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# The Arakawa–Kaneko zeta function

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**Abstract** We present a very natural generalization of the Arakawa–Kaneko zeta function introduced ten years ago by T. Arakawa and M. Kaneko. We give in particular a new expression of the special values of this function at integral points in terms of modified Bell polynomials. By rewriting Ohno's sum formula, we are able to deduce a new class of relations between Euler sums and the values of zeta.

**Keywords** Poly-Bernoulli numbers · Multiple zeta-star values · Euler sums · Zeta values

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## 1 Introduction

The Arakawa–Kaneko zeta function was introduced ten years ago by T. Arakawa and M. Kaneko in [1]. Let us recall that this is the function $\xi_k$ defined for any integer $k \geq 1$ by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t - 1} \mathrm{Li}_k\left(1 - e^{-t}\right) dt,$$

where $\mathrm{Li}_k$ denotes the $k$th polylogarithm $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. The integral converges for $\Re(s) > 0$ and the function $\xi_k$ can be analytically continued an entire function of

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the whole $s$-plane. For $k=1$, $\xi_1(s)$ is nothing else than $s\zeta(s+1)$ and for $s=1$, $\xi_k(1)=\zeta(k+1)$. In [1], Arakawa and Kaneko have expressed the special values of this function at *negative* integers with the help of generalized Bernoulli numbers $B_n^{(k)}$ called "poly-Bernoulli numbers". Introduced by M. Kaneko in [5], these numbers are defined by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-z})}{1-e^{-z}}=\sum_{n=0}^{\infty} B_n^{(k)}\frac{z^n}{n!}.$$

In the case where $k=1$, one finds again—apart from the sign for $B_1^{(1)}$—the classical Bernoulli numbers. Arakawa and Kaneko also provide in [1] (see their Corollary 10) a rather complex expression of the special values of the function at *positive* integers in terms of MZV (multiple zeta value) but, very soon afterwards, a simpler representation of the values of $\xi_k$ at positive integers in terms of MZSV (multiple zeta-star value) has been obtained by Y. Ohno. More precisely, transforming the original expression given in [1] by means of a duality theorem, Ohno establishes in [7] that

$$\xi_{k-1}(m)=\sum_{n_1\ge n_2\ge\cdots\ge n_m\ge 1}\frac{1}{n_1^k n_2\cdots n_m}=\zeta^\star(k,\underbrace{1,\ldots,1}_{m-1}),$$

where $\zeta^\star(k_1,k_2,\ldots,k_m)$ refers to the sum

$$\sum_{n_1\ge n_2\ge\cdots\ge n_m\ge 1}\frac{1}{n_1^{k_1}n_2^{k_2}\cdots n_m^{k_m}}.$$

Subsequently, this expression has found an important continuation in [8] where Ohno states and proves his remarkable *sum formula*

$$\sum_{m=0}^{k-2}\zeta^\star(k-m,\underbrace{1,\ldots,1}_{m})=2(k-1)\big(1-2^{1-k}\big)\zeta(k),$$

which is also the subject of an interesting commentary in [6].

In this article, we introduce a more general function $\xi_k(s,x)$ defined for $\Re(s)>0$ and $x>0$ by

$$\xi_k(s,x)=\frac{1}{\Gamma(s)}\int_0^{+\infty} e^{-xt}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^{s-1}\,dt$$

which is a very natural extension of the Arakawa–Kaneko zeta function in the same way as the Hurwitz zeta function $\zeta(s,x)$ generalizes the Riemann zeta function; one has $\xi_k(s,1)=\xi_k(s)$ and, in the case where $k=1$, the function $\xi_1(s,x)$ is nothing else than the classical $s\zeta(s+1,x)$.

Following the same pattern as in [1], we show that this function $\xi_k(s,x)$ can be analytically continued to the whole complex $s$-plane as an entire function of $s$, and we express its special values at negative integral points by means of generalized

Bernoulli polynomials $B_n^{(k)}(x)$ whose values at 0 are precisely the poly-Bernoulli numbers (cf. Theorem 2 and Remark 3). In this way, we show that

$$\xi_k(-m, x) = (-1)^m B_m^{(k)}(x) \quad (m = 0, 1, 2, \dots).$$

Regarding the special values of $\xi_k(s, x)$ at *positive* integers, we obtain the following representation (cf. Theorem 1)

$$\xi_k(m+1, x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^k x(x+1)\cdots(x+n)} P_m\bigl(h_n^{(1)}(x), \dots, h_n^{(m)}(x)\bigr),$$

where $P_m(x_1, \dots, x_m)$ denotes the $m$th modified Bell polynomial defined by the generating function

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m}\right) = \sum_{m=0}^{\infty} P_m(x_1, \dots, x_m) t^m \qquad (*)$$

and where

$$h_n^{(m)}(x) = \sum_{j=0}^{n} \frac{1}{(j+x)^m}. \qquad (**)$$

This "Hasse formula" extends the representation already given in [3] in the case $k = 1$ (cf. Remark 2). Specializing this expression at $x = 1$, we then deduce a new expression for the values of the Arakawa–Kaneko function at positive integers (cf. Corollary 1) from which follows the decomposition

$$\xi_{k-1}(m+1) = \zeta^\star(k, \underbrace{1, \dots, 1}_{m}) = \sum_{n=1}^{\infty} \frac{1}{n^k} P_m\bigl(H_n, H_n^{(2)}, \dots, H_n^{(m)}\bigr)$$

where $H_n, H_n^{(2)}, \dots, H_n^{(m)}$ denote the harmonic numbers (cf. Corollary 2). This leads us to the rewriting of Ohno's sum formula in the following form

$$\sum_{m=1}^{k-3} \sum_{n=1}^{\infty} \frac{1}{n^{k-m}} P_m\bigl(H_n, H_n^{(2)}, \dots, H_n^{(m)}\bigr) = \bigl[(k-2) - (k-1)2^{2-k}\bigr]\zeta(k),$$

which defines a new class of relations between Euler sums and the values of zeta (cf. Corollary 3 and Example 3). This class contains in particular (in the simplest case where $k = 4$), the famous relation $\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4)$ whose origin goes back to Euler and Goldbach (cf. [4]).

## 2 A generalized Arakawa–Kaneko function

**Proposition 1** *Let*

$$F(s, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} f(t) t^{s-1}\, dt$$

*be a Laplace–Mellin integral with*

$$f(t)=\sum_{n=0}^{\infty} a_{n+1}\big(1-e^{-t}\big)^n,$$

*where the coefficients $a_n$ are assumed to satisfy the condition $|a_n| = O(\frac{1}{n})$. The following properties hold*:

1. *The integral $F(s,x)$ converges for $\Re(s)>0$ and $x>0$.*
2. *If $m$ is a natural number and $s=m+1$ then*

$$F(m+1,x)=\sum_{n=0}^{\infty}\frac{n!a_{n+1}}{x(x+1)\cdots(x+n)}P_m\big(h_n^{(1)}(x),\dots,h_n^{(m)}(x)\big), \qquad (1)$$

*where $P_m$ and $h_n^{(m)}$ are respectively given by formulas (∗) and (∗∗).*

*Proof* By our assumption on $a_n$, there exists a constant $C>0$ and an integer $N\geq 1$ such that for all $t\geq 0$

$$\sum_{n=N}^{\infty}|a_n|\big(1-e^{-t}\big)^{n-1}\leq C\sum_{n=N}^{\infty}\frac{(1-e^{-t})^{n-1}}{n}\leq C\sum_{n=1}^{\infty}\frac{(1-e^{-t})^{n-1}}{n}=\frac{Ct}{1-e^{-t}},$$

which ensures the convergence of the integral and justifies the interchange of $\int$ and $\sum$

$$F(s,x)=\sum_{n=0}^{\infty}a_{n+1}\int_0^{+\infty}e^{-xt}\big(1-e^{-t}\big)^n\frac{t^{s-1}}{\Gamma(s)}\,dt.$$

Then, formula (1) results from the following Lemma 1. □

**Lemma 1** *For $x>t$, one has*

$$\int_0^{+\infty}e^{-x\xi}\big(1-e^{-\xi}\big)^n e^{\xi t}\,d\xi=\frac{n!}{x(x+1)\cdots(x+n)}\times\exp\left(\sum_{m=1}^{\infty}h_n^{(m)}(x)\frac{t^m}{m}\right).$$

*Proof* For $a>0$ and $b>0$, let us start from the classical Euler's relation

$$\mathrm{B}(a,b)=\int_0^1 u^{a-1}(1-u)^{b-1}\,du=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Putting $u=e^{-\xi}$, $a=x-t$ and $b=n+1$, one deduces

$$\int_0^{+\infty}e^{-x\xi}\big(1-e^{-\xi}\big)^n e^{\xi t}\,d\xi=\frac{n!}{(x-t)(1+x-t)\cdots(n+x-t)}.$$

Moreover, one has

$$\begin{aligned}
&\frac{n!}{(x-t)(1+x-t)\cdots(n+x-t)}\\
&\quad=\frac{n!}{x(x+1)\cdots(x+n)}\times\prod_{k=0}^{n}\left(1-\frac{t}{k+x}\right)^{-1}\\
&\quad=\frac{n!}{x(x+1)\cdots(x+n)}\times\exp\left(-\sum_{k=0}^{n}\ln\left(1-\frac{t}{k+x}\right)\right)\\
&\quad=\frac{n!}{x(x+1)\cdots(x+n)}\times\exp\left(\sum_{k=0}^{n}\sum_{m=1}^{\infty}\frac{t^m}{m(x+k)^m}\right)\\
&\quad=\frac{n!}{x(x+1)\cdots(x+n)}\times\exp\left(\sum_{m=1}^{\infty}h_n^{(m)}(x)\frac{t^m}{m}\right).
\end{aligned}$$

□

Applying now Proposition 1 with $a_{n+1}=\frac{1}{(n+1)^k}$, one has

$$f(t)=\sum_{n=0}^{\infty}\frac{(1-e^{-t})^n}{(n+1)^k}=\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}},$$

and we immediately obtain the following theorem:

**Theorem 1** *Consider an integer $k\geq 1$. The Laplace–Mellin integral*

$$\xi_k(s,x)=\frac{1}{\Gamma(s)}\int_0^{+\infty}e^{-xt}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^{s-1}\,dt$$

*converges for $\Re(s)>0$ and $x>0$. Moreover, one has*

$$\xi_k(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+1)^k}\int_0^{+\infty}e^{-xt}\left(1-e^{-t}\right)^n\frac{t^{s-1}}{\Gamma(s)}\,dt.$$

*In particular, if $m$ is a natural number and $s=m+1$, then*

$$\xi_k(m+1,x)=\sum_{n=0}^{\infty}\frac{n!}{(n+1)^k x(x+1)\cdots(x+n)}P_m\left(h_n^{(1)}(x),\ldots,h_n^{(m)}(x)\right). \tag{2}$$

*Remark 1* In the case $x=1$, one has

$$\begin{aligned}
\xi_k(s,1)&=\frac{1}{\Gamma(s)}\int_0^{+\infty}e^{-t}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^{s-1}\,dt\\
&=\frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{t^{s-1}}{e^t-1}\mathrm{Li}_k\left(1-e^{-t}\right)dt=\xi_k(s).
\end{aligned}$$

Thus, in this case, one finds again the original Arakawa–Kaneko zeta function introduced in [1].

*Remark 2* In the case $k = 1$, one has $\frac{\mathrm{Li}_1(1-e^{-t})}{1-e^{-t}} = \frac{t}{1-e^{-t}}$ from which it follows that

$$\xi_1(s-1,x) = \int_0^{+\infty} e^{-xt}\left(\frac{t}{1-e^{-t}}\right)\frac{t^{s-2}}{\Gamma(s-1)}\,dt = (s-1)\zeta(s,x).$$

Thus, in this case, identity (2) is nothing else than the representation for the values of the Hurwitz zeta function (called Hasse formula) given in [3]

$$(m+1)\zeta(m+2,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)x(x+1)\cdots(x+n)} P_m\big(h_n^{(1)}(x),\ldots,h_n^{(m)}(x)\big).$$

*Example 1*

$$\xi_k(1,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^k x(x+1)\cdots(x+n)};$$

$$\xi_k(2,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^k x(x+1)\cdots(x+n)} \sum_{i=0}^{n}\frac{1}{i+x};$$

$$2\xi_k(3,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^k x(x+1)\cdots(x+n)} \left[\left(\sum_{i=0}^{n}\frac{1}{i+x}\right)^2 + \sum_{i=0}^{n}\frac{1}{(i+x)^2}\right];$$

$$6\xi_k(4,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^k x(x+1)\cdots(x+n)} \times\left[\left(\sum_{i=0}^{n}\frac{1}{i+x}\right)^3 + 3\sum_{i=0}^{n}\frac{1}{(i+x)}\sum_{i=0}^{n}\frac{1}{(i+x)^2} + 2\sum_{i=0}^{n}\frac{1}{(i+x)^3}\right].$$

**Theorem 2** *For all integers $k \geq 1$ and real $x > 0$, the function $s \mapsto \xi_k(s,x)$ can be analytically continued to the whole complex $s$-plane and its values at negative integers are given by*

$$\xi_k(-m,x) = (-1)^m B_m^{(k)}(x) \quad (m = 0,1,2,\ldots), \tag{3}$$

*where $B_n^{(k)}(x)$ is defined by the generating function*

$$e^{-xz}\frac{\mathrm{Li}_k(1-e^{-z})}{1-e^{-z}} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{z^n}{n!}.$$

*Proof* We apply the classical method to analytically continue a function defined as a Mellin transform (cf. [2]). One splits up $\xi_k(s,x)$ as the sum of two integrals:

$$\xi_k(s,x) = \frac{1}{\Gamma(s)}\int_0^1 e^{-xt}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^{s-1}\,dt + \frac{1}{\Gamma(s)}\int_1^{+\infty} e^{-xt}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^{s-1}\,dt.$$

The second integral converges absolutely for all $s \in \mathbb{C}$ and $x > 0$ and cancels at negative integers (because $\frac{1}{\Gamma}(-m) = 0$ for $m = 0, 1, 2, \ldots$). For $\Re(s) > 0$, the first integral may be written as

$$\frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{B_n^{(k)}(x)}{n!} \times \frac{1}{n+s},$$

from which follows that

$$\lim_{s\to -m}\xi_k(s,x) = \left(\lim_{s\to -m}\frac{1}{\Gamma(s)(m+s)}\right)\frac{B_m^{(k)}(x)}{m!} = (-1)^m B_m^{(k)}(x). \qquad \square$$

*Remark 3* From the generating function which defines them, the polynomials $B_n^{(k)}(x)$ are given by

$$B_n^{(k)}(x) = \sum_{q=0}^{n}(-1)^{n-q}\binom{n}{q}B_q^{(k)}x^{n-q},$$

where $B_n^{(k)} = B_n^{(k)}(0)$ are the poly-Bernoulli number introduced by Kaneko in [5]. In particular, specializing identity (3) at $x = 1$, one finds again

$$\xi_k(-m,1) = (-1)^m B_m^{(k)}(1) = \sum_{q=0}^{m}(-1)^q\binom{m}{q}B_q^{(k)},$$

which is nothing else than the expression given by Arakawa and Kaneko (cf. [1], Theorem 6).

## 3 New expression of $\xi_k(m+1)$

Specializing identity (2) at $x = 1$, one obtains

**Corollary 1** *For all natural numbers $m \geq 0$ and integers $k \geq 1$, one has*

$$\xi_k(m+1) = \sum_{n=1}^{\infty}\frac{1}{n^{k+1}}P_m\left(H_n, H_n^{(2)}, \ldots, H_n^{(m)}\right) \quad \text{with } H_n^{(m)} = \sum_{j=1}^{n}\frac{1}{j^m}. \tag{4}$$

*Example 2*

$$\xi_k(1) = \zeta(k+1);$$

$$\xi_k(2) = \sum_{n=1}^{\infty} \frac{H_n}{n^{k+1}};$$

$$\xi_k(3) = \frac{1}{2}\left[\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^{k+1}} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{k+1}}\right];$$

$$\xi_k(4) = \frac{1}{6}\left[\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^{k+1}} + 3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^{k+1}} + 2\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^{k+1}}\right];$$

$$\xi_k(5) = \frac{1}{24}\left[\sum_{n=1}^{\infty} \frac{(H_n)^4}{n^{k+1}} + 6\sum_{n=1}^{\infty} \frac{(H_n)^2 H_n^{(2)}}{n^{k+1}} + 3\sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^{k+1}} + 8\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^{k+1}} + 6\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^{k+1}}\right].$$

## 4 Rewriting of Ohno's sum formula and application to Euler sums

From the comparison of (4) with the expression given by Ohno (cf. [7] Theorem 2, and [6] Paragraph 2),

$$\xi_{k-1}(m) = \zeta^\star(k, \underbrace{1, \ldots, 1}_{m-1}) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1^k n_2 \cdots n_m},$$

one can immediately deduce the following decomposition

**Corollary 2** *For all natural numbers $m \geq 0$ and integers $k \geq 2$,*

$$\zeta^\star(k, \underbrace{1, \ldots, 1}_{m}) = \sum_{n=1}^{\infty} \frac{1}{n^k} P_m\big(H_n, H_n^{(2)}, \ldots, H_n^{(m)}\big). \tag{5}$$

Rewriting now Ohno's sum formula (cf. [8], Theorem 8)

$$\sum_{m=0}^{k-2} \zeta^\star(k-m, \underbrace{1, \ldots, 1}_{m}) = 2(k-1)\big(1-2^{1-k}\big)\zeta(k),$$

thanks to the preceding decomposition (5), and taking into account that $\xi_1(k-1) + \xi_{k-1}(1) = k\zeta(k)$, one obtains the following formula which defines a new class of relations between Euler sums and the zeta values

**Corollary 3** *For all integers $k \geq 4$,*

$$\sum_{m=1}^{k-3} \sum_{n=1}^{\infty} \frac{1}{n^{k-m}} P_m\big(H_n, H_n^{(2)}, \ldots, H_n^{(m)}\big) = \big[(k-2) - (k-1)2^{2-k}\big]\zeta(k). \tag{6}$$

*Example 3*

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4) \quad \text{(Euler and Goldbach)};$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^4} + \frac{1}{2}\left[\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^3} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3}\right] = \frac{5}{2}\zeta(5);$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^5} + \frac{1}{2}\left[\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^4} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4}\right]$$
$$+ \frac{1}{6}\left[\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^3} + 3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} + 2\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3}\right] = \frac{59}{16}\zeta(6);$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^6} + \frac{1}{2}\left[\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^5} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5}\right]$$
$$+ \frac{1}{6}\left[\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^4} + 3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} + 2\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4}\right]$$
$$+ \frac{1}{24}\left[\sum_{n=1}^{\infty} \frac{(H_n)^4}{n^3} + 6\sum_{n=1}^{\infty} \frac{(H_n)^2 H_n^{(2)}}{n^3} + 3\sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3}\right.$$
$$\left. + 8\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3} + 6\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3}\right] = \frac{77}{16}\zeta(7).$$

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