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# On shifted Mascheroni series and hyperharmonic numbers



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## ABSTRACT

*Text.* In this article, we study the nature of the forward shifted series  $\sigma_r = \sum_{n>r} \frac{|b_n|}{n-r}$  where  $r$  is a positive integer and  $b_n$  are Bernoulli numbers of the second kind, expressing them in terms of the derivatives  $\zeta'(-k)$  of zeta at the negative integers and Euler's constant  $\gamma$ . These expressions may be inverted to produce new series expansions for the quotient  $\zeta(2k+1)/\zeta(2k)$ . Motivated by a theoretical interpretation of these series in terms of Ramanujan summation, we give an explicit formula for the Ramanujan sum of hyperharmonic numbers as an application of our results.

*Video.* For a video summary of this paper, please visit <https://youtu.be/uyLmgDh9JV8>.

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## 0. Introduction

The series

$$\frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362880} + \dots$$

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has a long history which goes back to the works of Lorenzo Mascheroni (1790) who was the first scholar to recognize Euler’s constant

$$\gamma = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right)$$

as being the sum of the series. This is the first known convergent series representation for Euler’s constant with rational coefficients only. The Mascheroni series was subsequently rediscovered several times during the next centuries (cf. [B], pp. 379–380 for more historical references).

With modern notations (cf. [J], p. 280), Mascheroni series is nothing else than

$$\sum_{n \geq 1} \frac{|b_n|}{n}$$

where  $b_n$  denotes Bernoulli numbers of the second kind (also called *Gregory coefficients* by some authors).

In this paper, we investigate shifted Mascheroni series

$$\sigma_r = \sum_{n > r} \frac{|b_n|}{n - r} \quad r = 0, 1, 2, \dots$$

for which very few results are known apart from  $\sigma_0 = \gamma$  and  $\sigma_1 = \frac{1}{2} \ln(2\pi) - \frac{\gamma}{2} - \frac{1}{2}$  ([Y1], Corollary 9).<sup>1</sup> More precisely, we show that for  $r \geq 2$ ,

$$\sigma_r = -\frac{1}{(r-1)!} \sum_{k=1}^{r-1} S_1(r-1, k) \zeta'(-k) + (-1)^r b_r \gamma - \frac{1}{(r-1)!} \sum_{k=1}^r S_1(r-1, k-1) \frac{B_k}{k^2},$$

where  $S_1(r, k)$  are (unsigned) Stirling numbers of the first kind and  $B_k$  are Bernoulli numbers (Proposition 2 and Example 1). We remark that for any positive integer  $r$  the sum of the negatively shifted series

$$\sum_{n \geq 0} \frac{(-1)^n b_n}{n+r} = \frac{1}{r} - \sum_{n \geq 1} \frac{|b_n|}{n+r}$$

is the natural logarithm of an explicit rational number (Proposition 1 below), and therefore is always a period in the sense of Kontsevich and Zagier [KZ]. It is conjectured, however, that  $\sigma_0 = \gamma$  is not a period, and it may likewise be reasonably supposed from the above formula that  $\sigma_r$  is not a period for any positive integer  $r$ .

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<sup>1</sup> Very recently, Blagouchine [B2] gave as well closed form expressions for  $\sigma_2$  and  $\sigma_3$ .

The above expression for  $\sigma_r$  admits the dual relation

$$\zeta'(-k) = \sum_{r=2}^{k+1} (-1)^{k-r} (r-1)! S_2(k, r-1) \sigma_r - \frac{B_{k+1}}{k+1} \gamma - \frac{B_{k+1}}{(k+1)^2},$$

for  $k \geq 1$ , where  $S_2(k, r)$  are Stirling numbers of the second kind (Proposition 3 and Example 2). This decomposition of  $\zeta'(-k)$  on the “basis” of  $\sigma_r$  enables to deduce in §3 a new series representation for the quotient  $\zeta(2k+1)/\zeta(2k)$  which is stated in Theorem 1. More precisely, we show that for  $k \geq 2$ ,

$$\frac{\zeta(2k+1)}{\zeta(2k)} = \frac{4}{B_{2k}} \left[ \sum_{n=2k+2}^{\infty} \frac{|b_n|(n-1)^2 U_k(n)}{(n-2) \dots (n-2k-1)} - C_k \right],$$

where  $U_k \in \mathbb{Z}[x]$  is a monic polynomial of degree  $2k-3$  and  $C_k$  is a rational constant. On the basis of advanced computations performed with PARI software, we conjecture that the constants  $C_k$  are always positive, and that the polynomials  $U_k$  are irreducible over  $\mathbb{Z}[x]$ . In addition, we can prove the irreducibility of  $U_k$  for a presumably infinite set of  $k$  values. More precisely, we prove that if  $p = 2k+1$  is an odd prime which is not a Wolstenholme prime, then the polynomial  $U_k$  is irreducible in  $\mathbb{Z}[x]$  (Theorem 2).

We conclude in §4 with a theoretical interpretation of these identities; we show that the sequence  $(\sigma_r)$  of shifted Mascheroni series is the dual sequence of the sequence of Ramanujan sums of hyperharmonic numbers. This enables us to give an explicit formula for the Ramanujan sum of hyperharmonic numbers in terms of known constants (Corollary 1); in particular the Ramanujan sum of the order  $q$  hyperharmonic numbers is a specific  $\mathbb{Q}$ -linear combination of  $\{1, \gamma, \ln(2\pi)\} \cup \{\zeta'(-k)\}_{k=1}^{q-1}$ , and this expression is consistent with the main result of [CGP].

## 1. Preliminaries

### 1.1. Bernoulli numbers and polynomials

The Bernoulli numbers  $(B_n)$  are defined by the generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$$

or by the equivalent recursion:

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n-1} \frac{B_k}{k!(n-k)!} = 0 \quad \text{for } n \geq 2.$$

The first values are

$$B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42},$$

and

$$B_{2k+1} = 0 \quad \text{for } k \geq 1.$$

The Bernoulli numbers of the second kind ( $b_n$ ) are determined by the generating function:

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} b_n t^n$$

or by the equivalent recursion:

$$b_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n-1} \frac{(-1)^k b_k}{n-k} = 0 \quad \text{for } n \geq 2.$$

The first values are

$$b_0 = 1, b_1 = \frac{1}{2}, b_2 = -\frac{1}{12}, b_3 = \frac{1}{24}, b_4 = -\frac{19}{720}, b_5 = \frac{3}{160} \text{ etc.}$$

An explicit expression is given by

$$n!b_n = \int_0^1 x(x-1)(x-2)\dots(x-n+1) dx.$$

More generally, let  $B_n^{(z)}(x)$  and  $b_n^{(z)}(x)$  be the order  $z$  Bernoulli polynomials of the first and second kind defined respectively by the generating functions

$$\left(\frac{t}{e^t - 1}\right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)}(x) \frac{t^n}{n!}$$

and

$$\left(\frac{t}{\ln(1+t)}\right)^z (1+t)^x = \sum_{n=0}^{\infty} b_n^{(z)}(x) t^n.$$

These are polynomials of degree  $n$  in  $x$  and of degree  $n$  in the order  $z$ . They satisfy the difference equations

$$B_n^{(z)}(x+1) - B_n^{(z)}(x) = nB_{n-1}^{(z-1)}(x), \quad b_n^{(z)}(x+1) - b_n^{(z)}(x) = b_{n-1}^{(z)}(x),$$

and are linked together by Carlitz’s identities ([Ca], eqs. (2.11), (2.12))

$$n!b_n^{(z)}(x) = B_n^{(n-z+1)}(x+1), \quad B_n^{(z)}(x) = n!b_n^{(n-z+1)}(x-1).$$

In particular,

$$B_n^{(n)}(1) = n!b_n^{(1)}(0) = n!b_n.$$

When  $x = 0$  or  $z = 1$  that part of the notation is usually suppressed.

### 1.2. Stirling numbers

The (unsigned) Stirling numbers of the first kind  $S_1(n, k)$  are defined by the generating function

$$\frac{[\ln(1+t)]^k}{k!} = \sum_{n=k}^{\infty} (-1)^{n-k} S_1(n, k) \frac{t^n}{n!}.$$

They also arise as the coefficients of the rising factorial

$$(x)_n := x(x+1) \dots (x+n-1) = \sum_{k=0}^n S_1(n, k) x^k,$$

and  $S_1(n, k) = 0$  for  $n < k$ .

The Stirling numbers of the second kind  $S_2(k, m)$  are defined by

$$S_2(k, m) = \frac{1}{m!} \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} j^k,$$

and  $S_2(k, m) = 0$  if  $k < m$ .

The Stirling numbers of the first and second kind satisfy the difference equations

$$S_1(n+1, k) = S_1(n, k-1) + nS_1(n, k), \quad S_2(n+1, k) = S_2(n, k-1) + kS_2(n, k),$$

the orthogonality relation

$$\sum_{k=m}^n (-1)^{n-k} S_1(n, k) S_2(k, m) = \sum_{k=m}^n (-1)^{n-k} S_2(n, k) S_1(k, m) = \delta_{m,n}$$

where  $\delta_{m,n}$  is the Kronecker symbol, and the duality relation

$$f_n = \sum_{m=0}^n (-1)^{n-m} S_1(n, m) g_m \iff g_n = \sum_{m=0}^n S_2(n, m) f_m.$$

### 1.3. Barnes multiple log gamma functions

For positive integer  $r$ ,  $\Re(s) > r$  and  $\Re(a) > 0$ , let  $\zeta_r(s, a)$  be the Barnes multiple zeta function of order  $r$  defined by

$$\begin{aligned} \zeta_r(s, a) &:= \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-at}t^{s-1}}{(1 - e^{-t})^r} dt \\ &= \sum_{m=0}^\infty \binom{m+r-1}{m} (m+a)^{-s}. \end{aligned}$$

When  $r = 1$  and  $a = 1$  we have the Riemann zeta function  $\zeta_1(s, 1) = \zeta(s)$ . The Barnes multiple log gamma function  $\Psi_r(a)$  of order  $r$  is defined for  $\Re(a) > 0$  by

$$\Psi_r(a) := \frac{\partial}{\partial s} \zeta_r(s, a) \Big|_{s=0}.$$

Let us remind (cf. [Y1], Corollary 4) that the function  $\Psi_r(a)$  admits the following expansion:

$$\Psi_r(a) = \sum_{\substack{n \geq 0 \\ n \neq r}} \frac{(-1)^n B_n^{(n)}(a)}{n!(n-r)} + \frac{(-1)^r B_r^{(r)}(a)}{r!} \gamma + \frac{(-1)^r}{r!} \left[ \frac{\partial}{\partial s} B_r^{(r+s)}(a) \right]_{s=0}.$$

## 2. Shifted Mascheroni series

We begin this section by disposing of the simpler case of the negatively shifted Mascheroni series, which may be evaluated using only natural logarithms of integers.

**Proposition 1.** *For all integers  $r \geq 1$  we have*

$$\sum_{n \geq 1} \frac{|b_n|}{n+r} = \frac{1}{r} + \sum_{k=1}^r (-1)^k \binom{r}{k} \ln(k+1).$$

**Proof.** We actually prove the more general assertion

$$\sum_{n=0}^\infty \frac{(-1)^n b_n(a-1)}{n+r} = \sum_{k=0}^r (-1)^{k-1} \binom{r}{k} \ln(k+a) \tag{1}$$

for  $\Re(a) > 0$  and positive integers  $r$ . The case  $r = 1$  of this identity is given in ([Y1], Corollary 8). Assuming the statement is true for  $r$ , we use the difference equation

$$b_n(x+1) - b_n(x) = b_{n-1}(x)$$

to compute

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n b_n(a-1)}{n+r+1} &= \sum_{n=0}^{\infty} \frac{(-1)^n b_{n+1}(a)}{n+r+1} - \sum_{n=0}^{\infty} \frac{(-1)^n b_{n+1}(a-1)}{n+r+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} b_n(a)}{n+r} - \sum_{n=0}^{\infty} \frac{(-1)^{n-1} b_n(a-1)}{n+r} \\ &= \sum_{k=0}^r (-1)^{k-1} \binom{r}{k} \ln(k+a) - \sum_{k=0}^r (-1)^{k-1} \binom{r}{k} \ln(k+a+1) \\ &= \sum_{k=0}^r (-1)^{k-1} \binom{r}{k} \ln(k+a) - \sum_{k=1}^{r+1} (-1)^k \binom{r}{k-1} \ln(k+a) \\ &= \sum_{k=0}^{r+1} (-1)^{k-1} \binom{r+1}{k} \ln(k+a), \end{aligned}$$

which proves the statement for  $r + 1$ . By induction, the assertion holds for all positive integers  $r$ ; the statement of the proposition is then obtained by taking  $a = 1$ .  $\square$

**Remark 1.** By integration of the well-known decomposition of the rational fraction  $\frac{1}{x(x+1)\dots(x+r)}$ , we deduce from (1) the following equivalent identity, for  $a > 0$  and positive integers  $r$ ,

$$\sum_{n=0}^{\infty} \frac{(-1)^n b_n(a-1)}{n+r} = (r-1)! \int_a^{a+1} \frac{dt}{t(t+1)\dots(t+r-1)},$$

which explicitly exhibits the negatively shifted Mascheroni series as periods in the sense of [KZ].

As observed in the introduction, it appears that the nature of the forward shifted series  $\sigma_r$  is not as simple as that of the negatively shifted Mascheroni series.

**Proposition 2.** For all nonnegative integers  $r$ , let

$$\sigma_r := \sum_{n=r+1}^{\infty} (-1)^{n-1} \frac{b_n}{n-r} = \sum_{n=r+1}^{\infty} \frac{|b_n|}{n-r},$$

then

$$\sigma_0 = \gamma, \quad \sigma_1 = \frac{1}{2} (\ln(2\pi) - \gamma - 1),$$

and for  $r \geq 2$ ,

$$\sigma_r = -\frac{1}{(r-1)!} \sum_{k=1}^{r-1} S_1(r-1, k) \zeta'(-k) + (-1)^r b_r \gamma + t_r,$$

with

$$t_r = -\frac{1}{(r-1)!} \sum_{k=1}^r S_1(r-1, k-1) \frac{B_k}{k^2}.$$

**Proof.** From Corollary 4 of [Y1] with  $a = 1$  we have

$$\Psi_r(1) = \sum_{\substack{n \geq 0 \\ n \neq r}} \frac{(-1)^n b_n}{n-r} + (-1)^r \gamma b_r + \frac{(-1)^r}{r!} \left[ \frac{\partial}{\partial s} B_r^{(r+s)}(1) \right]_{s=0},$$

which implies

$$\sigma_r = -\Psi_r(1) + (-1)^r \gamma b_r + t_r$$

where  $t_r \in \mathbb{Q}$ , by means of the recursion

$$\sum_{n=0}^{r-1} \frac{(-1)^n b_n}{n-r} = 0.$$

We now make use of the formula

$$\zeta_r(s, 1) = \sum_{m=0}^{\infty} \binom{m+r-1}{r-1} (m+1)^{-s}$$

([Y2], eq. (3.3)) for Barnes multiple zeta functions to write

$$\zeta_r(s, 1) = \frac{1}{(r-1)!} \sum_{k=1}^{r-1} S_1(r-1, k) \zeta(s-k),$$

from which it follows by differentiating at  $s = 0$  that

$$\Psi_r(1) = \frac{1}{(r-1)!} \sum_{k=1}^{r-1} S_1(r-1, k) \zeta'(-k).$$

Therefore we have

$$\sigma_r = -\frac{1}{(r-1)!} \sum_{k=1}^{r-1} S_1(r-1, k) \zeta'(-k) + (-1)^r \gamma b_r + t_r$$

where

$$t_r = \frac{(-1)^r}{r!} \left[ \frac{\partial}{\partial s} B_r^{(r+s)}(1) \right]_{s=0} = \frac{(-1)^r}{r!} \sum_{j=1}^r \binom{r}{j} \frac{(-1)^{j+1} B_j}{j} B_{r-j}^{(r)}(1)$$



by means of ([Y1], eq. (3.13)). By Carlitz’ identities ([Ca], eqs. (2.11), (2.12)) and comparison of the generating functions, we have

$$\begin{aligned} B_{r-j}^{(r)}(1) &= (r-j)!b_{r-j}^{(1-j)}(0) \\ &= (r-j)! \cdot \text{coefficient of } t^{r-j} \text{ in } \left(\frac{\log(1+t)}{t}\right)^{j-1} \\ &= (-1)^{r-j}(r-j)!(j-1)!S_1(r-1, j-1)/(r-1)!, \end{aligned}$$

from which the stated formula for  $t_r$  follows directly.  $\square$

**Example 1.**

$$\begin{aligned} \sigma_2 &= -\zeta'(-1) - \frac{1}{12}\gamma - \frac{1}{24} \\ \sigma_3 &= -\frac{1}{2}\zeta'(-1) - \frac{1}{2}\zeta'(-2) - \frac{1}{24}\gamma - \frac{1}{48} \\ \sigma_4 &= -\frac{1}{3}\zeta'(-1) - \frac{1}{2}\zeta'(-2) - \frac{1}{6}\zeta'(-3) - \frac{19}{720}\gamma - \frac{13}{960} \\ \sigma_5 &= -\frac{1}{4}\zeta'(-1) - \frac{11}{24}\zeta'(-2) - \frac{1}{4}\zeta'(-3) - \frac{1}{24}\zeta'(-4) - \frac{3}{160}\gamma - \frac{19}{1920} \\ \sigma_6 &= -\frac{1}{5}\zeta'(-1) - \frac{5}{12}\zeta'(-2) - \frac{7}{24}\zeta'(-3) - \frac{1}{12}\zeta'(-4) - \frac{1}{120}\zeta'(-5) - \frac{863}{60480}\gamma - \frac{5611}{725760}. \end{aligned}$$

This formula for  $\sigma_r$  may be inverted to produce the following formula for the derivatives  $\zeta'(-k)$ :

**Proposition 3.** For all integers  $k \geq 1$ ,  $\zeta'(-k)$  admits the following decomposition:

$$\zeta'(-k) = \sum_{r=2}^{k+1} (-1)^{k-r}(r-1)!S_2(k, r-1)\sigma_r - \frac{B_{k+1}}{k+1}\gamma - \frac{B_{k+1}}{(k+1)^2}.$$

**Proof.** By the duality of Stirling numbers, since  $\{S_1(r-1, k)\}$  gives the connection coefficients passing from the sequence  $\{\zeta'(-k)\}$  to  $\{-(r-1)!(\sigma_r - (-1)^r\gamma b_r - t_r)\}$ , the reverse connection is given by  $\{(-1)^{k-r+1}S_2(k, r-1)\}$ , that is,

$$\zeta'(-k) = \sum_{r=2}^{k+1} (-1)^{k-r}S_2(k, r-1)(r-1)!(\sigma_r - (-1)^r\gamma b_r - t_r).$$

By means of the  $a = 0$  case of identity (5.9) in [Y3], the coefficient of  $\gamma$  in the right side of this expression is given by

$$(-1)^{k-1} \sum_{r=1}^k r!S_2(k, r)b_{r+1} = \frac{(-1)^k B_{k+1}}{k+1}, \tag{2}$$

which equals  $-B_{k+1}/(k+1)$  if  $k > 0$ . Therefore we may write

$$\zeta'(-k) = \sum_{r=2}^{k+1} (-1)^{k-r} S_2(k, r-1)(r-1)! \sigma_r - \frac{B_{k+1}}{k+1} \gamma + u_k,$$

where  $u_k \in \mathbb{Q}$  may be calculated as

$$\begin{aligned} u_k &= \sum_{r=2}^{k+1} (-1)^{k-r+1} (r-1)! S_2(k, r-1) t_r \\ &= \sum_{r=2}^{k+1} (-1)^{k-r} S_2(k, r-1) \sum_{j=1}^r S_1(r-1, j-1) \frac{B_j}{j^2} \\ &= \sum_{j=1}^{k+1} \frac{B_j}{j^2} \sum_{r=j}^{k+1} (-1)^{k-r} S_2(k, r-1) S_1(r-1, j-1) \\ &= - \sum_{j=1}^{k+1} \frac{B_j}{j^2} \delta_{j-1, k} = - \frac{B_{k+1}}{(k+1)^2} \end{aligned}$$

by the orthogonality of Stirling numbers. This completes the proof.  $\square$

**Example 2.**

$$\begin{aligned} \zeta'(-1) &= -\sigma_2 - \frac{1}{12} \gamma - \frac{1}{24} \\ \zeta'(-2) &= \sigma_2 - 2\sigma_3 \\ \zeta'(-3) &= -\sigma_2 + 6\sigma_3 - 6\sigma_4 + \frac{1}{120} \gamma + \frac{1}{480} \\ \zeta'(-4) &= \sigma_2 - 14\sigma_3 + 36\sigma_4 - 24\sigma_5 \\ \zeta'(-5) &= -\sigma_2 + 30\sigma_3 - 150\sigma_4 + 240\sigma_5 - 120\sigma_6 - \frac{1}{252} \gamma - \frac{1}{1512} \\ \zeta'(-6) &= \sigma_2 - 62\sigma_3 + 540\sigma_4 - 1560\sigma_5 + 1800\sigma_6 - 720\sigma_7. \end{aligned}$$

**3. Rational series for the quotient  $\zeta(2k+1)/\zeta(2k)$**

Proposition 3 shows that for even  $k$  the derivative  $\zeta'(-k)$  is a specific integer linear combination of the shifted Mascheroni series  $\sigma_r$ . In this section we use that proposition to deduce a new series representation for the quotient  $\zeta(2k+1)/\zeta(2k)$ .

**Lemma.** For  $k \geq 1$ ,

$$\zeta'(-2k) = -\frac{1}{4} B_{2k} \frac{\zeta(2k+1)}{\zeta(2k)}.$$

**Proof.** Differentiation of the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right)$$

leads to the relation

$$\zeta'(-2k)\pi^{2k} = (-1)^k \frac{(2k)!}{2^{2k+1}}\zeta(2k+1)$$

which is equivalent to the stated relation by means of Euler’s formula

$$\zeta(2k) = -\frac{1}{2}(2i\pi)^{2k} \frac{B_{2k}}{(2k)!}. \quad \square$$

**Theorem 1.** For all integers  $k \geq 1$ , there is a monic polynomial  $P_k \in \mathbb{Z}[x]$  of degree  $2k-1$  and a rational constant  $C_k$  such that

$$\frac{\zeta(2k+1)}{\zeta(2k)} = \frac{4}{B_{2k}} \left[ \sum_{n=2k+2}^{\infty} \frac{|b_n|P_k(n)}{(n-2k-1)2k} - C_k \right],$$

and, moreover, the polynomial  $P_k(x)$  is divisible by  $(x-1)^2$  for all  $k \geq 2$ .

**Proof.** From Proposition 3, for  $k$  even and positive, we have

$$\begin{aligned} \zeta'(-k) &= \sum_{r=1}^k (-1)^{r-1} r! S_2(k, r) \sigma_{r+1} \\ &= \sum_{n>k+1} |b_n| \left[ \sum_{r=1}^k \frac{(-1)^{r-1} r! S_2(k, r)}{n-r-1} \right] + \tilde{C}_k \end{aligned}$$

with

$$\tilde{C}_k = \sum_{r=1}^{k-1} (-1)^{r-1} r! S_2(k, r) \sum_{i=r+2}^{k+1} \frac{|b_i|}{i-r-1}.$$

Considering the rational function

$$Q_k(n) = \sum_{r=1}^k \frac{(-1)^{r-1} r! S_2(k, r)}{n-r-1} = -\frac{\tilde{P}_k(n)}{(n-2)\cdots(n-k-1)},$$

it is clear that the polynomial  $\tilde{P}_k(n)$  is monic with integer coefficients and degree  $k-1$ . To complete the proof we must show that when  $k$  is even and  $k > 2$ ,  $\tilde{P}_k(n)$  is divisible by  $(n-1)^2$ , which is equivalent to showing that  $Q_k(1) = 0$  and  $Q'_k(1) = 0$ .

Since  $S_2(n + 1, k) = S_2(n, k - 1) + kS_2(n, k)$ , for any  $k > 0$  we can compute

$$\begin{aligned} Q_k(1) &= \sum_{r=1}^k (-1)^{r-1} (r - 1)! S_2(k, r) \\ &= \sum_{r=1}^k (-1)^{r-1} ((r - 1)! S_2(k - 1, r - 1) + r! S_2(k - 1, r)) \\ &= \sum_{r=1}^k (c_{k,r-1} - c_{k,r}) = 0 \end{aligned}$$

as a telescoping sum, where  $c_{k,r} = (-1)^r r! S_2(k - 1, r)$ .

By the same identity of Stirling numbers, for any  $k > 0$  we may compute

$$\begin{aligned} Q'_k(1) &= \sum_{r=1}^k \frac{(-1)^{r-1} (r - 1)! S_2(k, r)}{r} \\ &= \sum_{r=1}^k \frac{(-1)^{r-1} (r - 1)! S_2(k - 1, r - 1)}{r} + \sum_{r=1}^k (-1)^{r-1} (r - 1)! S_2(k - 1, r) \\ &= (-1)^{k-1} B_{k-1} + Q_{k-1}(1) = (-1)^{k-1} B_{k-1}, \end{aligned}$$

by means of ([Y3], eq. (2.7)). Therefore for even  $k > 2$ , we have  $Q'_k(1) = 0$ , as desired.

Then, using the previous lemma, we obtain the stated relation by writing  $C_k := \tilde{C}_{2k}$  and  $P_k := \tilde{P}_{2k}$ .  $\square$

**Remark 2.** From the proof above, the constant  $C_k$  may be evaluated by the explicit formula

$$\begin{aligned} C_k &= \sum_{m=1}^{2k-1} (-1)^{m-1} m! S_2(2k, m) \sum_{i=m+2}^{2k+1} \frac{|b_i|}{i - 1 - m} \\ &= \sum_{j=1}^{2k-1} (-1)^{j-1} j^{2k} \sum_{m=j}^{2k-1} \binom{m}{j} \sum_{i=m+2}^{2k+1} \frac{|b_i|}{i - 1 - m}, \end{aligned}$$

and the polynomial  $P_k$  is given by

$$P_k(x) = (x - 2k - 1)_{2k} \sum_{m=1}^{2k} \frac{(-1)^m m! S_2(2k, m)}{x - 1 - m}.$$

**Example 3.** a)

$$C_1 = |b_3| = \frac{1}{24} \quad \text{and} \quad P_1(x) = x - 1,$$

hence

$$\frac{\zeta(3)}{\zeta(2)} = 24 \sum_{n=4}^{\infty} \frac{|b_n|(n-1)}{(n-2)(n-3)} - 1.$$

b)

$$C_2 = (|b_3| + 1/2|b_4| + 1/3|b_5|) - 14(|b_4| + 1/2|b_5|) + 36|b_5| = \frac{113}{480}$$

and

$$P_2(x) = (x-1)^2(x+4),$$

hence

$$\frac{\zeta(5)}{\zeta(4)} = \frac{113}{4} - 120 \sum_{n=6}^{\infty} \frac{|b_n|(n-1)^2(n+4)}{(n-2)(n-3)(n-4)(n-5)}.$$

c)

$$\begin{aligned} C_3 &= (|b_3| + 1/2|b_4| + 1/3|b_5| + 1/4|b_6| + 1/5|b_7|) \\ &\quad - 62(|b_4| + 1/2|b_5| + 1/3|b_6| + 1/4|b_7|) + 540(|b_5| + 1/2|b_6| + 1/3|b_7|) \\ &\quad - 1560(|b_6| + 1/2|b_7|) + 1800|b_7| \\ &= \frac{99325}{36288} \end{aligned}$$

and

$$P_3(x) = (x-1)^2(x^3 + 39x^2 + 38x - 120)$$

hence

$$\frac{\zeta(7)}{\zeta(6)} = 168 \sum_{n=8}^{\infty} \frac{|b_n|(n-1)^2(n^3 + 39n^2 + 38n - 120)}{(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)} - \frac{99325}{216}.$$

**Remark 3.** Kellner [Ke] recently gave a similar expression of the quotients  $\zeta(2k+1)/\zeta(2k)$  as the values of a certain linear functional on a sequence of monic polynomials of degree  $2k$ . Our representation does not appear to be related to that of Kellner.

On the basis of advanced computations performed with PARI software, we raise the following conjectures.

**Conjecture 1.** *The constants  $C_k$  are always positive.*

**Conjecture 2.** For all  $k \geq 2$ , the polynomials  $U_k$  defined by  $U_k(x) := P_k(x)/(x - 1)^2$  are irreducible over  $\mathbb{Z}[x]$ .

The positivity of the constants  $C_k$  was verified numerically for  $k \leq 100$ , and the irreducibility of  $U_k$  verified for  $k \leq 50$ . In addition, we can prove the irreducibility of  $U_k$  for a presumably infinite set of  $k$  values.

A *Wolstenholme prime* is a prime  $p$  for which  $\binom{2p}{p} \equiv 2 \pmod{p^4}$ , or equivalently for which  $p^3$  divides the numerator of the  $(p - 1)$ -st harmonic number  $H_{p-1}$ , or equivalently for which  $p$  divides the numerator of the Bernoulli number  $B_{p-3}$ . The only known Wolstenholme primes are  $p = 16843$  and  $p = 2124679$ ; there are none other less than  $10^9$ , although there are conjecturally infinitely many. Conversely, the Wolstenholme primes form a thin subsequence of the irregular primes, and one also conjectures that infinitely many primes are non-Wolstenholme.

**Theorem 2.** Suppose that  $p = 2k + 1$  is an odd prime which is not a Wolstenholme prime. Then the polynomial  $U_k$  is irreducible in  $\mathbb{Z}[x]$ .

**Proof.** We will show that the shifted polynomial  $U_k(t + 1)$  is a  $p$ -Eisenstein polynomial in  $\mathbb{Z}[t]$  under the stated conditions. If  $p = 2k + 1$  is a prime, then  $j^{2k} \equiv 1$  for  $1 \leq j < p$  and therefore

$$m!S_2(2k, m) \equiv \sum_{j=1}^m (-1)^{m-j} \binom{m}{j} \equiv (-1)^{m-1} \pmod{p}$$

for  $1 \leq m \leq 2k$  by the definition. Therefore

$$\begin{aligned} P_k(t + 1) &= (t - 1) \cdots (t - 2k) \sum_{r=1}^{2k} \frac{(-1)^r r! S_2(2k, r)}{t - r} \\ &\equiv (t - 1) \cdots (t - 2k) \sum_{r=1}^{2k} \frac{-1}{t - r} \pmod{p\mathbb{Z}[t]} \\ &\equiv t^{2k-1} \pmod{p\mathbb{Z}[t]}. \end{aligned}$$

This last congruence follows from the partial fraction decomposition

$$\frac{t^{2k-1}}{(t - 1) \cdots (t - 2k)} = \sum_{r=1}^{2k} \frac{a_r}{t - r}$$

in which  $a_r = (-1)^r \binom{2k}{r} r^{2k} / (2k)! \equiv -1 \pmod{p\mathbb{Z}_{(p)}}$ . Therefore we deduce that the shifted polynomial  $U_k(t + 1) \equiv t^{2k-3} \pmod{p\mathbb{Z}[t]}$ . To complete the proof, we must show that the constant term  $U_k(1)$  of  $U_k(t + 1)$ , considered as an element of  $\mathbb{Z}[t]$ , is not divisible by  $p^2$ .

In a calculation similar to the proof of [Theorem 1](#), we compute

$$\begin{aligned} Q''_{2k}(1) &= 2 \sum_{r=1}^{2k} \frac{(-1)^r (r-1)! S_2(2k, r)}{r^2} \\ &= 2 \left( \sum_{r=1}^{2k} \frac{(-1)^r (r-1)! S_2(2k-1, r-1)}{r^2} + \sum_{r=1}^{2k} \frac{(-1)^r (r-1)! S_2(2k-1, r)}{r} \right) \\ &= 2 \left( \mathbb{B}_{2k-1}^{(2)} - Q'_{2k-1}(1) \right) = 2 \left( \mathbb{B}_{2k-1}^{(2)} - B_{2k-2} \right), \end{aligned}$$

where  $\mathbb{B}_{2k-1}^{(2)}$  denotes the second-order poly-Bernoulli number defined by Kaneko ([\[K\]](#), [Theorem 1](#)) and  $Q_k(x)$  is as in the proof of [Theorem 1](#). Kaneko also proved ([\[K\]](#), [Theorem 3\(1\)](#)) that for odd  $n$  we have  $\mathbb{B}_n^{(2)} = (2-n)B_{n-1}/4$ , and therefore we find that  $Q''_{2k}(1) = -(2k+1)B_{2k-2}/2$ . Elementary calculus shows that  $Q''_{2k}(1) = -P''_k(1)/(2k)!$  and  $P''_k(1) = 2U_k(1)$ , so we finally arrive at the formula

$$U_k(1) = \frac{(2k+1)!B_{2k-2}}{4}.$$

If  $p = 2k + 1$  is a prime, this gives  $U_k(1) = p!B_{p-3}/4$ , which is divisible by  $p^2$  if and only if  $p$  divides the numerator of  $B_{p-3}$ , that is,  $p$  is a Wolstenholme prime. Thus if  $p$  is a prime that is not a Wolstenholme prime,  $U_k(t + 1)$  is  $p$ -Eisenstein and therefore  $U_k$  is irreducible over  $\mathbb{Z}$ .  $\square$

#### 4. Link with the Ramanujan summation of divergent series

**Definition** (*Dual sequences*). Let  $a = (a_n)_{n \geq 0}$  be a sequence of complex numbers. The dual sequence of  $a$  is the sequence  $a^*$  defined for all  $n \geq 0$  by

$$a_n^* = \sum_{k=0}^n (-1)^k \binom{n}{k} a_k.$$

It is well known (cf. [\[S\]](#)) that  $(a^*)^* = a$ , *i.e.*

$$a_n = \sum_{k=0}^n (-1)^k \binom{n}{k} a_k^*.$$

**Definition** (*Hyperharmonic numbers*). For all integers  $n \geq 1$  and  $q \geq 0$ , let  $H_n^{(q)}$  be the numbers defined recursively by

$$H_n^{(0)} = \frac{1}{n} \quad \text{and} \quad H_n^{(q)} = \sum_{j=1}^n H_j^{(q-1)} \quad \text{for } q \geq 1.$$

The sequence  $(H_n^{(q)})_{n \geq 1}$  is the sequence of hyperharmonic numbers of order  $q$ .

**Example 4.**

$$H_n^{(1)} = \sum_{j=1}^n \frac{1}{j} = H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

and

$$H_n^{(2)} = \sum_{j=1}^n H_j = (n + 1)H_n - n.$$

More generally, for  $q \geq 2$ ,  $H_n^{(q)}$  admits the following expression ([Ka,DB]):

$$\begin{aligned} H_n^{(q)} &= \binom{n+q-1}{q-1} (H_{n+q-1} - H_{q-1}) \\ &= \frac{1}{(q-1)!} \left[ \sum_{k=1}^q S_1(q, k) n^{k-1} (H_n - H_{q-1}) + \sum_{k=1}^{q-1} S_1(q, k+1) k n^{k-1} \right]. \end{aligned} \quad (3)$$

A general definition of the Ramanujan summation of a series was given in [CGP], §2.1: If  $x \mapsto a(x)$  is an analytic function on the right half-plane  $\Re(x) > 0$  satisfying certain growth conditions, the Ramanujan sum  $\sum_{n \geq 1}^{\mathcal{R}} a(n)$  is defined to equal the value  $R(1)$ , where  $R$  is the unique analytic solution to the difference equation  $R(x) - R(x+1) = a(x)$  satisfying  $\int_1^2 R(t) dt = 0$ . Here, we use the following slightly restrictive definition which is more convenient for our purpose (cf. [CC2], §5.4, Definition 11).

**Definition** (*Ramanujan summation*). Let  $x \mapsto a(x)$  be an analytic function of moderate growth on the right half-plane  $\Re(x) > 0$ , and let

$$D(a)(n+1) = \sum_{j=0}^n (-1)^j \binom{n}{j} a(j+1) \quad \text{for all integers } n \geq 0.$$

Then the series  $\sum_{n \geq 0} |b_{n+1}| D(a)(n+1)$  converges, and the Ramanujan sum of the series  $\sum_{n \geq 1} a(n)$  is given by

$$\sum_{n \geq 1}^{\mathcal{R}} a(n) = \sum_{n=1}^{\infty} |b_n| D(a)(n).$$

**Proposition 4.** Let  $(\Phi_q)_{q \geq 0}$  be the sequence defined by

$$\Phi_q := \sum_{n \geq 1}^{\mathcal{R}} H_n^{(q)}.$$

Then  $(\Phi_q)_{q \geq 0}$  and  $(\sigma_r)_{r \geq 0}$  are dual sequences.



**Proof.** Let  $R, S$  and  $D$  be respectively the operators of shifting, summation and finite difference on the space of sequences defined in [CC1]. Let us remind (cf. [CC1], Corollary 1) that the operator  $D$  is auto-inverse and leaves invariant the sequence  $H^{(0)} := (H_n^{(0)})_{n \geq 1}$  i.e.  $D(D(a)) = a$  for all sequences and

$$D(H^{(0)})(n) = H_n^{(0)} = \frac{1}{n}.$$

Let us consider the  $r$ -times shifted sequence

$$R^r(H^{(0)}) := (\underbrace{0, \dots, 0}_r, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-r}, \dots).$$

The relation  $DR = (I - S)D$ , and the invariance of  $H^{(0)}$  by  $D$ , allows to write the following binomial expansion

$$DR^r(H^{(0)})(n) = (I - S)^r(H^{(0)})(n) = \sum_{q=0}^r (-1)^k \binom{r}{q} S^q(H^{(0)})(n) = \sum_{q=0}^r (-1)^q \binom{r}{q} H_n^{(q)}$$

since  $S^q(H^{(0)})(n) = H_n^{(q)}$  by definition of the  $q$  order hyperharmonic numbers. Now, the basic properties of the Ramanujan’s summation (cf. [CGP], §2.2) and the expansion above enable to write the following identities

$$\begin{aligned} \sigma_r &= \sum_{n=1}^{\infty} |b_n| R^r(H^{(0)})(n) \\ &= \sum_{n \geq 1}^{\mathcal{R}} D \left( R^r(H^{(0)}) \right) (n) \\ &= \sum_{n \geq 1}^{\mathcal{R}} \sum_{q=0}^r (-1)^q \binom{r}{q} H_n^{(q)} \\ &= \sum_{q=0}^r (-1)^q \binom{r}{q} \sum_{n \geq 1}^{\mathcal{R}} H_n^{(q)} \\ &= \sum_{q=0}^r (-1)^q \binom{r}{q} \Phi_q, \end{aligned}$$

and this shows that  $(\sigma_r)$  is the dual sequence of  $(\Phi_q)$ .  $\square$

Conversely,  $(\Phi_q)$  is the dual sequence of  $(\sigma_r)$ , and this leads to the following statement which is a direct consequence of Proposition 2 and Proposition 4.

**Corollary 1** (Ramanujan’s sum of the order  $q$  hyperharmonic numbers).

$$\begin{aligned} \sum_{n \geq 1}^{\mathcal{R}} H_n^{(q)} &= \sum_{r=0}^q (-1)^r \binom{q}{r} \sigma_r \\ &= -\frac{q}{2} \ln(2\pi) + \gamma \left[ \sum_{r=0}^q \binom{q}{r} b_r \right] + \frac{q}{2} \\ &\quad - \sum_{1 \leq k < r \leq q} (-1)^r \binom{q}{r} \frac{1}{(r-1)!} S_1(r-1, k) \zeta'(-k) \\ &\quad - \sum_{2 \leq k \leq r \leq q} (-1)^r \binom{q}{r} \frac{1}{(r-1)!} S_1(r-1, k-1) \frac{B_k}{k^2}. \end{aligned}$$

**Remark 4.** The coefficient of  $\gamma$  in this expression may also be written more simply in term of the Bernoulli polynomials of the second kind as

$$\sum_{r=0}^q \binom{q}{r} b_r = b_q(q).$$

**Example 5.**

$$\begin{aligned} \sum_{n \geq 1}^{\mathcal{R}} H_n^{(0)} &= \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n} = \sigma_0 = \gamma \\ \sum_{n \geq 1}^{\mathcal{R}} H_n^{(1)} &= \sum_{n \geq 1}^{\mathcal{R}} H_n = \sigma_0 - \sigma_1 = -\frac{1}{2} \ln(2\pi) + \frac{3}{2} \gamma + \frac{1}{2} \\ \sum_{n \geq 1}^{\mathcal{R}} H_n^{(2)} &= -\ln(2\pi) + \frac{23}{12} \gamma - \zeta'(-1) + \frac{23}{24} \\ \sum_{n \geq 1}^{\mathcal{R}} H_n^{(3)} &= -\frac{3}{2} \ln(2\pi) + \frac{55}{24} \gamma - \frac{5}{2} \zeta'(-1) + \frac{1}{2} \zeta'(-2) + \frac{67}{48} \\ \sum_{n \geq 1}^{\mathcal{R}} H_n^{(4)} &= -2 \ln(2\pi) + \frac{1901}{720} \gamma - \frac{13}{3} \zeta'(-1) + \frac{3}{2} \zeta'(-2) - \frac{1}{6} \zeta'(-3) + \frac{1747}{960}. \end{aligned}$$

**Remark 5.** Candelpergher et al. have shown ([CGP], Corollary 1) that for  $k \geq 1$ ,

$$\sum_{n \geq 1}^{\mathcal{R}} n^k H_n = \gamma \left( \frac{1 - B_{k+1}}{k+1} \right) - \frac{1}{2} \ln(2\pi) + \sum_{m=1}^k \binom{k}{m} (-1)^m \zeta'(-m) + r_k \quad \text{with } r_k \in \mathbb{Q}.$$

Our expression of the Ramanujan sum of the order  $q$  hyperharmonic numbers as a  $\mathbb{Q}$ -linear combination of  $\{1, \gamma, \ln(2\pi)\} \cup \{\zeta'(-k)\}_{k=1}^{q-1}$  is linked to this result since summing identity (3) allows to write

$$\begin{aligned} q! \Phi_{q+1} &= \sum_{k=0}^q S_1(q+1, k+1) \sum_{n \geq 1}^{\mathcal{R}} n^k H_n - \sum_{k=0}^q S_1(q+1, k+1) \sum_{n \geq 1}^{\mathcal{R}} n^k H_q \\ &\quad + \sum_{k=1}^q S_1(q+1, k+1) k \sum_{n \geq 1}^{\mathcal{R}} n^{k-1}, \end{aligned}$$

where  $\sum_{n \geq 1}^{\mathcal{R}} n^k = \zeta(-k) + \frac{1}{k+1}$  (cf. [CGP], Example 1). The demonstration that the formula for  $\Phi_q$  obtained in this way is in fact the same as the one in Corollary 1 requires some unusual identities involving Bernoulli and Stirling numbers such as

$$\frac{1}{(q-1)!} \sum_{k=1}^q S_1(q, k) \left( \frac{1 - B_k}{k} \right) = b_q(q).$$

This identity, which arises from equating the coefficients of  $\gamma$  in the two expressions, may be proved by means of the Stirling dual identity to (2), the identities in [Ca], and manipulations of the generating functions. We view our method to obtain  $\sum_{n \geq 1}^{\mathcal{R}} H_n^{(q)}$  as straightforward and more enlightening.

## References

- [B] I.V. Blagouchine, Expansions of generalized Euler's constants, *J. Number Theory* 158 (2016) 365–396.
- [B2] I.V. Blagouchine, Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to  $\pi^{-1}$ , *J. Math. Anal. Appl.* 442 (2016) 404–434.
- [Ca] L. Carlitz, A note on Bernoulli and Euler polynomials of the second kind, *Scripta Math.* 25 (1961) 323–330.
- [CC1] B. Candelpergher, M.-A. Coppo, Le produit harmonique des suites, *Enseign. Math.* 59 (2013) 39–72.
- [CC2] B. Candelpergher, M.-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.* 27 (2012) 305–328.
- [CGP] B. Candelpergher, H. Gadiyar, R. Padma, Ramanujan summation and the exponential generating function  $\sum_{k=0}^{\infty} \frac{z^k}{k!} \zeta'(-k)$ , *Ramanujan J.* 21 (2010) 99–122.
- [DB] A. Dil, K. Boyadzhiev, Euler sums of hyperharmonic numbers, *J. Number Theory* 147 (2015) 490–498.
- [J] C. Jordan, *Calculus of Finite Differences*, Chelsea, New York, 1965.
- [K] M. Kaneko, Poly-Bernoulli numbers, *J. Théor. Nombres Bordeaux* 9 (1997) 221–228.
- [Ka] K. Kamano, Dirichlet series associated with hyperharmonic numbers, *Mem. Osaka Inst. Tech.* 56 (2011) 11–15.
- [Ke] B. Kellner, On quotients of Riemann zeta values at odd and even integer arguments, *J. Number Theory* 133 (2013) 2684–2698.
- [KZ] M. Kontsevich, D. Zagier, Periods, in: *Mathematics Unlimited – 2001 and Beyond*, Springer, Berlin, 2001, pp. 771–808.
- [S] Z.-W. Sun, Combinatorial identities in dual sequences, *European J. Combin.* 24 (2003) 709–718.

- [Y1] P.T. Young, Rational series for multiple zeta and log gamma functions, *J. Number Theory* 133 (2013) 3995–4009.
- [Y2] P.T. Young, Symmetries of Bernoulli polynomial series and Arakawa–Kaneko zeta functions, *J. Number Theory* 143 (2014) 142–161.
- [Y3] P.T. Young, The  $p$ -adic Arakawa–Kaneko zeta functions, *J. Number Theory* 155 (2015) 13–35.