

The Arakawa–Kaneko zeta function

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Abstract We present a very natural generalization of the Arakawa–Kaneko zeta function introduced ten years ago by T. Arakawa and M. Kaneko. We give in particular a new expression of the special values of this function at integral points in terms of modified Bell polynomials. By rewriting Ohno’s sum formula, we are able to deduce a new class of relations between Euler sums and the values of zeta.

Keywords Poly-Bernoulli numbers · Multiple zeta-star values · Euler sums · Zeta values

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1 Introduction

The Arakawa–Kaneko zeta function was introduced ten years ago by T. Arakawa and M. Kaneko in [1]. Let us recall that this is the function ξ_k defined for any integer $k \geq 1$ by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t - 1} \text{Li}_k(1 - e^{-t}) dt,$$

where Li_k denotes the k th polylogarithm $\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. The integral converges for $\Re(s) > 0$ and the function ξ_k can be analytically continued an entire function of

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the whole s -plane. For $k = 1$, $\xi_1(s)$ is nothing else than $s\zeta(s + 1)$ and for $s = 1$, $\xi_k(1) = \zeta(k + 1)$. In [1], Arakawa and Kaneko have expressed the special values of this function at *negative* integers with the help of generalized Bernoulli numbers $B_n^{(k)}$ called “poly-Bernoulli numbers”. Introduced by M. Kaneko in [5], these numbers are defined by the generating function

$$\frac{\text{Li}_k(1 - e^{-z})}{1 - e^{-z}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{z^n}{n!}.$$

In the case where $k = 1$, one finds again—apart from the sign for $B_1^{(1)}$ —the classical Bernoulli numbers. Arakawa and Kaneko also provide in [1] (see their Corollary 10) a rather complex expression of the special values of the function at *positive* integers in terms of MZV (multiple zeta value) but, very soon afterwards, a simpler representation of the values of ξ_k at positive integers in terms of MZSV (multiple zeta-star value) has been obtained by Y. Ohno. More precisely, transforming the original expression given in [1] by means of a duality theorem, Ohno establishes in [7] that

$$\xi_{k-1}(m) = \sum_{n_1 \geq n_2 \geq \dots \geq n_m \geq 1} \frac{1}{n_1^k n_2 \dots n_m} = \zeta^*(k, \underbrace{1, \dots, 1}_{m-1}),$$

where $\zeta^*(k_1, k_2, \dots, k_m)$ refers to the sum

$$\sum_{n_1 \geq n_2 \geq \dots \geq n_m \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_m^{k_m}}.$$

Subsequently, this expression has found an important continuation in [8] where Ohno states and proves his remarkable *sum formula*

$$\sum_{m=0}^{k-2} \zeta^*(k - m, \underbrace{1, \dots, 1}_m) = 2(k - 1)(1 - 2^{1-k})\zeta(k),$$

which is also the subject of an interesting commentary in [6].

In this article, we introduce a more general function $\xi_k(s, x)$ defined for $\Re(s) > 0$ and $x > 0$ by

$$\xi_k(s, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} t^{s-1} dt$$

which is a very natural extension of the Arakawa–Kaneko zeta function in the same way as the Hurwitz zeta function $\zeta(s, x)$ generalizes the Riemann zeta function; one has $\xi_k(s, 1) = \xi_k(s)$ and, in the case where $k = 1$, the function $\xi_1(s, x)$ is nothing else than the classical $s\zeta(s + 1, x)$.

Following the same pattern as in [1], we show that this function $\xi_k(s, x)$ can be analytically continued to the whole complex s -plane as an entire function of s , and we express its special values at negative integral points by means of generalized

Bernoulli polynomials $B_n^{(k)}(x)$ whose values at 0 are precisely the poly-Bernoulli numbers (cf. Theorem 2 and Remark 3). In this way, we show that

$$\xi_k(-m, x) = (-1)^m B_m^{(k)}(x) \quad (m = 0, 1, 2, \dots).$$

Regarding the special values of $\xi_k(s, x)$ at positive integers, we obtain the following representation (cf. Theorem 1)

$$\xi_k(m + 1, x) = \sum_{n=0}^{\infty} \frac{n!}{(n + 1)^k x(x + 1) \cdots (x + n)} P_m(h_n^{(1)}(x), \dots, h_n^{(m)}(x)),$$

where $P_m(x_1, \dots, x_m)$ denotes the m th modified Bell polynomial defined by the generating function

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m}\right) = \sum_{m=0}^{\infty} P_m(x_1, \dots, x_m) t^m \tag{*}$$

and where

$$h_n^{(m)}(x) = \sum_{j=0}^n \frac{1}{(j + x)^m}. \tag{**}$$

This ‘‘Hasse formula’’ extends the representation already given in [3] in the case $k = 1$ (cf. Remark 2). Specializing this expression at $x = 1$, we then deduce a new expression for the values of the Arakawa–Kaneko function at positive integers (cf. Corollary 1) from which follows the decomposition

$$\xi_{k-1}(m + 1) = \zeta^*(k, \underbrace{1, \dots, 1}_m) = \sum_{n=1}^{\infty} \frac{1}{n^k} P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)})$$

where $H_n, H_n^{(2)}, \dots, H_n^{(m)}$ denote the harmonic numbers (cf. Corollary 2). This leads us to the rewriting of Ohno’s sum formula in the following form

$$\sum_{m=1}^{k-3} \sum_{n=1}^{\infty} \frac{1}{n^{k-m}} P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)}) = [(k - 2) - (k - 1)2^{2-k}] \zeta(k),$$

which defines a new class of relations between Euler sums and the values of zeta (cf. Corollary 3 and Example 3). This class contains in particular (in the simplest case where $k = 4$), the famous relation $\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4} \zeta(4)$ whose origin goes back to Euler and Goldbach (cf. [4]).

2 A generalized Arakawa–Kaneko function

Proposition 1 *Let*

$$F(s, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} f(t) t^{s-1} dt$$

be a Laplace–Mellin integral with

$$f(t) = \sum_{n=0}^{\infty} a_{n+1} (1 - e^{-t})^n,$$

where the coefficients a_n are assumed to satisfy the condition $|a_n| = O(\frac{1}{n})$. The following properties hold:

1. The integral $F(s, x)$ converges for $\Re(s) > 0$ and $x > 0$.
2. If m is a natural number and $s = m + 1$ then

$$F(m + 1, x) = \sum_{n=0}^{\infty} \frac{n! a_{n+1}}{x(x + 1) \cdots (x + n)} P_m(h_n^{(1)}(x), \dots, h_n^{(m)}(x)), \tag{1}$$

where P_m and $h_n^{(m)}$ are respectively given by formulas (*) and (**).

Proof By our assumption on a_n , there exists a constant $C > 0$ and an integer $N \geq 1$ such that for all $t \geq 0$

$$\sum_{n=N}^{\infty} |a_n| (1 - e^{-t})^{n-1} \leq C \sum_{n=N}^{\infty} \frac{(1 - e^{-t})^{n-1}}{n} \leq C \sum_{n=1}^{\infty} \frac{(1 - e^{-t})^{n-1}}{n} = \frac{Ct}{1 - e^{-t}},$$

which ensures the convergence of the integral and justifies the interchange of \int and \sum

$$F(s, x) = \sum_{n=0}^{\infty} a_{n+1} \int_0^{+\infty} e^{-xt} (1 - e^{-t})^n \frac{t^{s-1}}{\Gamma(s)} dt.$$

Then, formula (1) results from the following Lemma 1. □

Lemma 1 For $x > t$, one has

$$\int_0^{+\infty} e^{-x\xi} (1 - e^{-\xi})^n e^{\xi t} d\xi = \frac{n!}{x(x + 1) \cdots (x + n)} \times \exp\left(\sum_{m=1}^{\infty} h_n^{(m)}(x) \frac{t^m}{m}\right).$$

Proof For $a > 0$ and $b > 0$, let us start from the classical Euler’s relation

$$B(a, b) = \int_0^1 u^{a-1} (1 - u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}.$$

Putting $u = e^{-\xi}$, $a = x - t$ and $b = n + 1$, one deduces

$$\int_0^{+\infty} e^{-x\xi} (1 - e^{-\xi})^n e^{\xi t} d\xi = \frac{n!}{(x - t)(1 + x - t) \cdots (n + x - t)}.$$

Moreover, one has

$$\begin{aligned} & \frac{n!}{(x-t)(1+x-t)\cdots(n+x-t)} \\ &= \frac{n!}{x(x+1)\cdots(x+n)} \times \prod_{k=0}^n \left(1 - \frac{t}{k+x}\right)^{-1} \\ &= \frac{n!}{x(x+1)\cdots(x+n)} \times \exp\left(-\sum_{k=0}^n \ln\left(1 - \frac{t}{k+x}\right)\right) \\ &= \frac{n!}{x(x+1)\cdots(x+n)} \times \exp\left(\sum_{k=0}^n \sum_{m=1}^{\infty} \frac{t^m}{m(x+k)^m}\right) \\ &= \frac{n!}{x(x+1)\cdots(x+n)} \times \exp\left(\sum_{m=1}^{\infty} h_n^{(m)}(x) \frac{t^m}{m}\right). \end{aligned} \quad \square$$

Applying now Proposition 1 with $a_{n+1} = \frac{1}{(n+1)^k}$, one has

$$f(t) = \sum_{n=0}^{\infty} \frac{(1 - e^{-t})^n}{(n+1)^k} = \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}},$$

and we immediately obtain the following theorem:

Theorem 1 Consider an integer $k \geq 1$. The Laplace–Mellin integral

$$\xi_k(s, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} t^{s-1} dt$$

converges for $\Re(s) > 0$ and $x > 0$. Moreover, one has

$$\xi_k(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \int_0^{+\infty} e^{-xt} (1 - e^{-t})^n \frac{t^{s-1}}{\Gamma(s)} dt.$$

In particular, if m is a natural number and $s = m + 1$, then

$$\xi_k(m + 1, x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^k x(x+1)\cdots(x+n)} P_m(h_n^{(1)}(x), \dots, h_n^{(m)}(x)). \quad (2)$$

Remark 1 In the case $x = 1$, one has

$$\begin{aligned} \xi_k(s, 1) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-t} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t - 1} \text{Li}_k(1 - e^{-t}) dt = \xi_k(s). \end{aligned}$$

Thus, in this case, one finds again the original Arakawa–Kaneko zeta function introduced in [1].

Remark 2 In the case $k = 1$, one has $\frac{\text{Li}_1(1-e^{-t})}{1-e^{-t}} = \frac{t}{1-e^{-t}}$ from which it follows that

$$\xi_1(s - 1, x) = \int_0^{+\infty} e^{-xt} \left(\frac{t}{1 - e^{-t}} \right) \frac{t^{s-2}}{\Gamma(s - 1)} dt = (s - 1)\zeta(s, x).$$

Thus, in this case, identity (2) is nothing else than the representation for the values of the Hurwitz zeta function (called Hasse formula) given in [3]

$$(m + 1)\zeta(m + 2, x) = \sum_{n=0}^{\infty} \frac{n!}{(n + 1)x(x + 1) \cdots (x + n)} P_m(h_n^{(1)}(x), \dots, h_n^{(m)}(x)).$$

Example 1

$$\begin{aligned} \xi_k(1, x) &= \sum_{n=0}^{\infty} \frac{n!}{(n + 1)^k x(x + 1) \cdots (x + n)}; \\ \xi_k(2, x) &= \sum_{n=0}^{\infty} \frac{n!}{(n + 1)^k x(x + 1) \cdots (x + n)} \sum_{i=0}^n \frac{1}{i + x}; \\ 2\xi_k(3, x) &= \sum_{n=0}^{\infty} \frac{n!}{(n + 1)^k x(x + 1) \cdots (x + n)} \left[\left(\sum_{i=0}^n \frac{1}{i + x} \right)^2 + \sum_{i=0}^n \frac{1}{(i + x)^2} \right]; \\ 6\xi_k(4, x) &= \sum_{n=0}^{\infty} \frac{n!}{(n + 1)^k x(x + 1) \cdots (x + n)} \\ &\quad \times \left[\left(\sum_{i=0}^n \frac{1}{i + x} \right)^3 + 3 \sum_{i=0}^n \frac{1}{(i + x)} \sum_{i=0}^n \frac{1}{(i + x)^2} + 2 \sum_{i=0}^n \frac{1}{(i + x)^3} \right]. \end{aligned}$$

Theorem 2 For all integers $k \geq 1$ and real $x > 0$, the function $s \mapsto \xi_k(s, x)$ can be analytically continued to the whole complex s -plane and its values at negative integers are given by

$$\xi_k(-m, x) = (-1)^m B_m^{(k)}(x) \quad (m = 0, 1, 2, \dots), \tag{3}$$

where $B_n^{(k)}(x)$ is defined by the generating function

$$e^{-xz} \frac{\text{Li}_k(1 - e^{-z})}{1 - e^{-z}} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{z^n}{n!}.$$

Proof We apply the classical method to analytically continue a function defined as a Mellin transform (cf. [2]). One splits up $\xi_k(s, x)$ as the sum of two integrals:

$$\begin{aligned} \xi_k(s, x) &= \frac{1}{\Gamma(s)} \int_0^1 e^{-xt} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} t^{s-1} dt \\ &\quad + \frac{1}{\Gamma(s)} \int_1^{+\infty} e^{-xt} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} t^{s-1} dt. \end{aligned}$$

The second integral converges absolutely for all $s \in \mathbb{C}$ and $x > 0$ and cancels at negative integers (because $\frac{1}{\Gamma}(-m) = 0$ for $m = 0, 1, 2, \dots$). For $\Re(s) > 0$, the first integral may be written as

$$\frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n^{(k)}(x)}{n!} \times \frac{1}{n + s},$$

from which follows that

$$\lim_{s \rightarrow -m} \xi_k(s, x) = \left(\lim_{s \rightarrow -m} \frac{1}{\Gamma(s)(m + s)} \right) \frac{B_m^{(k)}(x)}{m!} = (-1)^m B_m^{(k)}(x). \quad \square$$

Remark 3 From the generating function which defines them, the polynomials $B_n^{(k)}(x)$ are given by

$$B_n^{(k)}(x) = \sum_{q=0}^n (-1)^{n-q} \binom{n}{q} B_q^{(k)} x^{n-q},$$

where $B_n^{(k)} = B_n^{(k)}(0)$ are the poly-Bernoulli number introduced by Kaneko in [5]. In particular, specializing identity (3) at $x = 1$, one finds again

$$\xi_k(-m, 1) = (-1)^m B_m^{(k)}(1) = \sum_{q=0}^m (-1)^q \binom{m}{q} B_q^{(k)},$$

which is nothing else than the expression given by Arakawa and Kaneko (cf. [1], Theorem 6).

3 New expression of $\xi_k(m + 1)$

Specializing identity (2) at $x = 1$, one obtains

Corollary 1 *For all natural numbers $m \geq 0$ and integers $k \geq 1$, one has*

$$\xi_k(m + 1) = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)}) \quad \text{with } H_n^{(m)} = \sum_{j=1}^n \frac{1}{j^m}. \quad (4)$$

Example 2

$$\begin{aligned} \xi_k(1) &= \zeta(k + 1); \\ \xi_k(2) &= \sum_{n=1}^{\infty} \frac{H_n}{n^{k+1}}; \\ \xi_k(3) &= \frac{1}{2} \left[\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^{k+1}} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{k+1}} \right]; \\ \xi_k(4) &= \frac{1}{6} \left[\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^{k+1}} + 3 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^{k+1}} + 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^{k+1}} \right]; \\ \xi_k(5) &= \frac{1}{24} \left[\sum_{n=1}^{\infty} \frac{(H_n)^4}{n^{k+1}} + 6 \sum_{n=1}^{\infty} \frac{(H_n)^2 H_n^{(2)}}{n^{k+1}} + 3 \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^{k+1}} \right. \\ &\quad \left. + 8 \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^{k+1}} + 6 \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^{k+1}} \right]. \end{aligned}$$

4 Rewriting of Ohno’s sum formula and application to Euler sums

From the comparison of (4) with the expression given by Ohno (cf. [7] Theorem 2, and [6] Paragraph 2),

$$\xi_{k-1}(m) = \zeta^*(k, \underbrace{1, \dots, 1}_{m-1}) := \sum_{n_1 \geq n_2 \geq \dots \geq n_m \geq 1} \frac{1}{n_1^k n_2 \dots n_m},$$

one can immediately deduce the following decomposition

Corollary 2 *For all natural numbers $m \geq 0$ and integers $k \geq 2$,*

$$\zeta^*(k, \underbrace{1, \dots, 1}_m) = \sum_{n=1}^{\infty} \frac{1}{n^k} P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)}). \tag{5}$$

Rewriting now Ohno’s sum formula (cf. [8], Theorem 8)

$$\sum_{m=0}^{k-2} \zeta^*(k - m, \underbrace{1, \dots, 1}_m) = 2(k - 1)(1 - 2^{1-k})\zeta(k),$$

thanks to the preceding decomposition (5), and taking into account that $\xi_1(k - 1) + \xi_{k-1}(1) = k\zeta(k)$, one obtains the following formula which defines a new class of relations between Euler sums and the zeta values

Corollary 3 For all integers $k \geq 4$,

$$\sum_{m=1}^{k-3} \sum_{n=1}^{\infty} \frac{1}{n^{k-m}} P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)}) = [(k-2) - (k-1)2^{2-k}] \zeta(k). \quad (6)$$

Example 3

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n}{n^3} &= \frac{5}{4} \zeta(4) \quad (\text{Euler and Goldbach}); \\ \sum_{n=1}^{\infty} \frac{H_n}{n^4} + \frac{1}{2} \left[\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^3} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} \right] &= \frac{5}{2} \zeta(5); \\ \sum_{n=1}^{\infty} \frac{H_n}{n^5} + \frac{1}{2} \left[\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^4} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4} \right] \\ &+ \frac{1}{6} \left[\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^3} + 3 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} + 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3} \right] = \frac{59}{16} \zeta(6); \\ \sum_{n=1}^{\infty} \frac{H_n}{n^6} + \frac{1}{2} \left[\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^5} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} \right] \\ &+ \frac{1}{6} \left[\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^4} + 3 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} + 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4} \right] \\ &+ \frac{1}{24} \left[\sum_{n=1}^{\infty} \frac{(H_n)^4}{n^3} + 6 \sum_{n=1}^{\infty} \frac{(H_n)^2 H_n^{(2)}}{n^3} + 3 \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} \right. \\ &\left. + 8 \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3} + 6 \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3} \right] = \frac{77}{16} \zeta(7). \end{aligned}$$

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