k,\ell} = \frac{2(2k+1)}{2\ell+1}$ in the first two sums is immediate, but we have to pay attention to the last sum since Γ is infinite at the non-positive integers. We fix some $n, q \in \mathbb{N}$ such that $(2q+1)(\ell-k) = n(2k+1)$ (hence $n \geqslant 1$) and denote

$$\sigma = n\frac{\gamma}{2-\gamma} - q - \frac{1}{2} \to 0,$$

so that $\gamma-\gamma_{k,\ell}=\sigma(2-\gamma_{k,\ell})^2/(2n)+\mathcal{O}(\sigma^2)$ as $\sigma\to 0$. It follows that

$$\begin{aligned}\frac{2q+1}{\gamma}+1&=\frac{2q+1}{\gamma_{k,\ell}}+1-\sigma(2q+1)\frac{(2-\gamma_{k,\ell})^2}{2n\gamma_{k,\ell}^2}+\mathcal{O}(\sigma^2)\\&=q+1+n+\frac{1}{2}-\sigma(2q+1)\frac{(2-\gamma_{k,\ell})^2}{2n\gamma_{k,\ell}^2}+\mathcal{O}(\sigma^2)\\&=q+n+\frac{3}{2}-\sigma\frac{2n}{2q+1}+\mathcal{O}(\sigma^2),\end{aligned}$$

since $2n/(2q+1)=(\ell-k)/(2k+1)=(2-\gamma_{k,\ell})/\gamma_{k,\ell}$, and that

$$(q+1/2)(1-2/\gamma)=-n+\sigma\frac{2-\gamma}{\gamma}=-n+\sigma\frac{2-\gamma_{k,\ell}}{\gamma_{k,\ell}}-\sigma^2\frac{(2-\gamma_{k,\ell})^2}{n\gamma_{k,\ell}^2}+\mathcal{O}(\sigma^3).$$

Then, by using the formula, for $m\in\mathbb{N}$ and $z\to -m$,

$$\Gamma(z)=\frac{(-1)^m}{m!}\left(\frac{1}{z+m}+(H_m-\gamma_0)+\mathcal{O}(z+m)\right),$$

where $H_m=\sum_{j=1}^m 1/j$ and $\gamma_0=\lim_{m\to+\infty}(H_m-\ln m)$ is Euler's constant, we deduce

$$\begin{aligned}c_q&=\frac{\Gamma((2q+1)/\gamma+1)\Gamma((q+1/2)(1-2/\gamma))}{\sqrt{\pi}(2q+1)q!}\\&=(-1)^n\frac{\Gamma(q+n+3/2)-\sigma\frac{2n}{2q+1}\Gamma'(q+n+3/2)+\mathcal{O}(\sigma^2)}{\sqrt{\pi}(2q+1)q!n!}\\&\quad\times\left(\frac{1}{\sigma(2-\gamma_{k,\ell})/\gamma_{k,\ell}-\sigma^2\frac{(2-\gamma_{k,\ell})^2}{n\gamma_{k,\ell}^2}}+(H_n-\gamma_0)+\mathcal{O}(\sigma)\right)\\&=\frac{(-1)^n\Gamma(q+n+3/2)}{\sqrt{\pi}(2q+1)q!n!\sigma}\frac{\gamma_{k,\ell}}{2-\gamma_{k,\ell}}+\mathcal{O}(1).\end{aligned}\tag{33}$$

Moreover, for $n\neq 0$, we have $2n/(2-\gamma)=n+q+1/2+\sigma$, thus

$$\begin{aligned}a_n&=-\frac{\sqrt{\pi}\Gamma\left(\frac{\gamma n}{\gamma-2}+\frac{1}{2}\right)}{2n\Gamma\left(\frac{2n}{\gamma-2}\right)n!}\\&=-\frac{\sqrt{\pi}\Gamma(-q-\sigma)}{2n\Gamma(-n-q-1/2-\sigma)n!}\\&=\frac{\sqrt{\pi}(1+\sigma\Gamma'(-n-q-1/2)/\Gamma(-n-q-1/2)+\mathcal{O}(\sigma^2))}{2n\Gamma(-n-q-1/2)n!}\frac{(-1)^q}{q!\sigma}\left(1-(H_q-\gamma_0)\sigma+\mathcal{O}(\sigma)\right)\\&=\frac{\sqrt{\pi}(-1)^q}{2n\Gamma(-n-q-1/2)n!q!\sigma}+\mathcal{O}(1).\end{aligned}\tag{34}$$

We may then check that the two singular terms in a_n and c_q cancel out when $\sigma\to 0$. Indeed, using the reflection formula, we infer

$$\frac{\pi}{\Gamma(-q-n-1/2)}=(-1)^{q+n+1}\Gamma(q+n+3/2),$$

and combining this with the fact that $2n/(2q+1)=(\ell-k)/(2k+1)=(2-\gamma_{k,\ell})/\gamma_{k,\ell}$, we deduce $a_n+c_q=\mathcal{O}(1)$ as $\sigma\to 0$. We shall now inspect the terms of order σ^0. First, we have

$$(b/b_0)^{2n\gamma/(2-\gamma)}=(b/b_0)^{2q+1+2\sigma}=(b/b_0)^{2q+1}+2\sigma(b/b_0)^{2q+1}\ln(b/b_0)+\mathcal{O}(\sigma^2).$$

Then, going back to Eq. (**??**) and Eq. (**??**), we obtain

$$\begin{aligned}
c_q &= \frac{(-1)^n\Gamma(q+n+3/2)}{\sqrt{\pi}(2q+1)q!n!\sigma}\frac{\gamma_{k,\ell}}{2-\gamma_{k,\ell}} - \frac{(-1)^n}{\sqrt{\pi}(2q+1)n!q!}\Gamma'(q+n+3/2) \\
&\quad + \frac{(-1)^n}{n!}\frac{\Gamma(q+n+3/2)}{\sqrt{\pi}(2q+1)q!}(H_n-\gamma_0)+\mathcal{O}(\sigma) \\
&= \frac{\sqrt{\pi}(-1)^{q+1}}{2n\Gamma(-n-q-1/2)n!q!\sigma} + \frac{(-1)^q\sqrt{\pi}}{(2q+1)n!q!\Gamma(-n-q-1/2)}\times\frac{\Gamma'(-q-n-1/2)}{\Gamma(-q-n-1/2)} \\
&\quad + \frac{\sqrt{\pi}(-1)^{q+1}}{2n\Gamma(-n-q-1/2)n!q!}\frac{2-\gamma_{k,\ell}}{\gamma_{k,\ell}}(H_n-\gamma_0+1/n)+\mathcal{O}(\sigma),
\end{aligned}$$

by using the reflection formula and its logarithmic derivative, and

$$a_n = \frac{\sqrt{\pi}(-1)^q}{2n\Gamma(-n-q-1/2)n!q!\sigma} + \frac{\sqrt{\pi}(-1)^q}{2n\Gamma(-n-q-1/2)n!q!}\left(\frac{\Gamma'(-n-q-1/2)}{\Gamma(-n-q-1/2)} - H_q + \gamma_0\right) + \mathcal{O}(\sigma).$$

Therefore, as $\sigma \to 0$,

$$\begin{aligned}
&a_n(b/b_0)^{\frac{2\gamma}{2-\gamma}n} + c_q(b/b_0)^{2q+1} \\
&= (b/b_0)^{2q+1}\Big(a_n + c_q + 2\sigma a_n\ln(b/b_0)+\mathcal{O}(\sigma)\Big) \\
&\to \frac{\sqrt{\pi}(-1)^q(b/b_0)^{2q+1}}{2n\Gamma(-n-q-1/2)n!q!} \\
&\quad\times\left(\frac{\Gamma'}{\Gamma}(-n-q-1/2)\frac{2}{\gamma_{k,\ell}} + \gamma_0 - H_q + \frac{2-\gamma_{k,\ell}}{\gamma_{k,\ell}}(\gamma_0 - H_n - 1/n) + 2\ln(b/b_0)\right).
\end{aligned}$$

This concludes in the exceptional cases.

3.3.3 $\gamma = 2$ *(iv)*

The case $\gamma = 2$ allows explicit computation and we see that it is a case where the attractive term is strong enough to form pairs when b is small. Of course, this will be also the case when $\gamma > 2$. Actually, when $\gamma = 2$, the behavior of the expression

$$W(r) = 1 - \frac{b^2}{r^2} + \frac{b_0^2}{r^2} = 1 - \frac{b^2-b_0^2}{r^2}$$

depends whether $b > b_0$ or $b < b_0$. If $b > b_0$, then W possesses $r_{min} = \sqrt{b^2-b_0^2}$ as unique positive zero, and we have the exact value

$$\phi(b/b_0) = \int_{r_{min}}^{+\infty}\frac{(b/r^2)\,dr}{\sqrt{1-r_{min}^2/r^2}} = \frac{b\pi}{2r_{min}} = \frac{\pi}{2\sqrt{1-b_0^2/b^2}}. \tag{35}$$

If $b \leqslant b_0$, then $W \geqslant 1$ has no zero. This means that the two particles will crash one onto the other in finite time with a spiraling motion. The integral in the right-hand side of Eq. (**??**) is then equal to $+\infty$, but the angle ϕ has then no geometrical meaning and the picture given in Fig. **??** is then no longer the good one. The parameter b_0 is then a threshold with the property that particles crash as soon as $b \leqslant b_0$.

3.3.4 $\gamma > 2$ *(iv)*

If $\gamma > 2$, the attractive term is strong enough to form pairs for sufficiently small b, and we shall explicit the threshold. Notice first that when $\gamma > 2$, the function $W(r) = 1 - b^2/r^2 + 2b_0^\gamma/r^\gamma$ decreases on $(0, r_*(b)]$ and increases on $[r_*(b), +\infty)$, with

$$r_*(b) = \left(\frac{\gamma b_0^\gamma}{2b^2}\right)^{\frac{1}{\gamma-2}}.$$

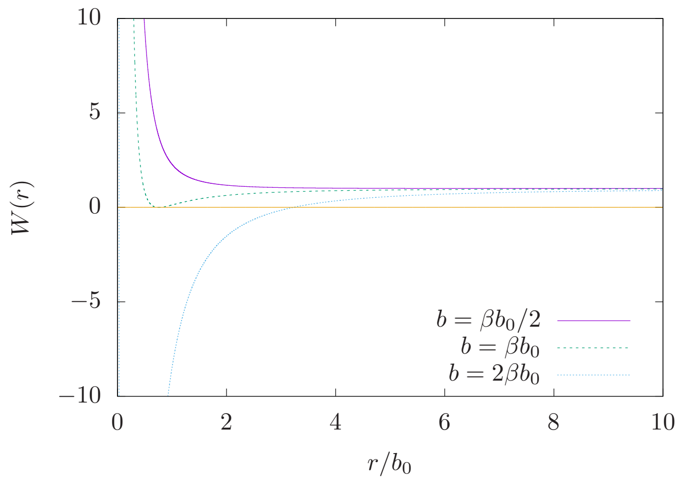

Figure 5: Graph of W as a function of r/b_0 for different values of b for $\gamma = 5/2$ for the attractive case. Observe that for $b = \beta b_0/2$ there is no root, $b = \beta b_0$ is the limiting case with a double root and for $b = 2\beta b_0$ there is one root.

Since $W(r_*(b)) = 1 - b^2/r_*^2(b) + b_0^\gamma/r_*^\gamma(b) = 1 - (b_0/b)^{-\frac{2\gamma}{\gamma-2}}[1-2/\gamma](\gamma/2)^{-\frac{2}{\gamma-2}}$, we may then easily check that if

$$b > \beta b_0, \tag{36}$$

where β is defined in (**??**), then W has a larger positive zero r_{min}, whereas if $b < \beta b_0$, the expression W is positive on $(0, +\infty)$, and if $b = \beta b_0$, the expression W has a double root at $r = r_*(\beta b_0) = b_0(\gamma/2-1)^{1/\gamma} > 0$, where $W(r_*(\beta b_0)) = 0$. These three behaviors are illustrated in Fig. **??**.

When $\gamma \to 2^+$, we have, as expected, $\beta = (\gamma/2)^{1/\gamma}(1-2/\gamma)^{\frac{2-\gamma}{2\gamma}} = (\gamma/2)^{1/\gamma}\exp((1/2)(1-2/\gamma)\ln(1-2/\gamma)) \to 1$, which is the threshold when $\gamma = 2$. If $b < \beta b_0$, the particles crash in finite time and ϕ has here again no physical or geometrical meaning, despite the fact that the integral

$$\int_0^{+\infty} \frac{(b/r^2)\,dr}{\sqrt{1 - b^2/r^2 + b_0^\gamma/r^\gamma}} = \int_0^{+\infty} \frac{dx}{\sqrt{1 - x^2 + (b_0/b)^\gamma x^\gamma}},$$

where r_{min} has been replaced by 0, converges.

When $b = \beta b_0$, the reduced particle remain asymptotically trapped on a circular orbit of radius $r_*(\beta b_0) > 0$. This phenomenon is called in the atomic physics literature *orbiting* (see e.g. [**?**]). The angle ϕ has once again no physical or geometrical meaning, and

$$\int_{r_*(\beta b_0)}^{+\infty} \frac{(b/r^2)\,dr}{\sqrt{1 - b^2/r^2 + b_0^\gamma/r^\gamma}} = +\infty$$

in view of the fact that $1 - b^2/r^2 + b_0^\gamma/r^\gamma \sim (r - r_*(\beta b_0))^2$ for r close to $r_*(\beta b_0)$.

Let us now consider the situation where we take $\gamma > 2$ and b slightly larger than βb_0, so that one expect a divergence in the integral ϕ. We have

$$\phi(b/b_0) = \int_{r_{min}}^{+\infty} \frac{b\,dr}{r^2\sqrt{W_b(r)}},$$

with $W_b(r) = 1 - b^2/r^2 + b_0^\gamma/r^\gamma$ (we have stressed the dependency on b since we are interested in the limit $b \to \beta b_0$). As b approaches βb_0, we have both $r_*(b) \to r_*(\beta b_0) = b_0(\gamma/2-1)^{1/\gamma} > 0$ ($r_*(b)$ is the minimum for W_b) and $r_{min} \to r_*(\beta b_0)$ (r_{min} is the largest zero of W_b). In the integral ϕ, the contributions for r close to $r_*(\beta b_0)$ will make the integral diverge since we shall have $W_b(r) \sim (r - r_*(\beta b_0))^2$ (we have a double root when $b = \beta b_0$), whereas the contributions for r much larger than $r_*(\beta b_0)$ will remain of order one. As a consequence, for any small length parameter $\ell > 0$, we have

$$\phi(b/b_0) \approx \int_{r_{min}}^{r_{min}+\ell} \frac{b\,dr}{r^2\sqrt{W_b(r)}},$$

and we may then replace $W_b(r)$ by its second order Taylor expansion near $r_*(b)$:

$$W_b(r) = W_b(r_*(b)) + (r - r_*(b))W_b'(r_*(b)) + \frac{1}{2}(r - r_*(b))^2 W_b''(r_*(b)) + \mathcal{O}((r - r_*(b))^3).$$

Since $W_b'(r_*(b)) = 0$ and

$$W_b''(r_*(b)) = \frac{\gamma(\gamma+1)b_0^\gamma}{r_*^{\gamma+2}(b)} - \frac{6b^2}{r_*^4(b)} \approx \frac{2(\gamma-2)b^2}{r_*(\beta b_0)^4} > 0, \tag{37}$$

this yields

$$\phi(b/b_0) \approx \int_{r_{min}}^{r_{min}+\ell} br^{-2}\,dr/\sqrt{W_b(r_*(b)) + (r - r_*(b))^2(W_b''(r_*(b))/2 + \mathcal{O}(r - r_*(b)))}.$$

We have $W_b(r_*(b)) < 0 < W_b''(r_*(b))$ with $W_b(r_*(b))$ small but $W_b''(r_*(b))$ of order one. The idea is then to use the substitution

$$z\sqrt{-W_b(r_*(b))} = (r - r_*(b))\sqrt{W_b''(r_*(b))/2 + \mathcal{O}(r - r_*(b))},$$

so that the expression in the square root in the integral becomes simply $-W_b(r_*(b))(z^2 - 1)$. This yields

$$\phi(b/b_0) \approx \frac{b}{\sqrt{-W_b(r_*(b))}} \int_1^{z_{max}} \frac{r(z)^{-2}\,dr/dz}{\sqrt{z^2-1}}\,dz, \tag{38}$$

where $z_{min} = 1$ and $z_{max} \approx Cte(\ell)/\sqrt{-W_b(r_*(b))} \gg 1$ are the corresponding values to r_{min} and $r_{min} + \ell$ in the z variable. The idea is now that, roughly speaking, $r(z) \approx r_*(b) \approx r_*(\beta b_0)$ and $dr/dz \approx \sqrt{-2W_b(r_*(b))/W_b''(r_*(b))}$, which implies

$$\phi(b/b_0) \approx \frac{b}{r_*(\beta b_0)^2}\sqrt{\frac{2}{W_b''(r_*(b))}} \int_1^{z_{max}} \frac{dz}{\sqrt{z^2-1}} \approx \sqrt{\frac{2b^2}{r_*(\beta b_0)^4 W_b''(r_*(\beta b_0))}}\ln(z_{max}) \approx -\frac{\ln|W_b(r_*(b))|}{2\sqrt{\gamma-2}}, \tag{39}$$

in view of Eq. (??) and the fact that $z_{max} \approx Cte(\ell)/\sqrt{-W_b(r_*(b))} \gg 1$. Finally, $W_b(r_*(b)) = 1 - (\beta b_0/b)^{-\frac{2\gamma}{\gamma-2}}$, and we end up with

$$\phi(b/b_0) \approx -\frac{\ln(1-\beta b_0/b)}{2\sqrt{\gamma-2}}. \tag{40}$$

For the sake of simplicity, we have included the mathematical details leading to Eq. (??) in Appendix ??.

4 Leading order expansions for hard regularized interactions

In this section we will present the details of the results for regularized hard interactions presented in Sect. ??, for the *Plummer potential* Eq. (??) and compact softening Eq. (??). A common feature for both of these potentials is that they fulfill the relation

$$v(r,\epsilon) = \frac{1}{\epsilon^\gamma}\mathcal{V}\left(\frac{r}{\epsilon}\right), \tag{41}$$

with

$$\mathcal{V}^{\mathrm{Pl}}(R) = \frac{1}{(R^2+1)^{\gamma/2}}$$

and

$$\mathcal{V}^{\mathrm{co}}(R) = \begin{cases} \dfrac{1}{R^\gamma} & \text{if } R \geqslant 1 \\ \mathrm{v}(R) & \text{if } 0 \leqslant R \leqslant 1. \end{cases}$$

We will show that the results presented below do not depend qualitatively on the explicit form of the regularization used. In what follow, we will study how the angle ϕ is modified by the regularization in the potential, first for repulsive interactions and then for attractive ones.

We recall the angle ϕ_ϵ corresponding to the regularized potential:

$$\phi_\epsilon(b, b_0) = \frac{b}{r_{min}} \int_0^1 \frac{dx}{\sqrt{1 - (\frac{bx}{r_{min}})^2 \pm \frac{b_0^\gamma}{\epsilon^\gamma}\mathcal{V}(\frac{r_{min}}{\epsilon x})}}. \tag{42}$$

4.1 Hard repulsive interactions with Plummer softening

Here $\mathcal{V}(R) = \mathcal{V}^{\mathrm{Pl}}(R) = (R^2+1)^{-\gamma/2}$. Then, the function $r \mapsto 1 - b^2/r^2 - b_0^\gamma/\left(r^2+\epsilon^2\right)^{\gamma/2}$ increases from $-\infty$ to 1 as r increases from 0^+ to $+\infty$, hence has a single positive zero r_{min}. It is easily checked that r_{min} is an increasing function of b and that the function $r \mapsto 1 - b_0^\gamma/\left(r^2+\epsilon^2\right)^{\gamma/2}$ possesses a positive zero if and only if $\epsilon < b_0$. Therefore, for small b,

$$r_{min} \approx r_0 = b_0\sqrt{1-(\epsilon/b_0)^2}$$

if $\epsilon/b_0 \leqslant 1$, and

$$r_{min} \approx \frac{b}{\sqrt{1-(\epsilon/b_0)^{-\gamma}}}$$

if $\epsilon/b_0 > 1$. This naturally leads us to distinguish the case $\epsilon < b_0$ and the case $\epsilon > b_0$.

4.1.1 The case $\epsilon < b_0$ (proof of Theorem ?? (i))

We assume $\epsilon/b_0 < 1$, so that $r_0 > 0$, $r_{min} = r_0(1+\mathcal{O}((b/b_0)^2))$, and consider here again the small parameter $\delta = (b/r_{min})^2 \ll 1$. Substituting

$$\frac{b_0^\gamma}{\epsilon^\gamma} = \frac{1 - b^2/r_{min}^2}{\mathcal{V}(r_{min}/\epsilon)} = \frac{1-\delta}{\mathcal{V}(r_{min}/\epsilon)}$$

yields

$$\phi_\epsilon(b,b_0) = \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{1-\delta x^2 - (1-\delta)\frac{\mathcal{V}(r_{min}/(\epsilon x))}{\mathcal{V}(r_{min}/\epsilon)}}} = \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{F(x, r_{min}/\epsilon) + \delta(1 - x^2 - F(x, r_{min}/\epsilon))}}, \tag{43}$$

where we have set

$$F(x, r_{min}/\epsilon) = 1 - \frac{\mathcal{V}(r_{min}/(\epsilon x))}{\mathcal{V}(r_{min}/\epsilon)}.$$

We prove in App. ?? that the function $x \mapsto \frac{1-x^2}{F(x, r_{min}/\epsilon)}$ is bounded on $[0,1]$ independently of b. This shows that we may apply the Taylor expansion in δ used in subsection ?? and write

$$\phi_\epsilon(b,b_0) = \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{F(x, r_{min}/\epsilon)}\sqrt{1+\delta(\frac{1-x^2}{F(x,r_{min}/\epsilon)}-1)}} = \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{F(x, r_{min}/\epsilon)}} + \mathcal{O}(\delta^{3/2}).$$

At this stage, since $r_{min} = r_0(1+\mathcal{O}((b/b_0)^2))$, one could legitimate the expansion

$$\int_0^1 \frac{dx}{\sqrt{F(x, r_{min}/\epsilon)}} = \int_0^1 \frac{dx}{\sqrt{F(x, r_0/\epsilon)}} + \mathcal{O}((b/b_0)^2).$$

Since $r_{min} = r_0(1+\mathcal{O}((b/b_0)^2))$, $\sqrt{\delta} = b/r_{min} = b/r_0(1+\mathcal{O}((b/b_0)^2))$, and thus, when $\epsilon/b_0 < 1$,

$$\phi_\epsilon^{\mathrm{Pl}}(b, b_0) = B_{\epsilon/b_0}^{\mathrm{Pl}}(\gamma)(b/b_0) + \mathcal{O}((b/b_0)^3), \tag{44}$$

where

$$B_{\epsilon/b_0}^{\mathrm{Pl}}(\gamma) = \frac{1}{\sqrt{1-(\epsilon/b_0)^2}} \int_0^1 \frac{dx}{\sqrt{1 - \dfrac{x^\gamma}{(1-(\epsilon/b_0)^2(1-x^2))^{\gamma/2}}}}. \tag{45}$$

Comparing Eq. (**??**) with the expression Eq. (**??**) of the angle of closest approach without softening, namely $\phi(b/b_0) = \alpha_1(\gamma)(b/b_0) + \mathcal{O}((b/b_0)^3) = -\frac{\sqrt{\pi}\Gamma(1+3/\gamma)}{3\Gamma(3/\gamma-1/2)}(b/b_0) + \mathcal{O}((b/b_0)^3)$, we observe that the linear dependence (at leading order) of ϕ with respect to b/b_0 is not modified, only the pre-factor changes. It is also easy to check that in the limit $\epsilon \to 0$ we have, as expected, $B_{\epsilon/b_0}^{\mathrm{Pl}}(\gamma) \to \alpha_1(\gamma)$. As expected, the new introduced scale is ϵ.

4.1.2 The case $\epsilon > b_0$ (proof of Theorem ?? (*iii*))

In the case $\epsilon > b_0$, we recall that, for b small,

$$r_{min} \approx b/\sqrt{1-(\epsilon/b_0)^{-\gamma}} \tag{46}$$

and that

$$\phi_\epsilon(b, b_0) = \frac{b}{r_{min}} \int_0^1 \frac{dx}{\sqrt{1-(bx/r_{min})^2 - (b_0^\gamma/\epsilon^\gamma)\mathcal{V}(r_{min}/(\epsilon x))}}.$$

Substituting $1 = b^2/r_{min}^2 + (\epsilon/b_0)^{-\gamma}\mathcal{V}(r_{min}/\epsilon)$ in the integral and considering the small parameter $\delta = r_{min}^2/\epsilon^2 \sim b^2/\epsilon^2$ gives

$$\phi_\epsilon(b, b_0) = \int_0^1 \frac{dx}{\sqrt{G_b(x)}},$$

where

$$G_b(x) = 1 - x^2 - \frac{r_{min}^2}{b^2(\epsilon/b_0)^\gamma}\left(\mathcal{V}(\sqrt{\delta}/x) - \mathcal{V}(\sqrt{\delta})\right).$$

In view of the fact that $r_{min}^2 \approx b^2/(1-(\epsilon/b_0)^{-\gamma})$ and $b_0 \leqslant \epsilon$, we expect

$$\phi_\epsilon(b, b_0) \approx \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

We also see that the situation is similar to the case studied in subsection **??**, but the dependency on the small parameter δ is more intricate. Actually, for the Plummer potential, we have $\mathcal{V}^{\mathrm{Pl}}(R) = (R^2+1)^{-\gamma/2}$, thus, for small R, $\mathcal{V}^{\mathrm{Pl}}(R) = 1 - \gamma R^2/2 + \mathcal{O}(R^4)$. Therefore, for fixed x and small δ, we obtain

$$G_b(x) = 1 - x^2 - \frac{\gamma\delta}{2((\epsilon/b_0)^\gamma - 1)}\left(\frac{1}{x^2} - 1\right) + \mathcal{O}(\delta^2),$$

which is a situation very similar to the case studied in subsection **??**, but unfortunately the function $x \mapsto (1/x^2-1)/(1-x^2) = -1/x^2$ being too singular near the origin, the power series expansion trick used there (see Subsect. **??**) breaks down.

We divide the correction $\phi_\epsilon(b, b_0) - \pi/2$ by δ and write it under the form

$$-\frac{1}{\delta}\left(\phi_\epsilon(b,b_0) - \frac{\pi}{2}\right) = \frac{r_{min}^2 b_0^\gamma \epsilon^2}{b^2}\int_0^1 g_\delta(x)\,dx \approx \frac{1}{(\epsilon/b_0)^\gamma - 1}\int_0^1 g_\delta(x)\,dx$$

by Eq. (**??**) and with

$$g_\delta(x) = \frac{\mathcal{V}(\sqrt{\delta}) - \mathcal{V}(\sqrt{\delta}/x)}{\delta\sqrt{G_b(x)}\sqrt{1-x^2}[\sqrt{G_b(x)} + \sqrt{1-x^2}]} \geqslant 0.$$

Clearly, as b/ϵ goes to 0, $\delta \ll 1$, $G_b(x) \approx 1 - x^2$ and we have

$$\int_0^1 g_\delta(x)\,dx \to \frac{\gamma}{4}\int_0^1 \frac{\frac{1}{x^2} - 1}{(1-x^2)^{3/2}}\,dx = +\infty,$$

due to the non integrable singularity at the origin. We shall prove that actually $\int_0^1 g_\delta(x)\,dx \sim \delta^{-1/2}$. As a first step, we get rid of the contribution for $1/2 \leqslant x \leqslant 1$. Indeed, $\int_0^1 g_\delta(x)\,dx \to +\infty$ whereas

$$\int_{1/2}^1 g_\delta(x)\,dx \to \frac{\gamma}{4}\int_{1/2}^1 \frac{\frac{1}{x^2} - 1}{(1-x^2)^{3/2}}\,dx < +\infty.$$

As a consequence, using the natural substitution $y = \sqrt{\delta}/x$,

$$\int_0^1 g_\delta(x)\,dx \approx \int_0^{1/2} g(x)\,dx = \frac{1}{\sqrt{\delta}}\int_{2\sqrt{\delta}}^{+\infty} \frac{\mathcal{V}(\sqrt{\delta}) - \mathcal{V}(y)}{D_b(y)}\,dy$$

where we have denoted

$$D_b(y) = y^2\sqrt{G_b\left(\frac{\sqrt{\delta}}{y}\right)\left(1 - \frac{\delta}{y^2}\right)}\left[\sqrt{G_b\left(\frac{\sqrt{\delta}}{y}\right)} + \sqrt{1 - \frac{\delta}{y^2}}\right].$$

When $\delta \to 0$, we have

$$G_b(\sqrt{\delta}/y) \to G^-_{\epsilon/b_0}(y) = 1 - \frac{1}{(\epsilon/b_0)^\gamma - 1}\left(\mathcal{V}(y) - \mathcal{V}(0)\right)$$

and one could rigorously justify that

$$\begin{aligned}\int_0^1 g_\delta(x)\,dx &\approx \frac{1}{\sqrt{\delta}}\int_0^{+\infty} \frac{\mathcal{V}(0) - \mathcal{V}(y)}{y^2\sqrt{G^-_{\epsilon/b_0}(y)}[\sqrt{G^-_{\epsilon/b_0}(y)} + 1]}\,dy \\ &= \frac{(\epsilon/b_0)^\gamma - 1}{\sqrt{\delta}}\int_0^{+\infty}\left(1 - \frac{1}{\sqrt{1 + \frac{1}{(\epsilon/b_0)^\gamma - 1}\left(\mathcal{V}(0) - \mathcal{V}(y)\right)}}\right)\frac{dy}{y^2}.\end{aligned}$$

The last integral is indeed convergent since: for large y, $\mathcal{V}(y) \to 0$, thus the integrand is $\sim 1/y^2$; for small y, $\mathcal{V}^{\rm Pl}(0) - \mathcal{V}^{\rm Pl}(y) = 1 - (1 + y^2)^{-\gamma/2} \approx \gamma/(2y^2)$, thus the integrand is continuous at the origin. It then follows that, for $b \ll \epsilon$:

$$\phi_\epsilon(b, b_0) = \frac{\pi}{2} - B^{\rm Pl}_{\epsilon/b_0}(\gamma) b/\epsilon + o(b/\epsilon), \tag{47}$$

with

$$\tilde{B}^{\rm Pl}_{\epsilon/b_0}(\gamma) = \frac{1}{\sqrt{1 - (\epsilon/b_0)^{-\gamma}}} \times \int_0^{+\infty}\left(1 - \frac{1}{\sqrt{1 + \frac{1}{(\epsilon/b_0)^\gamma - 1}\left(\mathcal{V}^{\rm Pl}(0) - \mathcal{V}^{\rm Pl}(y)\right)}}\right)\frac{dy}{y^2} > 0. \tag{48}$$

If $\epsilon \gg b_0$, we justify in Appendix **??** that

$$\tilde{B}^{\rm Pl}_{\epsilon/b_0}(\gamma) \approx (\epsilon/b_0)^{-\gamma}\sqrt{\pi}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{4\Gamma\left(\frac{\gamma}{2}\right)}. \tag{49}$$

We see here that, because $\epsilon \gg b_0$, the value of ϕ is completely different compared to the case $\epsilon \to 0$. As expected, in the limit $b \to 0$, $\phi \to \pi/2$, which means that the particle trajectory is unperturbed compared with the case without softening.

4.2 Hard repulsive interactions with compact softening

In this Subsection, we give the few modifications appearing in the asymptotic expansions when we consider a compact softening Eq. (**??**). The formula we shall obtain are qualitatively comparable to those in Subsect. **??** for the Plummer softening. The first step is to determine the asymptotic behavior of r_{min}, and here again, we shall distinguish the cases where ϵ/b_0 is small or large.

4.2.1 The case $\epsilon < b_0$ (proof of Theorem ?? (ii))

Assume that $\epsilon < b_0$. Then, the function $r \mapsto 1-b^2/r^2-b_0^\gamma/r^\gamma$ is increasing on $[\epsilon, +\infty)$ and $1-b^2/\epsilon^2-b_0^\gamma/\epsilon^\gamma < 0$ for $b/\epsilon \ll 1$. It follows that this function has a unique zero r_{min} on $[\epsilon, +\infty)$, which satisfies, for $b/b_0 \ll 1$,

$$r_{min} \approx b_0 > \epsilon.$$

In view of the fact that $r_{min} \approx b_0 > \epsilon$, the trajectory never enters into the region $\{r \leqslant \epsilon\}$ where the softening has an effect, hence we obtain the same asymptotics as in the case without softening (see Eq. (**??**)):

$$\phi_\epsilon(b, b_0) = \alpha_1(\gamma)(b/b_0) + \mathcal{O}((b/b_0)^3) = -\frac{\sqrt{\pi}\Gamma(1+3/\gamma)}{3\Gamma(3/\gamma - 1/2)}(b/b_0) + \mathcal{O}((b/b_0)^3). \tag{50}$$

4.2.2 The case $\epsilon > b_0(\max_{\mathbb{R}} \mathcal{V})^{1/\gamma}$ (proof of Theorem ?? (iii))

Assume now that $\epsilon > b_0(\max_{\mathbb{R}} \mathcal{V})^{1/\gamma}$, that is $(\epsilon/b_0)^\gamma > \max_{\mathbb{R}} \mathcal{V} = \max_{[0,1]} \mathcal{V} \geqslant 1$. The function $r \mapsto 1 - b^2/r^2 - b_0^\gamma/r^\gamma$ is then increasing on $[\epsilon, +\infty)$ from $1 - b^2/\epsilon^2 - (\epsilon/b_0)^{-\gamma}$ to 1. Since $\epsilon/b_0 > 1$, we have, for $b \ll \epsilon$, $1 - b^2/\epsilon^2 - (\epsilon/b_0)^{-\gamma} \approx 1 - (\epsilon/b_0)^{-\gamma} > 0$, hence $1 - b^2/r^2 - b_0^\gamma/r^\gamma$ is positive on $[\epsilon, +\infty)$. On $[0, \epsilon]$, the function $r \mapsto 1 - b^2/r^2 - (\epsilon/b_0)^{-\gamma}\mathcal{V}(r/\epsilon)$ is > 0 for $r = \epsilon$ and tends to $-\infty$ for $r \to 0$, thus has a largest root $r_{min} \leqslant \epsilon$. Moreover, since $b^2/r_{min}^2 = 1 - \mathcal{V}(r_{min}/\epsilon)(\epsilon/b_0)^{-\gamma} \geqslant 1 - (\epsilon/b_0)^{-\gamma} \max_{\mathbb{R}} \mathcal{V} > 0$ by our hypothesis, we have $r_{min} \lesssim b \ll \epsilon$, hence

$$r_{min} = \frac{b}{\sqrt{1 - \mathcal{V}(r_{min}/\epsilon)(\epsilon/b_0)^{-\gamma}}} \approx \frac{b}{\sqrt{1 - \mathcal{V}(0)(\epsilon/b_0)^{-\gamma}}}$$

that is close to Eq. (**??**). We may then carry out computations very similar to those leading to Eq. (**??**), provided v is $\mathcal{C}^2$ on $[0, 1]$, positive on $(0, 1]$ and $\mathrm{v}'(0) = 0$. This yields

$$\phi_\epsilon(b, b_0) = \frac{\pi}{2} - \tilde{B}^{\rm co}_{\epsilon/b_0}(\gamma)b/\epsilon + o(b/\epsilon), \tag{51}$$

with

$$\tilde{B}^{\rm co}_{\epsilon/b_0}(\gamma) = \frac{1}{\sqrt{1 - \mathcal{V}^{\rm co}(0)(\epsilon/b_0)^{-\gamma}}}\int_0^{+\infty}\left(1 - \frac{1}{\sqrt{1 + \frac{1}{(\epsilon/b_0)^\gamma - \mathcal{V}^{\rm co}(0)}\left(\mathcal{V}^{\rm co}(0) - \mathcal{V}^{\rm co}(y)\right)}}\right)\frac{dy}{y^2}. \tag{52}$$

Here, we do not claim that $B^{\rm co}_{\epsilon/b_0}(\gamma)$ is a positive constant. For instance, if $\mathrm{v}(0) = 0$, then $B^{\rm co}_{\epsilon/b_0}(\gamma) < 0$, whereas if $\mathrm{v}(x) = 1$ on $[0, 1]$, then $B^{\rm co}_{\epsilon/b_0}(\gamma) > 0$. For a general function v on $[0, 1]$, it may happen exceptionally that $B^{\rm co}_{\epsilon/b_0}(\gamma)$ vanishes, and in this case, the correction $\phi_\epsilon - \pi/2$ is not of order b/ϵ but smaller. This however does not happen for generic functions v.

4.3 Hard attractive interactions with a softening (proof of Theorem ??)

The function $r \mapsto 1 - b^2/r^2 + (\epsilon/b_0)^{-\gamma}\mathcal{V}(r/\epsilon)$ tends to 1 at infinity and to $-\infty$ at 0^+, hence possesses a largest zero r_{min}, but there may exist several zeros in general. Since $1 \leqslant 1 + (\epsilon/b_0)^{-\gamma}\mathcal{V}(r_{min}/\epsilon) = b^2/r_{min}^2$, we must have $r_{min} \leqslant b \ll \epsilon$, and this in turn implies, independently whether ϵ/b_0 is small or not,

$$r_{min} \approx \frac{b}{\sqrt{1+\mathcal{V}(0)(\epsilon/b_0)^{-\gamma}}} \tag{53}$$

(whereas without softening, we had $r_{min} \sim b^{2/(2-\gamma)}$).

Our small parameter here will be $\delta = r_{min}^2/\epsilon^2 \ll 1$ (by Eq. (**??**)). Substituting $1 = b^2/r_{min}^2 - (\epsilon/b_0)^{-\gamma}\mathcal{V}(r_{min}/\epsilon)$ in the integral gives

$$\phi_\epsilon(b, b_0) = \int_0^1 \frac{dx}{\sqrt{G_b(x)}},$$

where

$$G_b(x) = 1 - x^2 + \frac{r_{min}^2}{b^2(\epsilon/b_0)^\gamma}\left(\mathcal{V}(\sqrt{\delta}/x) - \mathcal{V}(\sqrt{\delta})\right).$$

Comparing with § **??**, the only difference is a change of sign. Therefore, similar computations to those in that paragraph yield

$$\phi_\epsilon(b, b_0) = \frac{\pi}{2} + C_{\epsilon/b_0}(\gamma)b/\epsilon + o(b/\epsilon), \tag{54}$$

where

$$C_{\epsilon/b_0}(\gamma) = \frac{1}{\sqrt{1+\mathcal{V}(0)(\epsilon/b_0)^{-\gamma}}}\int_0^{+\infty}\left(\frac{1}{\sqrt{1 - \frac{1}{(\epsilon/b_0)^\gamma + \mathcal{V}(0)}(\mathcal{V}(0) - \mathcal{V}(y))}} - 1\right)\frac{dy}{y^2}. \tag{55}$$

If $\epsilon \gg b_0$, we can show (as we have done for Eq. (**??**)) that

$$C_{\epsilon/b_0}(\gamma) \approx (\epsilon/b_0)^{-\gamma}\sqrt{\pi}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{4\Gamma\left(\frac{\gamma}{2}\right)}. \tag{56}$$

On the other hand, if $\gamma < 2$ and $\epsilon \ll b_0$, we can show that

$$C_{\epsilon/b_0}(\gamma) \approx \frac{(\epsilon/b_0)^{\gamma/2}}{\sqrt{\mathcal{V}(0)}}\int_0^{+\infty}\left(\sqrt{\frac{\mathcal{V}(0)}{\mathcal{V}(y)}} - 1\right)\frac{dy}{y^2}.$$

We have then a big difference with the case of repulsive interactions studied in Sect. **??** (and also in Sect. **??**), where $\phi_\epsilon \sim b/\max(\epsilon, b_0)$, displaying the characteristic length ϵ or b_0 depending which one is the largest one. Here, for attractive interactions, only the softening characteristic length ϵ appears in the first order term $\phi_\epsilon - \pi/2 \sim b/\epsilon$ in Eq. (**??**).

4.4 Computation of a threshold in ϵ for attractive potentials with $\gamma > 2$ (proof of Proposition ??)

When $\gamma > 2$ and without softening in the potential (formally, $\epsilon = 0$), the deflection angle ϕ diverges logarithmically to $+\infty$ when $b > \beta b_0$ approaches βb_0 (Eq. (**??**)). This divergence is due to the fact that $r_* \approx R = b_0(2-\gamma)^{1/\gamma}$ becomes a double root of the function W in this limit. The first paragraph of this Subsection is devoted to the proof of the existence of some threshold $\epsilon_*(b_0, \gamma) > 0$, for the Plummer softening, such that if $\epsilon < \epsilon_*(b_0, \gamma)$, then the angle ϕ_ϵ still diverges for some specific value of b (depending on b_0, γ and ϵ), whereas for $\epsilon > \epsilon_*(b_0, \gamma)$, the angle ϕ_ϵ no longer diverges and is a smooth function of b/b_0 for all positive values of b/b_0. This means that in order to remove the divergence in ϕ, one has to use a sufficiently large softening ϵ. In the first case, the divergence is here again due to the existence, for some critical value b, of some positive double root in r for the function

$$W_{b,\epsilon}(r) = 1 - \frac{b^2}{r^2} + (\epsilon/b_0)^{-\gamma}\mathcal{V}\left(\frac{r}{\epsilon}\right),$$

which means that we have some jump for r_{min} for this critical value b, whereas for $\epsilon > \epsilon_*(b_0,\gamma)$, the function $W_{b,\epsilon}(r)$ has no double root. In the second paragraph we will discuss the case of the compact softening.

4.4.1 The case of a Plummer softening

We now consider the Plummer softening $\mathcal{V}(R) = \mathcal{V}^{\rm Pl}(R) = (1+R^2)^{-\gamma/2}$ and are interested in determining under which condition on ϵ the function $W_{b,\epsilon}$ has a unique zero r_{min} for any $b > 0$. We have

$$W'_{b,\epsilon}(r) = \frac{\gamma b_0^\gamma}{r^3}\left(\frac{2b^2}{\gamma b_0^\gamma} - \frac{r^4}{(r^2+\epsilon^2)^{\gamma/2+1}}\right)$$

and, denoting $r = \epsilon R$,

$$\frac{r^4}{(r^2+\epsilon^2)^{\gamma/2+1}} = \epsilon^{2-\gamma}\frac{R^4}{(R^2+1)^{\gamma/2+1}}.$$

The function $R \mapsto R^4/(R^2+1)^{\gamma/2+1}$ is increasing on $[0, R_{max}]$ and decreasing on $[R_{max}, +\infty)$ (recall $\gamma > 2$), where $R_{max} = \sqrt{4/(\gamma-2)}$; its maximal value is $M(\gamma) = 16(\gamma-2)^{\frac{\gamma}{2}-1}(\gamma+2)^{-\frac{\gamma}{2}-1}$. Therefore, when $2b^2/(\gamma b_0^\gamma) < \epsilon^{2-\gamma}M(\gamma)$ (case 1), the function $W_{b,\epsilon}$ is increasing on $(0, r_1]$, decreasing on $[r_1, r_2]$ and increasing on $[r_2, +\infty)$; when $2b^2/(\gamma b_0^\gamma) > \epsilon^{2-\gamma}M(\gamma)$ (case 2), the function $W_{b,\epsilon}$ is increasing on $(0, +\infty)$. The two critical points r_1 and r_2 merge for $2b^2/(\gamma b_0^\gamma) = \epsilon^{2-\gamma}M(\gamma)$, and we shall see that the threshold is determined by the sign of $W_{b,\epsilon}$ at this merging point $r_1 = r_2$.

Let us now fix $\epsilon > 0$. For b very small, we are in case 1 and the two positive roots r_1 and r_2 of the equation $2b^2/(\gamma b_0^\gamma) = r^4/(r^2+\epsilon^2)^{\gamma/2+1}$ are r_1 (very small) and r_2 (very large). The function $W_{b,\epsilon}$ has then a local minimum $W_{b,\epsilon}(r_2) \approx 1$. When b increase, $W_{b,\epsilon}$ decrease, the two critical points r_1 and r_2 merge when $2b^2/(\gamma b_0^\gamma) = \epsilon^{2-\gamma}M(\gamma)$, and for larger b, $W_{b,\epsilon}$ is increasing on $(0, +\infty)$.

Let us consider the special value of b_{crit} where $2b_{crit}^2/(\gamma b_0^\gamma) = \epsilon^{2-\gamma}M(\gamma)$, for which the two critical points r_1 and r_2 merge: $r_1 = r_2 = r_{crit} = \epsilon R_{max}$. If $W_{b_{crit},\epsilon}(r_{crit}) > 0$, then by monotonicity in b, for any $b > 0$, the function $W_{b_{crit},\epsilon}$ has a largest positive zero r_{min} which is never a double root. If now $W_{b_{crit},\epsilon}(r_{crit}) < 0$, then, still by monotonicity in b, for b smaller, but close to b_{crit}, $W_{b,\epsilon}$ has two critical points $0 < r_1 < r_2$ with $0 > W_{b,\epsilon}(r_1) > W_{b,\epsilon}(r_2)$. As b decreases, the critical value $W_{b,\epsilon}(r_2)$ will be zero for some particular value of $b = b_\sharp$ for which r_2 has become a double root of $W_{b_\sharp,\epsilon}$, yielding a logarithmic divergence in ϕ_ϵ. As a consequence, we simply need to determine the sign of

$$W_{b_{crit},\epsilon}(r_{crit}) = 1 - \frac{b_{crit}^2}{\epsilon^2 R_{max}^2} + \frac{b_0^\gamma}{(\epsilon^2 R_{max}^2+\epsilon^2)^{\gamma/2}} = 1 - \frac{\epsilon^{-\gamma}M(\gamma)\gamma b_0^\gamma}{2R_{max}^2} + \frac{b_0^\gamma \epsilon^{-\gamma}}{(R_{max}^2+1)^{\gamma/2}} = 1 - (\epsilon_*(b_0,\gamma)/\epsilon)^\gamma,$$

where the threshold is given by

$$\epsilon_*(b_0,\gamma) = b_0\left(\frac{\gamma-2}{\gamma+2}\right)^{\frac{1}{2}+\frac{1}{\gamma}}. \tag{57}$$

It follows that if $\epsilon > \epsilon_*(b_0,\gamma)$, then ϕ_ϵ is a smooth function of b (see Fig. **??**), whereas if $\epsilon < \epsilon_*(b_0,\gamma)$, then ϕ_ϵ diverges as b approaches some value $b_\sharp = b_\sharp(\epsilon)$ corresponding to the case where $W_{b,\epsilon}$ has zero as a local minimum. By computations very similar to those in Sect. **??**, we see that the divergence is indeed logarithmic. One may also check that if $\epsilon = \epsilon_*(b_0,\gamma)$, then ϕ_ϵ is a diverging function of b for some $b_\sharp = b_\sharp(\epsilon)$. In other words, in order to regularize the divergence in the case $\gamma > 2$, we have to use a sufficiently large softening parameter, namely $\epsilon > \epsilon_*(b_0,\gamma)$.

Let us finally consider the case $\gamma = 2$. Notice that formally, $\epsilon_*(b_0,\gamma) \to 0$ as $\gamma \to 2$, hence we may think that ϕ_ϵ is a smooth function of b for any $\epsilon > 0$, and this is indeed the case. Actually, in the case $\gamma = 2$, the function $R \mapsto R^4/(R^2+1)^2$ is increasing on $[0,+\infty)$, and tends to 1 at infinity. Therefore, either $b/b_0 < 1$ and then the function $W_{b,\epsilon}$ is increasing on $(0, r_1]$ and decreasing on $[r_1, +\infty)$; either $b/b_0 \geqslant 1$ and then the function $W_{b,\epsilon}$ is increasing on $(0,+\infty)$. In any case $W_{b,\epsilon}$ has a single zero r_{min} and we never have a double root. It follows that ϕ_ϵ is a smooth function of b.

4.4.2 The case of a compact softening

For a general compact softening $\mathcal{V} = \mathcal{V}^{\rm co}$, computations are much less explicit. We first have

$$W'_{b,\epsilon}(r=\epsilon R) = \frac{b_0^\gamma}{R^3\epsilon^{\gamma+1}}\left(\frac{2b^2\epsilon^{\gamma-2}}{b_0^\gamma} + R^3\mathcal{V}'(R)\right),$$

and we then need to know the behavior of the function $R \mapsto -R^3\mathcal{V}'(R)$, which certainly has a positive maximum $M = M(\mathrm{v})$ attained at some $0 < R_{max} \leqslant 1$ since $\gamma > 2$. If, for instance, the function $R \mapsto -R^3\mathcal{V}'(R)$ is, for some $0 \leqslant R_+ \leqslant R_{max}$, nonpositive on $[0, R_+]$, then increasing on $[R_+, R_{max}]$ and then decreasing on $[R_{max}, +\infty)$, the behavior is the same as the one previously described for the Plummer softening. Since

$$\begin{aligned} W_{b_{crit},\epsilon}(r_{crit}) &= 1 - \frac{b_0^\gamma M(\mathrm{v})}{2\epsilon^\gamma R_{max}^2} + \frac{b_0^\gamma}{\epsilon^\gamma}\mathcal{V}(R_{max}) = 1 - \frac{b_0^\gamma}{2\epsilon^\gamma}\left(\frac{M(\mathrm{v})}{R_{max}^2} - 2\mathcal{V}(R_{max})\right) \\ &= 1 + \frac{b_0^\gamma}{2\epsilon^\gamma}\left(R_{max}\mathcal{V}'(R_{max}) + 2\mathcal{V}(R_{max})\right), \end{aligned}$$

there exists a threshold if and only if $M(\mathrm{v})/R_{max}^2 = -R_{max}\mathcal{V}'(R_{max}) > 2\mathcal{V}(R_{max})$, in which case the threshold is given by

$$\epsilon_*^{\rm co}(b_0,\gamma) = b_0\left(\frac{M(\mathrm{v})}{2R_{max}^2} - \mathcal{V}(R_{max})\right)^{1/\gamma} \tag{58}$$

and otherwise, we never have a double root for $W_{b,\epsilon}$ hence no divergence in ϕ_ϵ. The example below illustrates the first case.

If $\gamma = 3$ and $\mathrm{v}(R) = 21R^2 - 35R^3 + 15R^4$ for $0 \leqslant R \leqslant 1$, then $R \mapsto -R^3\mathcal{V}'(R)$ is decreasing and negative on $[0, \approx 0.474]$, increasing on $[\approx 0.474, \approx 0.984]$ and decreasing on $[\approx 0.984, +\infty)$, hence has maximum value $M(\mathrm{v}) \approx 3.023$ attained at $R_{max} \approx 0.984$. Moreover, $M(\mathrm{v})/R_{max}^2 - 2\mathcal{V}(R_{max}) \approx 1.023 > 0$, thus the variations of $W_{b,\epsilon}$ are the same as for the Plummer softening, with a threshold given by

$$\epsilon_*(b_0,\gamma) = b_0\left(\frac{M(\mathrm{v})}{2R_{max}^2} - \mathcal{V}(R_{max})\right)^{1/3} \approx 0.855 b_0.$$

4.5 Summary of the results and numerical checking

We summarize in table **??** the results obtained in this section for the Plummer softening. We have shown that the effect of the softening does not depend strongly on the form of the softening, obtaining the same qualitative results for the two softening considered — Plummer and compact one. There is an exception for repulsive interactions and $\epsilon < b_0$, in which case the compact softening does not modify the trajectory of the particles because they do not reach the region in which the potential is regularized.

In the case of repulsive interactions, we have seen that two different behaviours are shown depending whether ϵ/b_0 is larger than 1 or not. In the case $\epsilon/b_0 < 1$, the softening does not modify strongly the angle ϕ: it behaves linearly for $b \ll b_0$, only its slope is modified with ϵ. In the case in which $\epsilon/b_0 > 1$, hard collisions are radically modified, obtaining $\lim_{b/b_0\to 0}\phi_\epsilon = \pi/2$. The change of behaviour occurs sharply at $\epsilon/b_0 = 1$ as we show in Fig. **??**, in which ϕ is plotted as a function of ϵ at fixed b, for some values of γ. The range of validity in b of the linear correction is given by the *largest* value of b_0 and ϵ. In Fig. **??** (top) we show the comparison between the numerical integration of ϕ_ϵ in Eq. (**??**) with the asymptotic predictions Eqs. (**??**) and (**??**). We see a very good matching between the curves.

repulsive potential			attractive potential
$\phi_\epsilon \sim b/b_0$ when $b \ll b_0$	if	$\epsilon/b_0 < 1$	$\phi_\epsilon - \pi/2 \sim b/\epsilon$ when $b \ll \epsilon$
$\phi_\epsilon - \pi/2 \sim -b/\epsilon$ when $b \ll \epsilon$	if	$\epsilon/b_0 > 1$	

Table 1: Summary of the expansions of the angle ϕ_ϵ with a Plummer softening in the potential for hard collisions

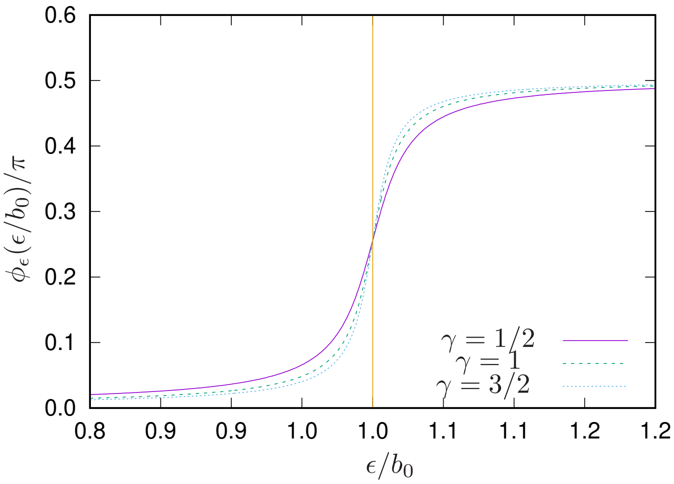

Figure 6: Value of ϕ for $b/b_0 = 10^{-2}$. The vertical line correspond to $\epsilon/b_0 = 1$.

For the case of attractive interactions, the range of validity in b of the linear correction is always given by ϵ. In Fig. **??** (bottom) we show a very good agreement matching between the exact integration Eq. (**??**) with the asymptotic predictions Eqs. (**??**) and (**??**).

We have also studied, for $\gamma > 2$, for which value of the softening, there is no formation of pairs for any value of b. We have seen that introducing a softening $\epsilon > 0$ but smaller than some critical one $\epsilon_*(b_0, \gamma)$, automatically regularizes the angle ϕ for any value b, except one, for which there is *orbiting*. If $\epsilon > \epsilon_*(b_0, \gamma)$ then the problem is completely regularized. In Fig. **??** we illustrate this behavior. The continuous red curve corresponds to the case in which $\epsilon > \epsilon_*(b_0, \gamma)$. In this case, ϕ_ϵ is a regular function of b, as it can be seen in the inset. The dashed green curve corresponds to the case in which $\epsilon < \epsilon_*(b_0, \gamma)$, for which ϕ_ϵ diverges for $b = b_\sharp(\epsilon)$, which is related to some jump for r_{min} at $b = b_\sharp(\epsilon)$.

5 Conclusions and applications

In this paper we have studied the scattering of two particles interacting with a central potential $v(r) \sim 1/r^\gamma$. This is a generalization of the Rutherford formula of the scattering of two particles interacting via a Coulomb or gravitational force. Unlike the original case, it is not possible to compute in general the deflection angle of the particles explicitly for general $\gamma \neq 1$. We have seen that the problem can be solved in form of power series, both for the attractive and repulsive case: one for the *weak scattering regime* ($b/b_0 > \beta$) and another one (or two) for the *hard scattering regime* ($b/b_0 < \beta$). We have also studied the case in which the exponent γ of the attractive potential is larger than 2, for which the angular momentum term cannot, in general, prevent the system to collapse and the particles crash. Studying the distance of closest approach r_{min} we have found two different behaviors whether γ is smaller or larger than 2:

- If $\gamma < 2$, in the limit $\gamma \to 2^-$ (for any b smaller than some critical value which we have calculated

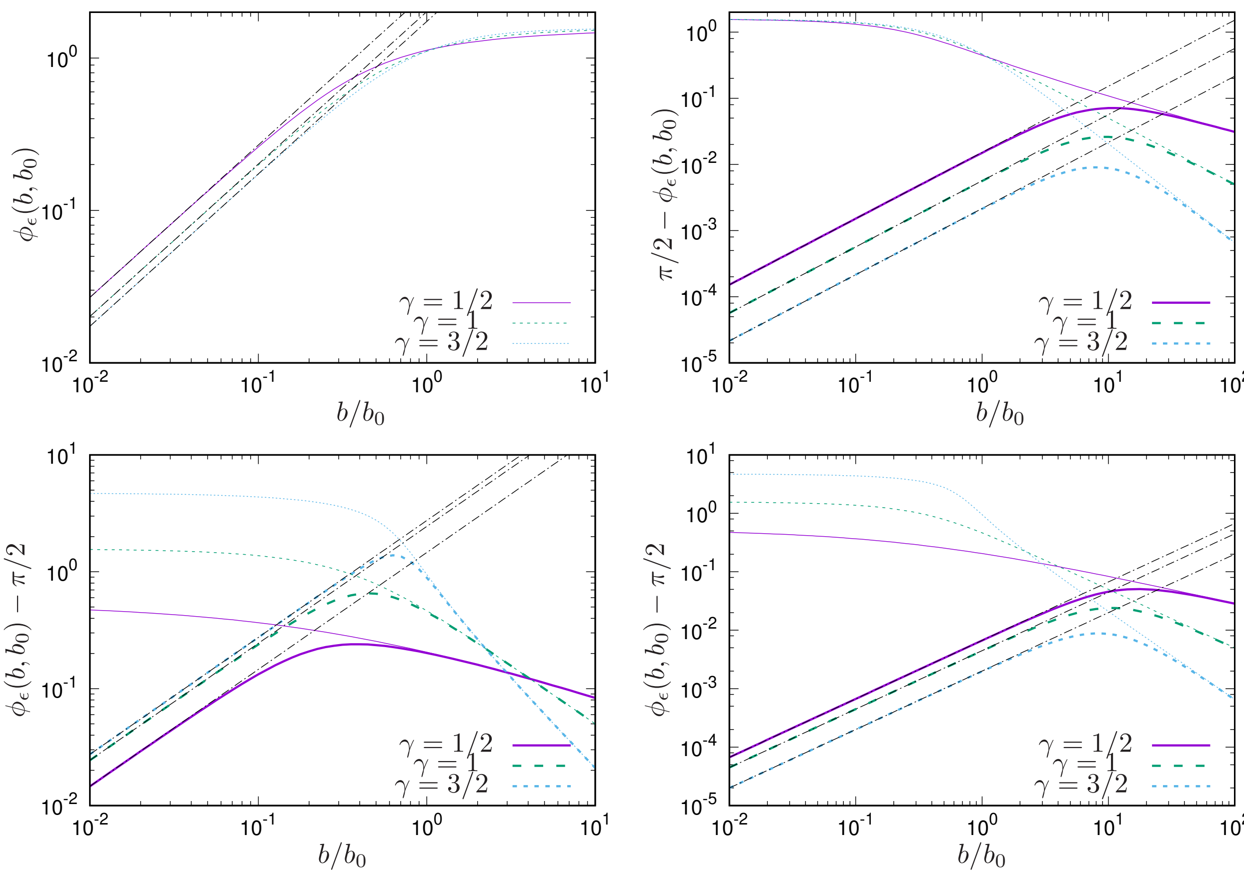

Figure 7: Top: Numerical computations for repulsive potentials with Plummer softening. Left: Graph of ϕ_ϵ as a function pf b/b_0 and of the leading order term (dashed-dotted line) given in Eq. (**??**), for different values of γ and $\epsilon/b_0 = 1/10$. Right: Graph of ϕ_ϵ for $\epsilon/b_0 = 10$ (thick lines) and the leading order expansion given in Eq. (**??**) (black dashed-dotted lines). The thin curves corresponds to ϕ_0. Bottom: Numerical computations for attractive potentials with Plummer softening (hard scattering). Left: Graphs of ϕ_ϵ (thick curves) and the theoretical prediction Eq. (**??**) (black dashed-dotted lines) as a function of b/b_0 for different values of γ and $\epsilon/b_0 = 1/10$. The thin curves corresponds to ϕ_0. Right: same quantity for $\epsilon/b_0 = 10$ and the theoretical prediction Eq. (**??**) (black dashed-dotted lines). The thin curves corresponds to ϕ_0.

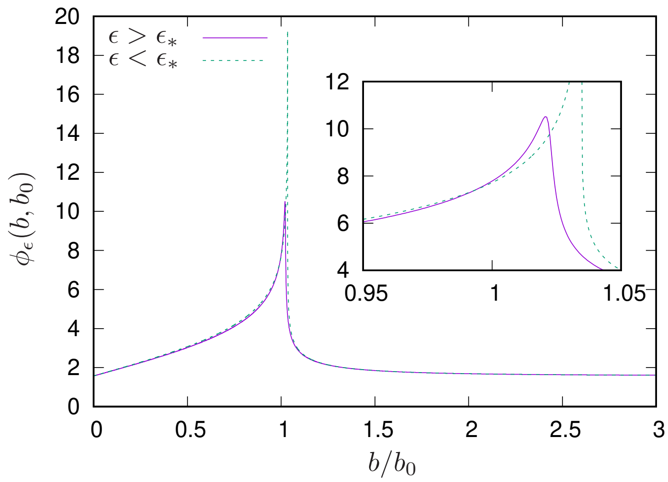

Figure 8: Plot of ϕ_ϵ as a function of b/b_0 for $\gamma = 5/2$ and two different values of the softening. The red continuous curve corresponds to a value of ϵ slightly larger than $\epsilon_*(b_0, \gamma)$ and the dashed green one to a value of ϵ slightly smaller than $\epsilon_*(b_0, \gamma)$.

explicitly), the value of r_{min} tends to 0. The trajectories in this limit is a succession of smaller and smaller loops embedded one in the other. An example of such trajectory was given in Fig. **??**.

- If $\gamma > 2$, the particles do not crash if the impact factor is larger than some critical value, which we have calculated. For impact factor slightly larger than this critical value, we have trajectories with $r_{min} \sim b_0$. The particles then *orbite* with distance r_{min} forming a binary, which will be destroyed in a finite time. We gave an example of such trajectories in Fig. **??**.

We have also studied the effect of introducing a regularization at small scales in the potential. The conclusions are detailed in Subsect. **??**.

One of the motivations of the paper was the computation of the Boltzmann collision operator Eq. (**??**). With the expressions given in the paper, knowing the velocity distribution function $\varphi(\mathbf{v}; t)$, it is straighforward to write a full series expansion of it in the case of pure power-law potentials.

In what follow we will give an example of application of the results for softened potentials, developed recently in [**?**, **?**]. In the context of astrophysics or plasma physics, it is natural to be interested in calculating the average change of velocity due to the collisions. It is classical (see e.g. [**?**]) to decompose the relative velocity of the particles before the collisions $\mathbf{V}$ as the sum of its component along the direction of the initial relative velocity $\mathbf{e}_\parallel$ and the component perpendicular to it $\mathbf{e}_\perp$, i.e.

$$\mathbf{V} = V_\perp \mathbf{e}_\perp + V_\parallel \mathbf{e}_\parallel. \tag{59}$$

It is possible to compute the average change of velocity $\Delta V_\perp$ and $\Delta V_\parallel$ after a collision has been completed integrating over all the impact factors b:

$$\frac{\Delta V_\perp}{V} = \sin(2\phi) \tag{60a}$$

$$\frac{\Delta V_\parallel}{V} = 1 + \cos(2\phi). \tag{60b}$$

One quantity of interest is the average change velocity square, which can be expressed by the integral over all the impact factors, i.e.

$$\langle \Delta V_\perp^2 \rangle \sim \int_0^R dbb^{d-2} \sin^2\left(2\phi_\epsilon\left(\frac{b}{b_0}\right)\right) \tag{61a}$$

$$\langle \Delta V_\parallel^2 \rangle \sim \int_0^R dbb^{d-2} \left[1+\cos\left(2\phi_\epsilon\left(\frac{b}{b_0}\right)\right)\right]^2, \tag{61b}$$

where $d > 1$ is the physical dimension and R the size of the system, which is the maximal impact factor available.

In astrophysical or cosmological N-body simulations, the goal is to simulate *collisionless* dynamics sampling a continuous distribution with macro-particles (see e.g. [**?**]). The softening used in these simulations is much larger than b_0 (in order to suppress collisional effects), and hence (see Sect. **??**), $\phi - \pi/2 \ll 1$. We can therefore write

$$\langle \Delta V_\perp^2 \rangle \sim 4\int_0^R dbb^{d-2} \left[\phi_\epsilon\left(\frac{b}{b_0}\right) - \frac{\pi}{2}\right]^2 \tag{62}$$

and $\langle \Delta V_\parallel^2 \rangle \ll \langle \Delta V_\perp^2 \rangle$. We can estimate Eq. (**??**) using the following approximate expression (see section **??** and Eq. (**??**)) for the angle ϕ_ϵ (we will consider explicitly attractive interactions with Plummer softening to simplify notations, the compact softening or repulsive case is analogous):

$$\phi_\epsilon - \frac{\pi}{2} \simeq \begin{cases} C_{\epsilon/b_0}(\gamma)\frac{b}{\epsilon} & \text{if } b < \epsilon \\ A(\gamma)\left(\frac{b_0}{b}\right)^\gamma & \text{if } b > \epsilon. \end{cases} \tag{63}$$

Using Eq. (**??**) to compute integral (**??**), considering softenings such that $b_0 \ll \epsilon \ll R$ we get the scaling, for $\gamma > (d-1)/2$,

$$\langle \Delta V_\perp^2 \rangle \sim b_0^{2\gamma} \epsilon^{d-1-2\gamma} \tag{64}$$

where we have used the asymptotic value of $C_\epsilon(\gamma)$ Eq. (**??**). Notice that impact factors smaller or larger than ϵ contributes to the scaling (**??**). In the limiting case $\gamma = (d-1)/2$, we get

$$\langle \Delta V_\perp^2 \rangle \sim b_0^2 \ln\left(\frac{R}{\epsilon}\right). \tag{65}$$

In this case contributions of collisions with $b < \epsilon$ are negligible. For $\gamma < (d-1)/2$, the effect of the softening is negligible because the main contribution to the change of velocity is given by impact factors $b \sim R$.

Acknowledgments

We thank M. Joyce for many useful discussions and comments.

A Derivation of the scattering formula

Let us consider the scattering of two isolated particles. It is convenient to use the center of mass frame to transform the two-particle problem in a one-particle one. Let us consider that particles have masses m_1 and m_2 and their position $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. We define their relative position as

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{66}$$

and fix the origin of the frame at the center of mass, i.e.

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{0}. \tag{67}$$

The relation between the position of the particles in the center of mass frame $\mathbf{r}$ and in the laboratory frame is, using Eqs. (**??**) and (**??**):

$$\mathbf{r}_1 = \frac{m}{m_1}\mathbf{r} \tag{68a}$$

$$\mathbf{r}_2 = -\frac{m}{m_2}\mathbf{r}, \tag{68b}$$

where we have defined the reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}. \tag{69}$$

In the center of mass frame, the collision occurs as depicted in Fig. **??**, in which appears the definition of the impact factor b, the angle of closest approach ϕ and the angle of deflection χ, which is $\chi = 2\phi$. In order to define the angles with the usual mathematical signs, the incident particle comes from $+\infty$. This picture assumes that the two particles are far away from each other for $t \to -\infty$ and for $t \to +\infty$. The angle ϕ can be calculated, as a function of the impact factor b, using the classical formula [**?**]

$$\phi(b) = \int_{r_{min}}^{\infty} \frac{(b/r^2)dr}{\sqrt{1 - (b/r)^2 - 2v(r)/(mu^2)}}, \tag{70}$$

where u is the asymptotic velocity of the incident particle at $+\infty$ ($u = |\dot{\mathbf{r}}|$). The quantity r_{min} is the largest positive root of the denominator, i.e. of

$$W(r) = 1 - (b/r)^2 - 2v(r)/mu^2. \tag{71}$$

We consider the pure power-law pair potential,

$$v(r) = \frac{g}{r^\gamma}, \qquad \gamma > 0, \tag{72}$$

with $g \neq 0$, where $g > 0$ corresponds to a repulsive interaction and $g < 0$ to an attractive one. We introduce the characteristic scale

$$b_0 = \left(\frac{2|g|}{mu^2}\right)^{1/\gamma}, \tag{73}$$

which allows us to rewrite Eq. (**??**) as

$$\phi(b) = \int_{r_{min}}^{\infty} \frac{(b/r^2)dr}{\sqrt{1 - (b/r)^2 \mp (b_0/r)^\gamma}}. \tag{74}$$

Now, the "minus" sign in the denominator corresponds to a repulsive interaction while the "plus" sign to an attractive one. By using the change of variables $r = b/x$ it is possible to rewrite Eq. (**??**) in the following form:

$$\phi(b/b_0) = \int_0^{x_{max}} \frac{dx}{\sqrt{1 - x^2 \mp (b_0/b)^\gamma x^\gamma}}, \tag{75}$$

where x_{max} is the smallest positive root of the denominator. Since x_{max} is a function of b/b_0 depending only on γ, Eq. (**??**) shows explicitly that ϕ is also a function of b/b_0 depending only on γ. Equation (**??**) can be solved explicitly only in few cases (e.g. gravity in $d = 3$ which is given by $\gamma = 1$), for the general case approximations or numerical computation of the integral should be used.

B Some technical mathematical details

In this appendix we give mathematical details of some derivations given in the paper.

B.1 Justification of the leading order expansion Eq. (??)

To completely justify the expansion Eq. (**??**), we have to pay attention to the z's close to z_{max}. Notice first that

$$dr/dz = \sqrt{-2W_b(r_*)/W_b''(r_*)}(1 + \mathcal{O}(z/z_{max}))$$

and that

$$r(z)^{-2} = (r_* + \mathcal{O}(z/z_{max}))^{-2},$$

hence the asymptotics $r(z) \approx r_* \approx r_*(\beta b_0)$ and $dr/dz \approx \sqrt{-2W_b(r_*)/W_b''(r_*)}$ are not completely true for $z \sim z_{max}$. We therefore split the right-hand side of Eq. (**??**) as

$$I_1 + I_2 = \frac{b}{\sqrt{-W_b(r_*)}} \int_1^{z_{max}/\ln(z_{max})} \frac{r(z)^{-2}\, dr/dz}{\sqrt{z^2-1}}\, dz + \frac{b}{\sqrt{-W_b(r_*)}} \int_{z_{max}/\ln(z_{max})}^{z_{max}} \frac{r(z)^{-2}\, dr/dz}{\sqrt{z^2-1}}\, dz.$$

In I_1, we have $0 \leqslant z/z_{max} \leqslant 1/|\ln z_{max}| = o(1)$, thus

$$dr/dz = \sqrt{-2W_b(r_*)/W_b''(r_*)}(1 + o(1))$$

and

$$r(z)^{-2} = (r_* + o(1))^{-2} = r_*(\beta b_0)^{-2} + o(1),$$

which yields

$$I_1 \approx b\sqrt{\frac{2}{W_b''(r_*(\beta b_0))}} \int_1^{z_{max}/\ln(z_{max})} \frac{r_*(\beta b_0)^{-2}\, dz}{\sqrt{z^2-1}} \approx \sqrt{\frac{2}{r_*(\beta b_0)^4 W_b''(r_*(\beta b_0))}} \ln(z_{max}).$$

Turning back to I_2, where $1 \ll z_{max}/\ln(z_{max}) \leqslant z \leqslant z_{max}$, we simply use that $r(z)^{-2} = \mathcal{O}(1)$ and that $dr/dz = \sqrt{-2W_b(r_*)}\mathcal{O}(1)$, thus

$$I_2 = \mathcal{O}\left(\int_{z_{max}/\ln(z_{max})}^{z_{max}} \frac{dz}{z}\right) = \mathcal{O}(\ln(\ln z_{max})) \ll \ln(z_{max}).$$

This concludes the justification of Eq. (**??**).

B.2 Bounding the function $\frac{1-x^2}{F(x, r_{min}/\epsilon)}$

We prove here that the function $x \mapsto \frac{1-x^2}{F(x, r_{min}/\epsilon)}$ is bounded on $[0, 1]$, independently of $b \ll b_0$ (for the Plummer softening). We recall that for the regime ($\epsilon < b_0$ and $b \ll b_0$) we are studying, $r_{min} \approx b_0\sqrt{1-(\epsilon/b_0)^2}$, thus $r_{min}/\epsilon \approx (\epsilon/b_0)^{-1}\sqrt{1-(\epsilon/b_0)^2}$.

Let us first work on the interval $[0, 1/2]$. Then, $F(x, r_{min}/\epsilon) = 1 - \mathcal{V}^{\mathrm{Pl}}(r_{min}/(\epsilon x))/\mathcal{V}^{\mathrm{Pl}}(r_{min}/\epsilon)$ is decreasing with respect to x since $\mathcal{V}^{\mathrm{Pl}}(R) = (1+R^2)^{-\gamma/2}$ is decreasing on $[0, +\infty)$, hence, for $0 \leqslant x \leqslant 1/2$,

$$0 \leqslant \frac{1-x^2}{F(x, r_{min}/\epsilon)} \leqslant \frac{1}{F(x, r_{min}/\epsilon)} \leqslant \frac{1}{F(1/2, r_{min}/\epsilon)}.$$

The right-hand side does not depend on x and is equal to

$$\left(1 - \frac{\mathcal{V}^{\mathrm{Pl}}(2r_{min}/\epsilon)}{\mathcal{V}^{\mathrm{Pl}}(r_{min}/\epsilon)}\right)^{-1} \approx \left(1 - \frac{\mathcal{V}^{\mathrm{Pl}}(2(\epsilon/b_0)^{-1}\sqrt{1-(\epsilon/b_0)^2})}{\mathcal{V}^{\mathrm{Pl}}((\epsilon/b_0)^{-1}\sqrt{1-(\epsilon/b_0)^2})}\right)^{-1},$$

which gives the desired upper bound on $[0, 1/2]$.

We now work on $[1/2,1]$, and use that $\frac{d}{dx}\mathcal{V}^{\rm Pl}(r_{min}/(\epsilon x)) = -(r_{min}/(\epsilon x^2))(\mathcal{V}^{\rm Pl})'(r_{min}/(\epsilon x)) \geqslant m$ for some positive constant $m = m(\epsilon/b_0)$ independent of b, since $\mathcal{V}^{\rm Pl}$ is decreasing on $[0,+\infty)$. As a consequence of the mean value theorem we get

$$0 \leqslant \frac{1-x^2}{F(x, r_{min}/\epsilon)} = \frac{(1+x)(1-x)}{F(x, r_{min}/\epsilon) - F(1, r_{min}/\epsilon)} \leqslant \frac{2}{m}.$$

This concludes the proof of the upper bound on $[0,1/2]$.

B.3 Justification of the relation Eq. (??)

If $\epsilon/b_0 \gg 1$, we may use for instance the Taylor expansion of the square root to deduce

$$\begin{aligned}
\tilde{B}_{\epsilon/b_0}(\gamma) &\approx \int_0^{+\infty} \left(1 - \frac{1}{\sqrt{1 + \frac{1}{(\epsilon/b_0)^\gamma - 1}\left(\mathcal{V}^{\rm Pl}(0) - \mathcal{V}^{\rm Pl}(y)\right)}}\right) \frac{dy}{y^2} \approx \frac{1}{(\epsilon/b_0)^\gamma - 1} \int_0^{+\infty} \frac{\mathcal{V}^{\rm Pl}(0) - \mathcal{V}^{\rm Pl}(y)}{2y^2}\, dy \\
&\approx \frac{1}{(\epsilon/b_0)^\gamma} \int_0^{+\infty} \frac{1-(1+y^2)^{-\gamma/2}}{2y^2}\, dy = \frac{\gamma}{4(\epsilon/b_0)^\gamma} \int_0^{+\infty} (1+y^2)^{-\gamma/2-1}\, dy \\
&= \frac{\gamma}{4(\epsilon/b_0)^\gamma} \int_0^{\pi/2} \cos^\gamma(\vartheta)\, d\vartheta = \frac{\sqrt{\pi}}{4(\epsilon/b_0)^\gamma} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)},
\end{aligned}$$

by first integration by parts and then the use of the substitution $y = \tan\vartheta$.

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