A note on some alternating series involving zeta and multiple zeta values

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Abstract In this article, we study a class of conditionally convergent alternating series including, as a special case, the famous series $\sum_{n \geq 2} (-1)^n \frac{\zeta(n)}{n}$ which links Euler’s constant $\gamma$ to special values of the Riemann zeta function at positive integers. We give several new relations of the same kind. Among other things, we show the existence of a similar relation for the Apostol-Vu harmonic zeta function which have never been noticed before. We also highlight a deep connection with the Ramanujan summation of certain divergent series which originally motivated this work.

Keywords Riemann zeta function; harmonic zeta function; Stirling numbers of the first kind; Stirling numbers of the second kind; Bernoulli numbers; Harmonic numbers; Gregory coefficients of higher order; multiple zeta values; Ramanujan summation of divergent series.

Mathematics Subject Classification (2010) 11B73; 11B75; 11M06; 11M32; 40G99; 41A58.

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Introduction

The first part of this article is devoted to the study of the conditionally convergent alternating series $\nu_k$ defined by

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k},$$

where $\zeta(s)$ is the Riemann zeta function and $k$ denotes an integral parameter. By a classical result (cf. [8, p. 66], [15, p. 62]), it is well known that $\nu_0$ is Euler’s constant

$$\gamma = \lim_{n \to \infty} \left\{ \sum_{j=1}^{n} \frac{1}{j} - \ln n \right\} = 0.5772156649 \ldots .$$

This remarkable connection between $\gamma$ and the special values at positive integers of the Riemann zeta function goes back to Euler’s early works on harmonic series (cf. [13]). Less famous but yet fairly well-known (cf. [8, p. 93], [16, Eq. (5.1)], [17, Eq. (1.5)]) is the relation

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2} \ln 2\pi + 1$$

sometimes called Suryanarayana formula. Recently, Blagouchine ([5, p. 413]) gave a general expression of these series $\nu_k$ in the case where $k$ is a positive integer:

$$\nu_k = \frac{\gamma}{2} - \frac{\ln 2\pi}{k+1} + \frac{1}{k} + \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^r \binom{k}{2r-1} \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) + \sum_{r=1}^{\lfloor \frac{k+1}{2} \rfloor - 1} (-1)^r \binom{k}{2r} \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r + 1). \tag{1}$$

This formula seems quite cumbersome but can be much simplified using the functional equation of $\zeta$. After some elementary transformations, we show that equation (1) can be reduced to the following equivalent (but much more pleasant) expression:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k, \tag{2}$$

where $C_k$ is a rational number (see Proposition 1). Moreover, this expression allows to highlight a deep connection between $\nu_{2k}$ and the sum (in the sense of the Ramanujan summation of divergent series) of the series $\sum_{n \geq 1} n^{2k} H_n$, where $H_n$ is the $n$th harmonic number (see Remark 2).

Next, in a second part, we introduce a generalization of these series series $\nu_k$ replacing the zeta values by certain multiple zeta values. A natural extension may
be defined as follows: for all integers \( k \geq -1 \) and \( p \geq 0 \), we consider the class of series \((\nu_{k,p})\) with
\[
\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n,1,\ldots,1),
\]
where
\[
\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},
\]
so that the previous series \( \nu_k \) become \( \nu_{k,0} \). Then we establish (see Proposition 2) the following identity which is the main result of this work:
\[
\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G^{(k+1)}_n|}{n^{p+1}},
\]
where \( G^{(k)}_n \) denotes the Gregory coefficients of higher order recently introduced by Blagouchine (cf. [5, 6]). They are defined by
\[
G^{(k)}_n := \frac{1}{n!} \sum_{j=1}^{n} \frac{s(n,j)}{j+k} \quad (k \geq 0, n \geq 1),
\]
where \( s(n,j) \) are the Stirling numbers of the first kind. Comprehensive informations on the Stirling numbers of the first and the second kind may be found in [1, 5, 12, 15, 18]. One can prove easily (see Lemma 4) that \( G^{(k)}_n = (-1)^{n+1}|G^{(k)}_n| \), so that the rationals numbers \( G^{(k)}_n \) alternate in sign. As a special case of equation (3), we derive the following result:
\[
\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G^{(k)}_n|}{n} \quad (k \geq 0).
\]
In the case \( k = 1 \), we recover the classical Mascheroni’s series for \( \gamma \) (cf. [5, p. 406], [15, p. 280]):
\[
\gamma = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362880} + \cdots.
\]
Another notable consequence of formula (3) is the deduction of this nice formula (see Example 3):
\[
\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2} \gamma^2 + \frac{\pi^2}{12},
\]
where \( \zeta_H(s) \) denotes the Apostol-Vu harmonic zeta function (cf. [2, 3, 4]) and \( \gamma_1 \) is the first Stieltjes constant (cf. [7, 8]).

Finally, in the last section, we highlight a relation between the series \( \nu_k \), the Stirling numbers of the second kind \( S(n,k) \), and the shifted Mascheroni series \( \sigma_r \), whose study was the main subject of [12] (see Proposition 3 and Example 4).
1 The case of a positive integer

In this section, we focus on the case of a positive integer \( k \) and give two independent proofs of our formula (2). More precisely, we prove the following proposition:

**Proposition 1.** For any positive integer \( k \), we have

\[
\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k
\]

with

\[
C_k = \frac{1}{k} + \sum_{n=1}^{k} \binom{k}{n} \frac{B_n H_{n-1}}{k+1-n}, \tag{6}
\]

where \( H_n \) are the harmonic numbers,

\[
H_0 = 0, \quad H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \geq 1),
\]

and \( B_n \) are the Bernoulli numbers defined by means of the exponential generating function

\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).
\]

In particular, \( B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_{2r+1} = 0 \) for \( r \geq 1 \).

**Proof.** We can quite easily derive (2) from (1). Differentiation of the functional equation

\[
\zeta(s) = 2(2\pi)^{s-1} \Gamma(1-s)\zeta(1-s) \sin \frac{\pi s}{2}
\]

(cf. [3, Eq. (25.4.2)]), leads to the two relations

\[
(-1)^r \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r + 1) = \zeta'(-2r) \quad (r \geq 1),
\]

and

\[
(-1)^r \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) = -\zeta'(1 - 2r) + \frac{B_{2r}}{2r} \left(H_{2r-1} - \gamma - \ln 2\pi\right) \quad (r \geq 1).
\]

Substituting these relations into (1) and grouping together the terms under the two symbols \( \Sigma \), leads to the expression

\[
\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k,
\]
where the rational constant $C_k$ is given by equation (6).

Another alternative proof of (2), independant from (1), may be deduced from the expansion in powers of $z$ of the relation

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \zeta^R(j - k) = (1 - e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \zeta'(-k)$$

$$+ (1 - e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \frac{1}{(k+1)^2}$$

$$+ \int_0^1 \ln(t + 1)e^{-zt} dt$$

(cf. [8, p. 93]), with

$$\zeta^R(j - k) = \begin{cases} \gamma & \text{if } j = k + 1 \\ \zeta(j - k) - \frac{1}{j-k+1} & \text{otherwise.} \end{cases}$$

Rewriting the series $\nu_k$ as

$$\nu_k = \sum_{j=k+2}^{\infty} \frac{(-1)^{j-k}}{j} \zeta(j - k),$$

and using the well-known relations

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(1 - 2r) = -\frac{B_{2r}}{2r}, \quad \text{and} \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi \quad \text{(cf. [3, p. 605])},$$

then a careful identification of the terms in $z^k$ in the previous development leads again, after some simplifications, to formula (2), and provides in addition another equivalent expression for the constant $C_k$:

$$C_k = \frac{H_k}{k+1} - \sum_{j=1}^{k} \frac{B_j}{j(k+1-j)}. \quad (7)$$

An unexpected consequence of this equivalence is the curious identity

$$\sum_{j=1}^{k} \frac{B_j}{k+1-j} \left\{ \frac{1}{j} + \binom{k}{j} H_{j-1} \right\} = \frac{H_k}{k+1} - \frac{1}{k} \quad (k \geq 1)$$

whose direct proof does not seem obvious. □
**Example 1.** For the first values of $k$, we have the following relations:

\[
\begin{align*}
\nu_1 &= \frac{\gamma}{2} - \frac{1}{2} \ln 2\pi + 1, \\
\nu_2 &= \frac{\gamma}{3} - \frac{1}{2} \ln 2\pi - 2\zeta'(-1) + \frac{2}{3}, \\
\nu_3 &= \frac{\gamma}{4} - \frac{1}{2} \ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}, \\
\nu_4 &= \frac{\gamma}{5} - \frac{1}{2} \ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}, \\
\nu_5 &= \frac{\gamma}{6} - \frac{1}{2} \ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}.
\end{align*}
\]

**Remark 1.** Starting from the Maclaurin series expansion

\[
\psi(x + 1) + \gamma = \sum_{n=2}^{\infty} (-1)^n \zeta(n)x^{n-1} \quad (|x| < 1)
\]

(cf. [3, Eq. (25.8.5)]), where $\psi(x)$ denotes the digamma function (i.e. the logarithmic derivative of the $\Gamma$-function), and multiplying each side by $x^k$ (with $k \geq 1$), then an integration between 0 and 1 gives

\[
\nu_k = \frac{\gamma}{k + 1} + \int_0^1 x^k \psi(x + 1) \, dx.
\]

Thus, it follows from formula (2) that

\[
\int_0^1 x^k \psi(x + 1) \, dx = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k \quad (k \geq 1).
\]

**Remark 2** (Link with the Ramanujan summation: part I). Candelpergher et al. ([10, Corollary 1], see also [8, p. 82]) established that

\[
\sum_{n \geq 1} \frac{\mathcal{R} H_n}{n^p} = \frac{3}{2} \gamma - \frac{1}{2} \ln 2\pi + \frac{1}{2},
\]

and for any positive integer $p$,

\[
\sum_{n \geq 1} n^p H_n = \left(1 - B_{p+1} \right) \gamma - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{p} (-1)^j \binom{p}{j} \zeta'(-j) + R_p \quad \text{with} \quad R_p \in \mathbb{Q},
\]

where the symbol $\sum$ denotes the sum of the series in the sense of the Ramanujan summation of divergent series (cf. [7, 8, 9, 10]). For $p = 2k$ (with $k \geq 1$), we have
\( B_{p+1} = 0 \) and \( R_p = C_p - \frac{B_p}{2^p} + \frac{B_p}{2} \), then, in view of formula (2), these relations may be translated into the following identities:

\[
\sum_{n \geq 1}^R H_n = \nu_1 + \gamma - \frac{1}{2},
\]

and for \( k \geq 1 \),

\[
\sum_{n \geq 1}^R n^{2k} H_n = \nu_{2k} + \frac{1 - 2k}{2} \zeta(1 - 2k) = \nu_{2k} + \zeta'(2k) + (2k - 1) \frac{B_{2k}}{4k} .
\] (8)

In particular, we have

\[
\sum_{n \geq 1}^R n^2 H_n = \nu_2 + \zeta'(-2) + \frac{B_2}{4} = \nu_2 - \frac{\zeta(3)}{4\pi^2} + \frac{1}{24} .
\]

2 The case \( k = -1 \)

The case \( k = -1 \) behaves differently from the previous case and must be studied separately. We recall the identities

\[
\nu_{-1} = \int_{0}^{1} \frac{\psi(x + 1) + \gamma}{x} \, dx = \sum_{n=1}^{\infty} \frac{\ln(m + 1)}{m(m + 1)} = - \sum_{n=2}^{\infty} \zeta'(n) = 1.2577468869 \ldots
\]

(cf. [8, p. 105], [11, p. 142]). Another interesting representation (communicated by I. V. Blagouchine) is

\[
\nu_{-1} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3/2 + ix)}{(1/2 + ix) \cosh(\pi x)} \, dx.
\]

Moreover, we can write yet another relation which will be useful in the next section: let \( \kappa_1 \) be the constant

\[
\kappa_1 := \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = 0.5290529699 \ldots ,
\]

where the rational numbers \( b_n \) are the Bernoulli numbers of the second kind defined by means of their generating function

\[
\frac{x}{\ln(x + 1)} = 1 + \sum_{n=1}^{\infty} b_n x^n \quad (|x| < 1).
\]

These numbers \( b_n \) were introduced and studied by Jordan ([15, p. 265 et seq.]). Note that several authors quoted here use different notations: \( b_n \) are denoted by
\(G_n\) (and called \textit{Gregory coefficients}) in [5, 6, 7], and they are denoted by \(\frac{\beta_n}{n!}\) in [8]. The coefficients \(n! b_n\) are sometimes called \textit{Cauchy numbers} (cf. [9]). The constants \(\kappa_1\) and \(\nu_{-1}\) are linked by the relation

\[
\kappa_1 + \frac{1}{2} \zeta(2) = \nu_{-1} + \gamma_1 + \frac{1}{2} \gamma^2
\]

(cf. [7, Eq. (37)], [8, Eq. (3.23) p. 105]), where \(\gamma_1\) denotes the first Stieltjes constant (cf. [3, 7, 8])

\[
\gamma_1 = \lim_{n \to \infty} \left\{ \sum_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2} \ln^2 n \right\} = -0.07281584548\ldots.
\]

In terms of the Ramanujan summation, \(\gamma_1\) is \(\sum_{n \geq 1} \frac{\ln n}{n}\) (cf. [8, p. 67]), whereas \(\kappa_1\) is \(\sum_{n \geq 1} \frac{H_n}{n}\) (cf. [8, Eq. (4.29) p. 133]).

### 3 Alternating series involving multiple zeta values

In this section, we consider a more general class of series of the previous type replacing zeta values with certain multiple zeta values. We prove our formula (3) and deduce some interesting consequences.

**Proposition 2.** For all integers \(p \geq 0\) and \(k \geq -1\), let

\[
\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n,1,\ldots,1);
\]

then

\[
\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{np+1},
\]

where the rational numbers \(G_n^{(k)}\) are defined by equation (4).

**Corollary 1.** In particular, for \(p = 0\), we have

\[
\nu_{k-1,0} = \nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \quad (k \geq 0);
\]

and since \(G_n^{(1)} = b_n\), for \(k = 0\), we have also

\[
\nu_{0,p} = \kappa_p := \sum_{n=1}^{\infty} \frac{|b_n|}{np+1} \quad (p \geq 0).
\]
In order to prove Proposition 2, we begin by stating the following lemmas:

**Lemma 1.** For all integers $j \geq 1$ and $p \geq 0$, we have
\[
\int_0^1 \frac{\ln^j(1-x)}{x} \ln^p(x) \frac{dx}{x} = (-1)^{j+p} j! p! \zeta(j+1, \frac{1}{p}, \ldots, \frac{1}{p}). \tag{10}
\]

*Proof.* This follows directly from [18, Eq. (2.27), (2.28)]. \qed

**Lemma 2.** The Stirling numbers of the first kind $s(n,j)$ with fixed $j \geq 1$ admit the (vertical) exponential generating function (cf. [1, Eq. (2.8)])
\[
\ln^j(1+x) = \sum_{n=j}^{\infty} s(n,j) \frac{x^n}{n!} \quad (|x| < 1). \tag{11}
\]

**Lemma 3.** For all integers $n \geq 1$ and $p \geq 0$, we have
\[
(-1)^p \int_0^1 x^{n-1} \ln^p(x) \frac{dx}{x} = \frac{p!}{n^{p+1}} \tag{12}
\]

*Proof.* This is nothing else than [7, Eq. (41)] in the case where $p$ is an integer. \qed

**Lemma 4.** For all integers $n \geq 1$ and $k \geq 0$, we have
\[
G^{(k)}_n = \frac{(-1)^{n+1}}{n!} \int_0^1 x^k (1-x)^{n-1} dx,
\]
where $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$ is the Pochhammer symbol. In particular, this implies that
\[
G^{(k)}_n = (-1)^{n+1} |G^{(k)}_n|. \tag{13}
\]

*Proof.* Integration between 0 and 1 of the expansion
\[
x^{k-1}x(x-1)\cdots(x-n+1) = \sum_{j=1}^{n} s(n,j) x^{j+k-1}
\]
gives the required result. \qed

*Proof of Proposition 2.* Using successively formulas (10)–(13) above, we can write
the following equalities:

\[ \nu_{k,p} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j+k+1} \zeta(j+1,1,\ldots,1) \]

\[ = \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_{0}^{1} \frac{\ln^{j}(1-x) \ln^{p}(x)}{j!} \, dx \]

\[ = \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_{0}^{1} \left( \sum_{n=j}^{\infty} (-1)^{n} s(n,j) \frac{j^{n}}{n!} \right) \frac{\ln^{p}(x)}{x} \, dx \]

\[ = \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \sum_{n=j}^{\infty} (-1)^{n} s(n,j) \frac{j^{n}}{n!} \int_{0}^{1} x^{n-1} \ln^{p}(x) \, dx \]

\[ = \sum_{j=1}^{\infty} \frac{1}{j+k+1} \sum_{n=j}^{\infty} (-1)^{n} s(n,j) \frac{j^{n}}{n!} n^{p+1} \]

\[ = \sum_{j=1}^{\infty} \frac{1}{j+k+1} \sum_{n=j}^{\infty} \frac{s(n,j)}{n!} \frac{1}{n^{p+1}} \]

\[ = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} C_{n}^{(k+1)}}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{|G_{n}^{(k+1)}|}{n^{p+1}}. \]

This completes the proof. \(\square\)

**Example 2.** For the first values of \(k \geq -1\), we have the following expansions in series containing only positive rational terms:

\[ \nu_{-1} = \sum_{n=1}^{\infty} \frac{|G_{n}^{(0)}|}{n} = 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \cdots, \]

\[ \nu_{0} = \sum_{n=1}^{\infty} \frac{|G_{n}^{(1)}|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362880} + \cdots, \]

\[ \nu_{1} = \sum_{n=1}^{\infty} \frac{|G_{n}^{(2)}|}{n} = \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25200} + \frac{731}{725760} + \cdots, \]

\[ \nu_{2} = \sum_{n=1}^{\infty} \frac{|G_{n}^{(3)}|}{n} = \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100800} + \frac{5849}{10886400} + \cdots, \]

\[ \nu_{3} = \sum_{n=1}^{\infty} \frac{|G_{n}^{(4)}|}{n} = \frac{1}{5} + \frac{1}{120} + \frac{1}{420} + \frac{83}{80640} + \frac{59}{108000} + \frac{397}{1209600} + \cdots. \]
Example 3. Let $\zeta_H$ be the Apostol-Vu harmonic zeta function (cf. [2, 3, 4, 10]) defined for $\text{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s}.$$

We recall that $\zeta_H$ is analytic in the half-plane $\text{Re}(s) > 1$ and can be extended meromorphically with poles at the integers $1, 0, -1, -3, -5, \ldots$. The special values at negative even integers are $\zeta_H(-2k) = B_{2k}/2 - B_{2k}/4k$. The special values at positive integers are also well-known: the first values are $\zeta_H(2) = 2\zeta(3), \zeta_H(3) = 5/4\zeta(4),$ and more generally, they may be computed by means of the following beautiful formula (first discovered by Euler (cf. [14]) and several times rediscovered afterwards):

$$2\zeta_H(n) = (n + 2)\zeta(n + 1) - \sum_{r=1}^{n-2} \zeta(r + 1)\zeta(n - r) \quad (n \geq 3).$$

Otherwise, by Proposition 2 above, we can write

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \nu_{0,1} - \nu_{-1} + \zeta(2) = \kappa_1 - \nu_{-1} + \zeta(2),$$

and thus, from equation (9), we derive the following elegant evaluation:

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12} = 0.916240149\ldots. \quad (14)$$

Another expression of this constant is $\zeta''(0) + \frac{1}{4}\ln^2(2\pi) + \frac{\pi^2}{8}$ (cf. [3, Eq. (25.6.12)]).

Remark 3 (Link with the Ramanujan summation: part II). For $s \in \mathbb{C}$, let $\zeta^R_H$ be the function $s \mapsto \sum_{n \geq 1}^{\mathbb{R}} H_n n^{-s}$ where $\sum_{n \geq 1}^{\mathbb{R}}$ stands for the Ramanujan summation. The function $\zeta^R_H$ is an entire function linked to the harmonic zeta function $\zeta_H$ by the relation

$$\zeta^R_H(s) = \zeta_H(s) - \int_1^{\infty} x^{-s} (\psi(x + 1) + \gamma) \, dx \quad \text{for } \text{Re}(s) > 1$$

(cf. [10, Eq. (84)]). We have the identities

$$\zeta^R_H(1) = \nu_{0,1} = \kappa_1, \quad \zeta^R_H(0) = \nu_1 + \gamma - \frac{1}{2},$$

and formula (8) may be nicely rewritten

$$\zeta^R_H(-2k) = \zeta_H(-2k) + \zeta'(-2k) + \nu_{2k}. $$

11
4 Link with the shifted Mascheroni series

Let us consider now the forward shifted Mascheroni series which are defined by

\[ \sigma_r := \sum_{n=1}^{\infty} \frac{|b_{n+r}|}{n}, \quad \text{for } r = 0, 1, 2, \ldots. \]

We have in particular \( \sigma_0 = \nu_0 = \gamma. \) The study of these series \( \sigma_r \) was the main subject of [12]. Among other things, we have established the following decomposition of \( \zeta'(-j) \) on the “basis” of \( \sigma_r \) (cf. [12, Proposition 3]):

\[
\zeta'(-j) = \sum_{r=2}^{j+1} (-1)^{j-r}(r-1)!S(j, r-1)\sigma_r - \frac{B_{j+1}}{j+1} \gamma - \frac{B_{j+1}}{(j+1)^2}, \quad \text{for } j = 1, 2, 3, \ldots,
\]

where \( S(j, r) \) are Stirling numbers of the second kind; moreover, for \( j = 0, \) we have also a similar relation:

\[
\frac{1}{2} \ln 2\pi = -\zeta'(0) = \sigma_1 + \frac{\gamma}{2} + \frac{1}{2}.
\]

Then, substituting these relations into (2) enables us to write each series \( \nu_k \) with \( k \geq 1 \) as an integral linear combination of \( \gamma, \sigma_1, \sigma_2, \ldots, \sigma_k \) plus a rational number \( D_k \) which is closely linked to \( C_k. \) In this combination, the coefficient of \( \gamma \) is zero since it is equal to \( \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j \) which vanishes by a well-known property of the Bernoulli numbers. Finally, equation (2) may be nicely rewritten in terms of \( \sigma_r \) as follows:

**Proposition 3.** For all integers \( k \geq 1, \) we have the relation

\[
\nu_k = D_k - \sigma_1 + \sum_{r=2}^{k} (-1)^{r-1}(r-1)! \left( \sum_{j=r-1}^{k-1} \binom{k}{j} S(j, r-1) \right) \sigma_r \tag{15}
\]

with

\[
D_k = C_k - \frac{1}{2} + \sum_{n=2}^{k} \binom{k}{n} \frac{B_n}{n(k+1-n)} = \frac{1}{k} + \sum_{n=1}^{k} \frac{1}{\binom{k}{n}} \frac{B_n H_n}{k+1-n}.
\]
Example 4. For the first values of \( k \), we have the following relations:

\[
\begin{align*}
\nu_1 &= \frac{1}{2} - \sigma_1 , \\
\nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2 , \\
\nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3 , \\
\nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4 , \\
\nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5 , \\
\nu_6 &= \frac{23}{180} - \sigma_1 + 62\sigma_2 - 420\sigma_3 + 1560\sigma_4 - 1800\sigma_5 + 720\sigma_6 .
\end{align*}
\]

Acknowledgments

The author gratefully acknowledges the referee for his careful reading, constructive comments and helpful suggestions.

References


[10] B. Candelpergher, H. Gadiyar, and R. Padma, Ramanujan summation and the exponential generating function $\sum_{k=0}^{\infty} \frac{z^k}{k!} \zeta'(-k)$, *Ramanujan J.* **21** (2010), 99–122.


