A note on some alternating series involving zeta and multiple zeta values

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Abstract In this article, we study a class of conditionally convergent alternating series including, as a special case, the famous series $\sum_{n \geq 2} (-1)^n \zeta(n)/n$ which links Euler’s constant $\gamma$ to special values of the Riemann zeta function at positive integers. We give several new relations of the same kind. Among other things, we show the existence of a similar relation for the Apostol-Vu harmonic zeta function which have never been noticed before. We also highlight a deep connection with the Ramanujan summation of certain divergent series which originally motivated this work.

Keywords Riemann zeta function; harmonic zeta function; Stirling numbers of the first kind; Stirling numbers of the second kind; Bernoulli numbers; Bernoulli numbers of the second kind; harmonic numbers; Gregory coefficients of higher order; multiple zeta values; Ramanujan summation of divergent series.

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Introduction

The first part of this article is devoted to the study of the conditionally convergent alternating series \( \nu_k \) defined by

\[
\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n + k},
\]

where \( \zeta(s) \) is the Riemann zeta function and \( k \) denotes an integral parameter. By a classical result (cf. [8, p. 66], [15, p. 62]), it is well known that \( \nu_0 \) is Euler’s constant

\[
\gamma = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} - \ln n \right\} = 0.5772156649 \ldots.
\]

This remarkable connection between \( \gamma \) and the special values at positive integers of the Riemann zeta function goes back to Euler’s early works on harmonic series (cf. [13]). Less famous but yet fairly well-known (cf. [8, p. 93], [16, Eq. (5.1)], [17, Eq. (1.5)]) is the relation

\[
\nu_1 = \frac{\gamma}{2} - \frac{1}{2} \ln 2\pi + 1
\]

sometimes called Suryanarayana formula. Recently, Blagouchine ([5, p. 413]) gave a general expression of these series \( \nu_k \) in the case where \( k \) is a positive integer:

\[
\nu_k = \frac{\gamma}{2} - \ln 2\pi + \frac{1}{k+1} + \frac{1}{k} + \sum_{r=1}^{[\frac{k}{2}]} (-1)^r \left( \frac{k}{2r-1} \right) \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) + \sum_{r=1}^{[\frac{k+1}{2}]-1} (-1)^r \left( \frac{k}{2r} \right) \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1). \tag{1}
\]

This formula seems quite cumbersome but can be much simplified using the functional equation of \( \zeta \). After some elementary transformations, we show that equation (1) can be reduced to the following equivalent (but much more pleasant) expression:

\[
\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k, \tag{2}
\]

where \( C_k \) is a rational number (see Proposition 1). Moreover, this expression allows to highlight a deep connection between \( \nu_{2k} \) and the sum (in the sense of the Ramanujan summation of divergent series) of the series \( \sum_{n \geq 1} n^{2k} H_n \), where \( H_n \) is the \( n \)th harmonic number (see Remark 2).

Next, in a second part, we introduce a generalization of these series series \( \nu_k \) replacing the zeta values by certain multiple zeta values. A natural extension may
be defined as follows: for all integers $k \geq -1$ and $p \geq 0$, we consider the class of series $(\nu_{k,p})$ with

$$
\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n, 1, \ldots, 1) ,
$$

where

$$
\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} ,
$$

so that the previous series $\nu_k$ become $\nu_{k,0}$. Then we establish (see Proposition 2) the following identity which is the main result of this work:

$$
\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}} ,
$$

where $G_n^{(k)}$ denotes the Gregory coefficients of higher order recently introduced by Blagouchine (cf. [5, 6]). They are defined by

$$
G_n^{(k)} := \frac{1}{n!} \sum_{j=1}^{n} \frac{s(n, j)}{j+k} \quad (k \geq 0, n \geq 1),
$$

where $s(n, j)$ are the Stirling numbers of the first kind. Comprehensive informations on the Stirling numbers of the first and the second kind may be found in [1, 5, 12, 15, 18]. One can prove easily (see Lemma 4) that $G_n^{(k)} = (-1)^{n+1} |G_n^{(k)}|$, so that the rationals numbers $G_n^{(k)}$ alternate in sign. As a special case of equation (3), we derive the following result:

$$
\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \quad (k \geq 0) .
$$

In the case $k = 1$, we recover the classical Mascheroni’s series for $\gamma$ (cf. [5, p. 406], [15, p. 280]):

$$
\gamma = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362880} + \cdots .
$$

Another notable consequence of formula (3) is the deduction of this nice formula (see Example 3):

$$
\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2} \gamma^2 + \frac{\pi^2}{12} ,
$$

where $\zeta_H(s)$ denotes the Apostol-Vu harmonic zeta function (cf. [2, 3, 4]) and $\gamma_1$ is the first Stieltjes constant (cf. [7, 8]).

Finally, in the last section, we highlight a relation between the series $\nu_k$, the Stirling numbers of the second kind $S(n, k)$, and the shifted Mascheroni series $\sigma_r$ whose study was the main subject of [12] (see Proposition 3 and Example 4).
1 The case of a positive integer

In this section, we focus on the case of a positive integer \( k \) and give two independent proofs of our formula (2). More precisely, we prove the following proposition:

**Proposition 1.** For any positive integer \( k \), we have

\[
\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k
\]

with

\[
C_k = \frac{1}{k} + \sum_{n=1}^{k} \binom{k}{n} B_n H_{n-1} \frac{k+1-n}{k+1},
\]

where \( H_n \) are the harmonic numbers,

\[
H_0 = 0, \quad H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \geq 1),
\]

and \( B_n \) are the Bernoulli numbers defined by means of the exponential generating function

\[
\frac{x}{e^x - 1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!} \quad (|x| < 2\pi).
\]

In particular, \( B_0 = 1, \ B_1 = -\frac{1}{2}, \ B_2 = \frac{1}{6}, \ B_{2r+1} = 0 \) for \( r \geq 1 \).

**Proof.** We can quite easily derive (2) from (1). Differentiation of the functional equation

\[
\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s) \sin \frac{\pi s}{2}
\]

(cf. [3, Eq. (25.4.2)]), leads to the two relations

\[
(-1)^r \frac{(2r)!}{(2\pi)^{2r}} \zeta(2r+1) = \zeta'(-2r) \quad (r \geq 1),
\]

and

\[
(-1)^r \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r} (H_{2r-1} - \gamma - \ln 2\pi) \quad (r \geq 1).
\]

Substituting these relations into (1) and grouping together the terms under the two symbols \( \Sigma \), leads to the expression

\[
\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k.
\]
where the rational constant \( C_k \) is given by equation (6).

Another alternative proof of (2), independent from (1), may be deduced from the expansion in powers of \( z \) of the relation

\[
\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \zeta^R(j-k) = (1 - e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \zeta'(-k) + (1 - e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \frac{1}{(k+1)^2}
\]

\[
+ \int_0^1 \ln(t+1)e^{-zt} \, dt
\]

(cf. [8, p. 93]), with

\[
\zeta^R(j-k) = \begin{cases} 
\gamma & \text{if } j = k + 1 \\
\zeta(j-k) - \frac{1}{j-k+1} & \text{otherwise.}
\end{cases}
\]

Rewriting the series \( \nu_k \) as

\[
\nu_k = \sum_{j=k+2}^{\infty} \frac{(-1)^{j-k}}{j} \zeta(j-k),
\]

and using the well-known relations

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta(1-2r) = -\frac{B_{2r}}{2^{2r}}, \quad \text{and} \quad \zeta'(0) = -\frac{1}{2} \ln 2 \pi \quad (\text{cf. [3, p. 605]}),
\]

then a careful identification of the terms in \( z^k \) in the previous development leads again, after some simplifications, to formula (2), and provides in addition another equivalent expression for the constant \( C_k \):

\[
C_k = \frac{H_k}{k+1} - \sum_{j=1}^{k} \frac{B_j}{j(k+1-j)}.
\]

(7)

An unexpected consequence of this equivalence is the curious identity

\[
\sum_{j=1}^{k} \frac{B_j}{k+1-j} \left\{ \frac{1}{j} + \binom{k}{j} H_{j-1} \right\} = \frac{H_k}{k+1} - \frac{1}{k} \quad (k \geq 1)
\]

whose direct proof does not seem obvious. \( \square \)
Example 1. For the first values of $k$, we have the following relations:

\[
\begin{align*}
\nu_1 &= \gamma - \frac{1}{2} \ln 2\pi + 1, \\
\nu_2 &= \gamma - \frac{1}{2} \ln 2\pi - 2\zeta'(-1) + \frac{2}{3}, \\
\nu_3 &= \gamma - \frac{1}{2} \ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}, \\
\nu_4 &= \gamma - \frac{1}{2} \ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}, \\
\nu_5 &= \gamma - \frac{1}{2} \ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}.
\end{align*}
\]

Remark 1. Starting from the Maclaurin series expansion

\[
\psi(x + 1) + \gamma = \sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1} \quad (|x| < 1)
\]

(cf. [3, Eq. (25.8.5)]), where $\psi(x)$ denotes the digamma function (i.e. the logarithmic derivative of the $\Gamma$-function), and multiplying each side by $x^k$ (with $k \geq 1$), then an integration between 0 and 1 gives

\[
\nu_k = \frac{\gamma}{k+1} + \int_0^1 x^k \psi(x + 1) \, dx.
\]

Thus, it follows from formula (2) that

\[
\int_0^1 x^k \psi(x + 1) \, dx = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k \quad (k \geq 1).
\]

Remark 2 (Link with the Ramanujan summation: part I). Candelpergher et al. ([10, Corollary 1], see also [8, p. 82]) established that

\[
\sum_{n \geq 1} R^\mathcal{R} H_n = \frac{3}{2} \gamma - \frac{1}{2} \ln 2\pi + \frac{1}{2},
\]

and for any positive integer $p$,

\[
\sum_{n \geq 1} n^p H_n = \left( 1 - \frac{B_{p+1}}{p+1} \right) \gamma - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{p} (-1)^j \binom{p}{j} \zeta'(-j) + R_p \quad \text{with } R_p \in \mathbb{Q},
\]

where the symbol $\sum$ denotes the sum of the series in the sense of the Ramanujan summation of divergent series (cf. [7, 8, 9, 10]). For $p = 2k$ (with $k \geq 1$), we have
$B_{p+1} = 0$ and $R_p = C_p - \frac{B_p}{2p} + \frac{B_p}{T}$, then, in view of formula (2), these relations may be translated into the following identities:

$$\sum_{n \geq 1}^\infty H_n = \nu_1 + \gamma - \frac{1}{2},$$

and for $k \geq 1$,

$$\sum_{n \geq 1}^\infty n^{2k} H_n = \nu_{2k} + \zeta'(-2k) + \frac{1 - 2k}{2} \zeta(1 - 2k) = \nu_{2k} + \zeta'(-2k) + (2k - 1) \frac{B_{2k}}{4k}.$$ (8)

In particular, we have

$$\sum_{n \geq 1}^\infty n^2 H_n = \nu_2 + \zeta'(2) + \frac{B_2}{4} = \nu_2 - \frac{\zeta(3)}{4\pi^2} + \frac{1}{24}.$$

### 2 The case $k = -1$

The case $k = -1$ behaves differently from the previous case and must be studied separately. We recall the identities

$$\nu_{-1} = \int_0^1 \frac{\psi(x + 1) + \gamma}{x} dx = \sum_{n=1}^\infty \frac{\ln(m + 1)}{m(m + 1)} = - \sum_{n=2}^\infty \zeta'(n) = 1.2577468869 \ldots$$

(cf. [8, p. 105], [11, p. 142]). Another interesting representation (communicated by I. V. Blagouchine) is

$$\nu_{-1} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3/2 + ix)}{(1/2 + ix) \cosh(\pi x)} \, dx.$$  

Moreover, we can write yet another relation which will be useful in the next section: let $\kappa_1$ be the constant

$$\kappa_1 := \sum_{n=1}^\infty \frac{|b_n|}{n^2} = 0.5290529699 \ldots,$$

where the rational numbers $b_n$ are the **Bernoulli numbers of the second kind** defined by means of their generating function

$$\frac{x}{\ln(x + 1)} = 1 + \sum_{n=1}^\infty b_n x^n \quad (|x| < 1).$$

These numbers $b_n$ were introduced and studied by Jordan ([15, p. 265 et seq.]). Note that several authors quoted here use different notations: $b_n$ are denoted by
$G_n$ (and called Gregory coefficients) in [5, 6, 7], and they are denoted by $\frac{\beta_n}{n!}$ in [8]. The coefficients $n! b_n$ are sometimes called Cauchy numbers (cf. [9]). The constants $\kappa_1$ and $\nu_{-1}$ are linked by the relation

$$\kappa_1 + \frac{1}{2}\zeta(2) = \nu_{-1} + \gamma_1 + \frac{1}{2}\gamma^2$$  \hspace{1cm} (9)

(cf. [7, Eq. (37)], [8, Eq. (3.23) p. 105]), where $\gamma_1$ denotes the first Stieltjes constant (cf. [3, 7, 8])

$$\gamma_1 = \lim_{n \to \infty} \left\{ \sum_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2} \ln^2 n \right\} = -0.07281584548 \ldots .$$

In terms of the Ramanujan summation, $\gamma_1$ is $\sum_{n \geq 1} \frac{\ln n}{n}$ (cf. [8, p. 67]), whereas $\kappa_1$ is $\sum_{n \geq 1} \frac{H_n}{n}$ (cf. [8, Eq. (4.29) p. 133]).

3 Alternating series involving multiple zeta values

In this section, we consider a more general class of series of the previous type replacing zeta values with certain multiple zeta values. We prove our formula (3) and deduce some interesting consequences.

Proposition 2. For all integers $p \geq 0$ and $k \geq -1$, let

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n + k} \zeta(n, 1, \ldots, 1);$$

then

$$\nu_{k,p} := \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{np+1},$$

where the rational numbers $G_n^{(k)}$ are defined by equation (4).

Corollary 1. In particular, for $p = 0$, we have

$$\nu_{k-1,0} = \nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \quad (k \geq 0);$$

and since $G_n^{(1)} = b_n$, for $k = 0$, we have also

$$\nu_0,p = \kappa_p := \sum_{n=1}^{\infty} \frac{|b_n|}{np+1} \quad (p \geq 0).$$
In order to prove Proposition 2, we begin by stating the following lemmas:

**Lemma 1.** For all integers $j \geq 1$ and $p \geq 0$, we have

\[
\int_0^1 \frac{\ln^j(1-x) \ln^p(x)}{x} \, dx = (-1)^{j+p} j! p! \zeta(j+1, 1, \ldots, 1) .
\] (10)

**Proof.** This follows directly from [18, Eq. (2.27), (2.28)].

**Lemma 2.** The Stirling numbers of the first kind $s(n, j)$ with fixed $j \geq 1$ admit the (vertical) exponential generating function (cf. [1, Eq. (2.8)])

\[
\frac{\ln^j(1+x)}{j!} = \sum_{n=j}^{\infty} s(n, j) \frac{x^n}{n!} \quad (|x| < 1).
\] (11)

**Lemma 3.** For all integers $n \geq 1$ and $p \geq 0$, we have

\[
(-1)^p \int_0^1 x^{n-1} \ln^p(x) \, dx = \frac{p!}{n^{p+1}}.
\] (12)

**Proof.** This is nothing else than [7, Eq. (41)] in the case where $p$ is an integer.

**Lemma 4.** For all integers $n \geq 1$ and $k \geq 0$, we have

\[
G^{(k)}_n = \frac{(-1)^{n+1}}{n!} \int_0^1 x^k (1-x)^{n-1} \, dx,
\]

where $(z)_n = z(z+1)(z+2) \cdots (z+n-1)$ is the Pochhammer symbol. In particular, this implies that

\[
G^{(k)}_n = (-1)^{n+1} |G^{(k)}_n|.
\] (13)

**Proof.** Integration between 0 and 1 of the expansion

\[
x^{k-1}x(x-1) \cdots (x-n+1) = \sum_{j=1}^{n} s(n, j) x^{j+k-1}
\]
gives the required result.

**Proof of Proposition 2.** Using successively formulas (10)–(13) above, we can write
the following equalities:

\[ \nu_{k,p} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j+k+1} \zeta(j+1,1,\ldots,1) \]

\[ = \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_0^1 \frac{\ln^j(1-x) \ln^p(x)}{x} \, dx \]

\[ = \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_0^1 \left( \sum_{n=j}^{\infty} (-1)^n s(n,j) \frac{x^n}{n!} \right) \ln^p(x) \, dx \]

\[ = \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \sum_{n=j}^{\infty} (-1)^n s(n,j) \frac{x^n}{n!} \int_0^1 x^{n-1} \ln^p(x) \, dx \]

\[ = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{s(n,j)}{n+k+1} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \frac{1}{n^{p+1}} \]

\[ = \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{n} \sum_{j=1}^{n} s(n,j) \right) \frac{1}{n^{p+1}} \]

\[ = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{G_n^{(k+1)}}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}. \]

This completes the proof. \( \square \)

**Example 2.** For the first values of \( k \geq -1 \), we have the following expansions in series containing only positive rational terms:

\[ \nu_{-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(0)}|}{n} = 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \cdots, \]

\[ \nu_0 = \sum_{n=1}^{\infty} \frac{|G_n^{(1)}|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362880} + \cdots, \]

\[ \nu_1 = \sum_{n=1}^{\infty} \frac{|G_n^{(2)}|}{n} = \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25200} + \frac{731}{725760} + \cdots, \]

\[ \nu_2 = \sum_{n=1}^{\infty} \frac{|G_n^{(3)}|}{n} = \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100800} + \frac{5849}{10886400} + \cdots, \]

\[ \nu_3 = \sum_{n=1}^{\infty} \frac{|G_n^{(4)}|}{n} = \frac{1}{5} + \frac{1}{120} + \frac{1}{420} + \frac{83}{80640} + \frac{59}{108000} + \frac{397}{1209600} + \cdots. \]
Example 3. Let $\zeta_H$ be the Apostol-Vu harmonic zeta function (cf. [2, 3, 4, 10]) defined for $\text{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s}.$$ 

We recall that $\zeta_H$ is analytic in the half-plane $\text{Re}(s) > 1$ and can be extended meromorphically with poles at the integers $1, 0, -1, -3, -5, \ldots$. The special values at negative even integers are $\zeta_H(-2k) = B_{2k}/2 - B_{2k}/4k$. The special values at positive integers are also well-known: the first values are $\zeta_H(2) = 2\zeta(3)$, $\zeta_H(3) = \frac{5}{4}\zeta(4)$, and more generally, they may be computed by means of the following beautiful formula (first discovered by Euler (cf. [14]) and several times rediscovered afterwards):

$$2\zeta_H(n) = (n + 2)\zeta(n + 1) - \sum_{r=1}^{n-2} \zeta(r + 1)\zeta(n - r) \quad (n \geq 3).$$

Otherwise, by Proposition 2 above, we can write

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \nu_{0,1} - \nu_{-1} + \zeta(2) = \kappa_{1} - \nu_{-1} + \zeta(2),$$

and thus, from equation (9), we derive the following elegant evaluation:

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12} = 0.916240149\ldots.$$ (14)

Another expression of this constant is $\zeta''(0) + \frac{1}{2}\ln^2(2\pi) + \frac{\pi^2}{6}$ (cf. [3, Eq. (25.6.12)]).

Remark 3 (Link with the Ramanujan summation: part II). For $s \in \mathbb{C}$, let $\zeta^R_H$ be the function $s \mapsto \sum_{n \geq 1}^{R} H_n n^{-s}$ where $\sum_{R}^{\infty}$ stands for the Ramanujan summation. The function $\zeta^R_H$ is an entire function linked to the harmonic zeta function $\zeta_H$ by the relation

$$\zeta^R_H(s) = \zeta_H(s) - \int_1^{\infty} x^{-s} (\psi(x+1) + \gamma) \, dx \quad \text{for } \text{Re}(s) > 1$$

(cf. [10, Eq. (84)]). We have the identities

$$\zeta^R_H(1) = \nu_{0,1} = \kappa_{1}, \quad \zeta^R_H(0) = \nu_{1} + \gamma - \frac{1}{2},$$

and formula (8) may be nicely rewritten

$$\zeta^R_H(-2k) = \zeta_H(-2k) + \zeta'(-2k) + \nu_{2k}.$$
4 Link with the shifted Mascheroni series

Let us consider now the forward shifted Mascheroni series which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|b_{n+r}|}{n}, \quad \text{for } r = 0, 1, 2, \ldots.$$  

We have in particular $\sigma_0 = \nu_0 = \gamma$. The study of these series $\sigma_r$ was the main subject of [12]. Among other things, we have established the following decomposition of $\zeta'(-j)$ on the “basis” of $\sigma_r$ (cf. [12, Proposition 3]):

$$\zeta'(-j) = \sum_{r=2}^{j+1} (-1)^{j-r}(r-1)!S(j,r-1)\sigma_r - B_{j+1} \frac{B_{j+1}}{j+1} \gamma - \frac{B_{j+1}}{(j+1)^2}, \quad \text{for } j = 1, 2, 3, \ldots,$$

where $S(j,r)$ are Stirling numbers of the second kind; moreover, for $j = 0$, we have also a similar relation:

$$\frac{1}{2} \ln 2\pi = -\zeta'(0) = \sigma_1 + \frac{\gamma}{2} + \frac{1}{2}.$$

Then, substituting these relations into (2) enables us to write each series $\nu_k$ with $k \geq 1$ as an integral linear combination of $\gamma, \sigma_1, \sigma_2, \ldots, \sigma_k$ plus a rational number $D_k$ which is closely linked to $C_k$. In this combination, the coefficient of $\gamma$ is zero since it is equal to $\frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j}B_j$ which vanishes by a well-known property of the Bernoulli numbers. Finally, equation (2) may be nicely rewritten in terms of $\sigma_r$ as follows:

**Proposition 3.** For all integers $k \geq 1$, we have the relation

$$\nu_k = D_k - \sigma_1 + \sum_{r=2}^{k} (-1)^r(r-1)! \left( \sum_{j=r-1}^{k-1} \binom{k}{j} S(j,r-1) \right) \sigma_r$$  

with

$$D_k = C_k - \frac{1}{2} + \sum_{n=2}^{k} \binom{k}{n} \frac{B_n}{n(k+1-n)} = \frac{1}{k} + \sum_{n=1}^{k} \binom{k}{n} \frac{B_n H_n}{k+1-n}.$$  

12
Example 4. For the first values of $k$, we have the following relations:

\[
\begin{align*}
\nu_1 &= \frac{1}{2} - \sigma_1, \\
\nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2, \\
\nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3, \\
\nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4, \\
\nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5, \\
\nu_6 &= \frac{23}{180} - \sigma_1 + 62\sigma_2 - 420\sigma_3 + 1560\sigma_4 - 1800\sigma_5 + 720\sigma_6.
\end{align*}
\]

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References


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