A note on some alternating series involving zeta and multiple zeta values

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Abstract In this article, we study a class of conditionally convergent alternating series including, as a special case, the famous series $\sum_{n\geqslant 2}(-1)^n\frac{\zeta(n)}{n}$ which links Euler's constant γ to special values of the Riemann zeta function at positive integers. We give several new relations of the same kind. Among other things, we show the existence of a similar relation for the Apostol-Vu harmonic zeta function which have never been noticed before. We also highlight a deep connection with the Ramanujan summation of certain divergent series which originally motivated this work.

Keywords Riemann zeta function; harmonic zeta function; Stirling numbers of the first kind; Stirling numbers of the second kind; Bernoulli numbers; Bernoulli numbers of the second kind; harmonic numbers; Gregory coefficients of higher order; multiple zeta values; Ramanujan summation of divergent series.

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Introduction

The first part of this article is devoted to the study of the conditionally convergent alternating series ν_k defined by

$$\nu_k := \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n+k}\,,$$

where $\zeta(s)$ is the Riemann zeta function and k denotes an integral parameter. By a classical result (cf. [8, p. 66], [15, p. 62]), it is well known that ν_0 is Euler's constant

$$\gamma = \lim_{n\to\infty}\left\{\sum_{j=1}^{n}\frac{1}{j} - \ln n\right\} = 0.5772156649\ldots .$$

This remarkable connection between γ and the special values at positive integers of the Riemann zeta function goes back to Euler's early works on harmonic series (cf. [13]). Less famous but yet fairly well-known (cf. [8, p. 93], [16, Eq. (5.1)], [17, Eq. (1.5)]) is the relation

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1$$

sometimes called Suryanarayana formula. Recently, Blagouchine ([5, p. 413]) gave a general expression of these series ν_k in the case where k is a positive integer:

$$\begin{aligned}\nu_k &= \frac{\gamma}{2} - \frac{\ln 2\pi}{k+1} + \frac{1}{k}\\ &\quad + \sum_{r=1}^{[\frac{k}{2}]}(-1)^r\binom{k}{2r-1}\frac{(2r)!}{r(2\pi)^{2r}}\zeta'(2r) + \sum_{r=1}^{[\frac{k+1}{2}]-1}(-1)^r\binom{k}{2r}\frac{(2r)!}{2(2\pi)^{2r}}\zeta(2r+1)\,. \quad (1)\end{aligned}$$

This formula seems quite cumbersome but can be much simplified using the functional equation of ζ. After some elementary transformations, we show that equation (1) can be reduced to the following equivalent (but much more pleasant) expression:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + C_k\,, \quad (2)$$

where C_k is a rational number (see Proposition 1). Moreover, this expression allows to highlight a deep connection between ν_{2k} and the sum (in the sense of the Ramanujan summation of divergent series) of the series $\sum_{n\geqslant 1} n^{2k}H_n$, where H_n is the nth harmonic number (see Remark 2).

Next, in a second part, we introduce a generalization of these series series ν_k replacing the zeta values by certain multiple zeta values. A natural extension may be defined as follows: for all integers $k \geq -1$ and $p \geq 0$, we consider the class of series $(\nu_{k,p})$ with

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n, \underbrace{1, \ldots, 1}_{p}),$$

where

$$\zeta(s_1, s_2, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},$$

so that the previous series ν_k become $\nu_{k,0}$. Then we establish (see Proposition 2) the following identity which is the main result of this work:

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}, \tag{3}$$

where $G_n^{(k)}$ denotes the *Gregory coefficients of higher order* recently introduced by Blagouchine (cf. [5, 6]). They are defined by

$$G_n^{(k)} := \frac{1}{n!} \sum_{j=1}^{n} \frac{s(n,j)}{j+k} \qquad (k \geq 0, n \geq 1), \tag{4}$$

where $s(n,j)$ are the Stirling numbers of the first kind. Comprehensive informations on the Stirling numbers of the first and the second kind may be found in [1, 5, 12, 15, 18]. One can prove easily (see Lemma 4) that $G_n^{(k)} = (-1)^{n+1}|G_n^{(k)}|$, so that the rationals numbers $G_n^{(k)}$ alternate in sign. As a special case of equation (3), we derive the following result:

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \qquad (k \geq 0). \tag{5}$$

In the case $k = 1$, we recover the classical Mascheroni's series for γ (cf. [5, p. 406], [15, p. 280]):

$$\gamma = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \cdots.$$

Another notable consequence of formula (3) is the deduction of this nice formula (see Example 3):

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12},$$

where $\zeta_H(s)$ denotes the Apostol-Vu harmonic zeta function (cf. [2, 3, 4]) and γ_1 is the first Stieltjes constant (cf. [7, 8]).

Finally, in the last section, we highlight a relation between the series ν_k, the Stirling numbers of the second kind $S(n,k)$, and the shifted Mascheroni series σ_r whose study was the main subject of [12] (see Proposition 3 and Example 4).

1 The case of a positive integer

In this section, we focus on the case of a positive integer k and give two independant proofs of our formula (2). More precisely, we prove the following proposition:

Proposition 1. For any positive integer k, we have

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

with

$$C_k = \frac{1}{k} + \sum_{j=1}^{k} \binom{k}{j} \frac{B_j\, H_{j-1}}{k+1-j}\,, \tag{6}$$

where H_n are the harmonic numbers,

$$H_0 = 0\,, \quad H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \geq 1)\,,$$

and B_n are the Bernoulli numbers defined by means of the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| < 2\pi)\,.$$

In particular, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_{2r+1} = 0$ for $r \geq 1$.

Proof. We can quite easily derive (2) from (1). Differentiation of the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2}$$

(cf. [3, Eq. (25.4.2)]), leads to the two relations

$$(-1)^r \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1) = \zeta'(-2r) \qquad (r \geq 1),$$

and

$$(-1)^r \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r}\left(H_{2r-1} - \gamma - \ln 2\pi\right) \qquad (r \geq 1)\,.$$

Substituting these relations into (1) and grouping together the terms under the two symbols Σ, leads to the expression

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j} \zeta'(-j) + C_k\,,$$

where the rational constant C_k is given by equation (6).

Another alternative proof of (2), independant from (1), may be deduced from the expansion in powers of z of the relation

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\zeta^{\mathcal{R}}(j-k) &= (1-e^z)\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\zeta'(-k)\\ &+(1-e^z)\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\frac{1}{(k+1)^2}\\ &+\int_0^1 \ln(t+1)e^{-zt}\,dt\end{aligned}$$

(cf. [8, p. 93]), with

$$\zeta^{\mathcal{R}}(j-k)=\begin{cases}\gamma & \text{if } j=k+1\\ \zeta(j-k)-\frac{1}{j-k-1} & \text{otherwise.}\end{cases}$$

Rewriting the series ν_k as

$$\nu_k=\sum_{j=k+2}^{\infty}\frac{(-1)^{j-k}}{j}\zeta(j-k)\,,$$

and using the well-known relations

$$\zeta(0)=-\frac{1}{2}\,,\quad \zeta(1-2r)=-\frac{B_{2r}}{2r}\,,\quad \text{and}\quad \zeta'(0)=-\frac{1}{2}\ln 2\pi\quad \text{(cf. [3, p. 605])},$$

then a careful identification of the terms in z^k in the previous development leads again, after some simplifications, to formula (2), and provides in addition another equivalent expression for the constant C_k:

$$C_k=\frac{H_k}{k+1}-\sum_{j=1}^{k}\frac{B_j}{j(k+1-j)}\,. \tag{7}$$

An unexpected consequence of this equivalence is the curious identity

$$\sum_{k=2}^{n}\frac{B_k}{n+1-k}\left\{\frac{1}{k}+\binom{n}{k}H_{k-1}\right\}=\frac{H_n}{n+1}+\frac{1}{2n}\qquad (n\geq 2)$$

whose direct proof does not seem obvious. □

Example 1. For the first values of k, we have the following relations:

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1\,,$$
$$\nu_2 = \frac{\gamma}{3} - \frac{1}{2}\ln 2\pi - 2\zeta'(-1) + \frac{2}{3}\,,$$
$$\nu_3 = \frac{\gamma}{4} - \frac{1}{2}\ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}\,,$$
$$\nu_4 = \frac{\gamma}{5} - \frac{1}{2}\ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}\,,$$
$$\nu_5 = \frac{\gamma}{6} - \frac{1}{2}\ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}\,.$$

Remark 1. Starting from the Maclaurin series expansion

$$\psi(x+1) + \gamma = \sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1} \qquad (|x| < 1)$$

(cf. [3, Eq. (25.8.5)]), where $\psi(x)$ denotes the digamma function (i.e. the logarithmic derivative of the Γ-function), and multiplying each side by x^k (with $k \geq 1$), then an integration between 0 and 1 gives

$$\nu_k = \frac{\gamma}{k+1} + \int_0^1 x^k \psi(x+1)\,dx.$$

Thus, it follows from formula (2) that

$$\int_0^1 x^k \psi(x+1)\,dx = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k \qquad (k \geq 1).$$

Remark 2 (Link with the Ramanujan summation: part I)**.** Candelpergher et al. ([10, Corollary 1], see also [8, p. 82]) established that

$$\sum_{n \geqslant 1}^{\mathcal{R}} H_n = \frac{3}{2}\gamma - \frac{1}{2}\ln 2\pi + \frac{1}{2}\,,$$

and for any positive integer p,

$$\sum_{n \geqslant 1}^{\mathcal{R}} n^p H_n = \left(\frac{1 - B_{p+1}}{p+1}\right)\gamma - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{p} (-1)^j \binom{p}{j} \zeta'(-j) + R_p \quad \text{with } R_p \in \mathbb{Q},$$

where the symbol $\sum^{\mathcal{R}}$ denotes the sum of the series in the sense of the Ramanujan summation of divergent series (cf. [7, 8, 9, 10]). For $p = 2k$ (with $k \geq 1$), we have

$B_{p+1} = 0$ and $R_p = C_p - \frac{B_p}{2p} + \frac{B_p}{2}$, then, in view of formula (2), these relations may be translated into the following identities:

$$\sum_{n\geqslant 1}^{\mathcal{R}} H_n = \nu_1 + \gamma - \frac{1}{2},$$

and for $k \geq 1$,

$$\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k} H_n = \nu_{2k} + \zeta'(-2k) + \frac{1-2k}{2}\zeta(1-2k) = \nu_{2k} + \zeta'(-2k) + (2k-1)\frac{B_{2k}}{4k}. \quad (8)$$

In particular, we have

$$\sum_{n\geqslant 1}^{\mathcal{R}} n^2 H_n = \nu_2 + \zeta'(-2) + \frac{B_2}{4} = \nu_2 - \frac{\zeta(3)}{4\pi^2} + \frac{1}{24}.$$

2 The case $k = -1$

The case $k = -1$ behaves differently from the previous case and must be studied separately. We recall the identities

$$\nu_{-1} = \int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = \sum_{m=1}^{\infty} \frac{\ln(m+1)}{m(m+1)} = -\sum_{n=2}^{\infty} \zeta'(n) = 1.2577468869\ldots$$

(cf. [8, p. 105], [11, p. 142]). Another interesting representation (communicated by I. V. Blagouchine) is

$$\nu_{-1} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta(3/2 + ix)}{(1/2 + ix)\cosh(\pi x)}\,dx.$$

Moreover, we can write yet another relation which will be useful in the next section: let κ_1 be the constant

$$\kappa_1 := \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = 0.5290529699\ldots,$$

where the rational numbers b_n are the *Bernoulli numbers of the second kind* defined by means of their generating function

$$\frac{x}{\ln(x+1)} = 1 + \sum_{n=1}^{\infty} b_n\, x^n \qquad (|x| < 1).$$

These numbers b_n were introduced and studied by Jordan ([15, p. 265 et seq.]). Note that several authors quoted here use different notations: b_n are denoted by

G_n (and called *Gregory coefficients*) in [5, 6, 7], and they are denoted by $\frac{\beta_n}{n!}$ in [8]. The coefficients $n!\,b_n$ are sometimes called *Cauchy numbers* (cf. [9]). The constants κ_1 and ν_{-1} are linked by the relation

$$\kappa_1 + \frac{1}{2}\zeta(2) = \nu_{-1} + \gamma_1 + \frac{1}{2}\gamma^2 \tag{9}$$

(cf. [7, Eq. (37)], [8, Eq. (3.23) p. 105]), where γ_1 denotes the first Stieljes constant (cf. [3, 7, 8])

$$\gamma_1 = \lim_{n\to\infty}\left\{\sum_{j=1}^{n}\frac{\ln j}{j} - \frac{1}{2}\ln^2 n\right\} = -0.07281584548\ldots .$$

In terms of the Ramanujan summation, γ_1 is $\sum\limits_{n\geqslant 1}^{\mathcal{R}} \frac{\ln n}{n}$ (cf. [8, p. 67]), whereas κ_1 is $\sum\limits_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n}$ (cf. [8, Eq. (4.29) p. 133]).

3 Alternating series involving multiple zeta values

In this section, we consider a more general class of series of the previous type replacing zeta values with certain multiple zeta values. We prove our formula (3) and deduce some interesting consequences.

Proposition 2. For all integers $p \geq 0$ and $k \geq -1$, let

$$\nu_{k,p} := \sum_{n=2}^{\infty}\frac{(-1)^n}{n+k}\zeta(n,\underbrace{1,\ldots,1}_{p})\,;$$

then

$$\nu_{k,p} = \sum_{n=1}^{\infty}\frac{|G_n^{(k+1)}|}{n^{p+1}}\,,$$

where the rational numbers $G_n^{(k)}$ are defined by equation (4).

Corollary 1. In particular, for $p = 0$, we have

$$\nu_{k-1,0} = \nu_{k-1} = \sum_{n=1}^{\infty}\frac{|G_n^{(k)}|}{n} \qquad (k \geq 0);$$

and since $G_n^{(1)} = b_n$, for $k = 0$, we have also

$$\nu_{0,p} = \kappa_p := \sum_{n=1}^{\infty}\frac{|b_n|}{n^{p+1}} \qquad (p \geq 0)\,.$$

In order to prove Proposition 2, we begin by stating the following lemmas:

Lemma 1. For all integers $j \geq 1$ and $p \geq 0$, we have

$$\int_0^1 \frac{\ln^j(1-x)\,\ln^p(x)}{x}\,dx = (-1)^{j+p}\,j!\,p!\,\zeta(j+1,\underbrace{1,\ldots,1}_{p})\,. \tag{10}$$

Proof. This follows directly from [18, Eq. (2.27), (2.28)]. □

Lemma 2. The Stirling numbers of the first kind $s(n,j)$ with fixed $j \geq 1$ admit the (vertical) exponential generating function (cf. [1, Eq. (2.8)])

$$\frac{\ln^j(1+x)}{j!} = \sum_{n=j}^{\infty} s(n,j)\frac{x^n}{n!} \qquad (|x|<1)\,. \tag{11}$$

Lemma 3. For all integers $n \geq 1$ and $p \geq 0$, we have

$$(-1)^p \int_0^1 x^{n-1}\ln^p(x)\,dx = \frac{p!}{n^{p+1}} \tag{12}$$

Proof. This is nothing else than [7, Eq. (41)] in the case where p is an integer. □

Lemma 4. For all integers $n \geq 1$ and $k \geq 0$, we have

$$G_n^{(k)} = \frac{(-1)^{n+1}}{n!}\int_0^1 x^k(1-x)_{n-1}\,dx\,,$$

where $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$ is the Pochhammer symbol. In particular, this implies that

$$G_n^{(k)} = (-1)^{n+1}|G_n^{(k)}|\,. \tag{13}$$

Proof. Integration between 0 and 1 of the expansion

$$x^{k-1}x(x-1)\cdots(x-n+1) = \sum_{j=1}^{n} s(n,j)x^{j+k-1}$$

gives the required result. □

Proof of Proposition 2. Using successively formulas (10)–(13) above, we can write the following equalities:

$$\begin{aligned}
\nu_{k,p} &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j+k+1} \zeta(j+1, \underbrace{1, \ldots, 1}_{p}) \\
&= \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_0^1 \frac{\ln^j(1-x)}{j!} \frac{\ln^p(x)}{x} \, dx \\
&= \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_0^1 \left(\sum_{n=j}^{\infty} (-1)^n s(n,j) \frac{x^n}{n!} \right) \frac{\ln^p(x)}{x} \, dx \\
&= \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \sum_{n=j}^{\infty} (-1)^n \frac{s(n,j)}{n!} \int_0^1 x^{n-1} \ln^p(x) \, dx \\
&= -\sum_{j=1}^{\infty} \frac{1}{j+k+1} \sum_{n=j}^{\infty} (-1)^n \frac{s(n,j)}{n! \, n^{p+1}} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n! \, n^{p+1}} \sum_{j=1}^{n} \frac{s(n,j)}{j+k+1} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n!} \sum_{j=1}^{n} \frac{s(n,j)}{j+k+1} \right) \frac{1}{n^{p+1}} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{G_n^{(k+1)}}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}} \, .
\end{aligned}$$

This completes the proof. □

Example 2. For the first values of $k \geq -1$, we have the following expansions in series containing only positive rational terms:

$$\begin{aligned}
\nu_{-1} &= \sum_{n=1}^{\infty} \frac{|G_n^{(0)}|}{n} = 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \cdots \, , \\
\nu_{0} &= \sum_{n=1}^{\infty} \frac{|G_n^{(1)}|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \cdots \, , \\
\nu_{1} &= \sum_{n=1}^{\infty} \frac{|G_n^{(2)}|}{n} = \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25\,200} + \frac{731}{725\,760} + \cdots \, , \\
\nu_{2} &= \sum_{n=1}^{\infty} \frac{|G_n^{(3)}|}{n} = \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100\,800} + \frac{5849}{10\,886\,400} + \cdots \, , \\
\nu_{3} &= \sum_{n=1}^{\infty} \frac{|G_n^{(4)}|}{n} = \frac{1}{5} + \frac{1}{120} + \frac{1}{420} + \frac{83}{80\,640} + \frac{59}{108\,000} + \frac{397}{1\,209\,600} + \cdots \, .
\end{aligned}$$

Example 3. Let ζ_H be the Apostol-Vu harmonic zeta function (cf. [2, 3, 4, 10]) defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s}\,.$$

We recall that ζ_H is analytic in the half-plane $\mathrm{Re}(s) > 1$ and can be extended meromorphically with poles at the integers $1, 0, -1, -3, -5, \cdots$. The special values at negative even integers are $\zeta_H(-2k) = B_{2k}/2 - B_{2k}/4k$. The special values at positive integers are also well-known: the first values are

$$\zeta_H(2) = 2\zeta(3),\ \zeta_H(3) = \frac{5}{4}\zeta(4),$$

and more generally, they may be computed by means of the following beautiful formula (first discovered by Euler (cf. [14]) and several times rediscovered afterwards):

$$2\zeta_H(n) = (n+2)\zeta(n+1) - \sum_{r=1}^{n-2} \zeta(r+1)\zeta(n-r) \qquad (n \geq 3).$$

Otherwise, by Proposition 2 above, we can write

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \nu_{0,1} - \nu_{-1} + \zeta(2) = \kappa_1 - \nu_{-1} + \zeta(2)\,,$$

and thus, from equation (9), we derive the following elegant evaluation:

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12} = 0.916240149\ldots\,. \tag{14}$$

Another expression of this constant is $\zeta''(0) + \frac{1}{2}\ln^2(2\pi) + \frac{\pi^2}{8}$ (cf. [3, Eq. (25.6.12)]).

Remark 3 (Link with the Ramanujan summation: part II)**.** For $s \in \mathbb{C}$, let $\zeta_H^{\mathcal{R}}$ be the function $s \mapsto \sum_{n\geqslant 1}^{\mathcal{R}} H_n\, n^{-s}$ where $\sum^{\mathcal{R}}$ stands for the Ramanujan summation. The function $\zeta_H^{\mathcal{R}}$ is an entire function linked to the harmonic zeta function ζ_H by the relation

$$\zeta_H^{\mathcal{R}}(s) = \zeta_H(s) - \int_1^{\infty} x^{-s}\,(\psi(x+1) + \gamma)\;dx \quad \text{ for } \mathrm{Re}(s) > 1$$

(cf. [10, Eq. (84)]). We have the identities

$$\zeta_H^{\mathcal{R}}(1) = \nu_{0,1} = \kappa_1\,, \quad \zeta_H^{\mathcal{R}}(0) = \nu_1 + \gamma - \frac{1}{2}\,,$$

and formula (8) may be nicely rewritten

$$\zeta_H^{\mathcal{R}}(-2k) = \zeta_H(-2k) + \zeta'(-2k) + \nu_{2k}\,.$$

4 Link with the shifted Mascheroni series

Let us consider now the forward shifted Mascheroni series which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|b_{n+r}|}{n}, \qquad \text{for } r = 0, 1, 2, \cdots .$$

We have in particular $\sigma_0 = \nu_0 = \gamma$. The study of these series σ_r was the main subject of [12]. Among other things, we have established the following decomposition of $\zeta'(-j)$ on the "basis" of σ_r (cf. [12, Proposition 3]):

$$\zeta'(-j) = \sum_{r=2}^{j+1} (-1)^{j-r}(r-1)! S(j, r-1)\sigma_r - \frac{B_{j+1}}{j+1}\gamma - \frac{B_{j+1}}{(j+1)^2}, \qquad \text{for } j = 1, 2, 3, \cdots,$$

where $S(j, r)$ are Stirling numbers of the second kind; moreover, for $j = 0$, we have also a similar relation:

$$\frac{1}{2} \ln 2\pi = -\zeta'(0) = \sigma_1 + \frac{\gamma}{2} + \frac{1}{2}.$$

Then, substituting these relations into (2) enables us to write each series ν_k with $k \geq 1$ as an integral linear combination of $\gamma, \sigma_1, \sigma_2, \cdots, \sigma_k$ plus a rational number D_k which is closely linked to C_k. In this combination, the coefficient of γ is zero since it is equal to $\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j$ which vanishes by a well-known property of the Bernoulli numbers. Finally, equation (2) may be nicely rewritten in terms of σ_r as follows:

Proposition 3. For all integers $k \geq 1$, we have the relation

$$\nu_k = D_k - \sigma_1 + \sum_{r=2}^{k} (-1)^r (r-1)! \left(\sum_{j=r-1}^{k-1} \binom{k}{j} S(j, r-1) \right) \sigma_r \tag{15}$$

with

$$D_k = C_k - \frac{1}{2} + \sum_{n=2}^{k} \binom{k}{n} \frac{B_n}{n(k+1-n)} = \frac{1}{k} + \sum_{n=1}^{k} \binom{k}{n} \frac{B_n H_n}{k+1-n}.$$

Example 4. For the first values of k, we have the following relations:

$$\begin{aligned}
\nu_1 &= \frac{1}{2} - \sigma_1\,,\\
\nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2\,,\\
\nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3\,,\\
\nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4\,,\\
\nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5\,,\\
\nu_6 &= \frac{23}{180} - \sigma_1 + 62\sigma_2 - 420\sigma_3 + 1560\sigma_4 - 1800\sigma_5 + 720\sigma_6\,.
\end{aligned}$$

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References

[1] T. Agoh, K. Dilcher, Recurrence relations for Nörlund numbers and Bernoulli numbers of the second kind, *Fibonacci Quarterly* **48** (2010), 4–12.

[2] E. Alkan, Approximation by special values of harmonic zeta function and log-sine integrals, *Commun. Number Theory Phys.* **7** (2013), 515–550.

[3] T. M. Apostol, *Zeta and Related Functions*, NIST Handbook of Mathematical Functions, Chapter 25, Cambridge University Press, 2010.

[4] T. M. Apostol, T. H. Vu, Dirichlet series related to the Riemann zeta function, *J. Number Theory* **19** (1984), 85–102.

[5] I. V. Blagouchine, Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to π^{-1}, *J. Math. Anal. Appl.* **442** (2016), 404–434.

[6] I. V. Blagouchine, Three notes on Ser's and Hasse's representations for the zeta-functions, *Integers* **18A** (2018), 1–45.

[7] I. V. Blagouchine, M-A. Coppo, A note on some constants related to the zeta function, *Ramanujan J.* **47** (2018), 457–473.

[8] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. Series, vol. 2185, Springer, 2017.

[9] B. Candelpergher, M-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.* **27** (2012), 305–328.

[10] B. Candelpergher, H. Gadiyar, and R. Padma, Ramanujan summation and the exponential generating function $\sum_{k=0}^{\infty} \frac{z^k}{k!}\zeta'(-k)$, *Ramanujan J.* **21** (2010), 99–122.

[11] H. Cohen, *Number Theory, Volume II: Analytic and Modern Tools*, Graduate Texts in Math., vol. 240, Springer, 2007.

[12] M-A. Coppo, P. T. Young, Shifted Mascheroni series and hyperharmonic numbers, *J. Number Theory* **169** (2016), 1–20.

[13] L. Euler, De progressionibus harmonicis observationes (1734), Eneström-Number E43. *Opera Omnia*, Series 1, vol. 14, pp. 87–100.

[14] L. Euler, Meditationes circa singulare serierum genus (1775), Eneström-Number E477. *Opera Omnia*, Series 1, vol. 15, pp. 217–267.

[15] C. Jordan, *Calculus of Finite Differencies*, Chelsea, New York, 1965.

[16] H. M. Srivastava, Sums of certain series of the Riemann zeta function, *J. Math. Anal. Appl.* **134** (1988), 129-140.

[17] R. J. Singh, V. P. Verma, Some series involving Riemann zeta function, *Yokohama Math. J.* **31** (1983), 1–4.

[18] C. Xu, Multiple zeta values and Euler sums, *J. Number Theory* **177** (2017), 443–478.