Alternating series involving multizeta values

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Main reference

The main reference for this talk is my recent article:

A note on some alternating series involving zeta and multiple zeta values
Journal of Mathematical Analysis and Applications, **475** (2019), 1831–1841.

Available on my website: math.unice.fr/$\sim$coppo/

Euler's constant

The famous Euler-Mascheroni constant

$$\gamma := \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{1}{j} - \ln n \right\} = 0.5772156649\ldots$$

was first introduced by Euler and computed with high accuracy since the middle of the 18th century.

One of the main reasons of the importance of this constant lies in its close relation with the Riemann zeta function.

Stieltjes constants

The Riemann zeta-function

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \qquad \Re(s) > 1$$

is a meromorphic function in the entire complex plane with one simple pole at $s = 1$. The Laurent expansion of the Riemann zeta-function at $s = 1$ is given by

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k (s-1)^k}{k!} \gamma_k$$

The coefficients γ_k $(k = 1, 2, \ldots)$ are called the Stieltjes constants.

Stieltjes constants

A classical expression of the Stieltjes constants is

$$\gamma_k = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{\ln^k j}{j} - \frac{\ln^{k+1} n}{k+1} \right\}$$

In the specific case $k = 1$,

$$\gamma_1 = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2} \ln^2 n \right\} = -0.07281584548\ldots$$

The constant γ_1 will reappear at the end of this presentation.

Euler's and Mascheroni's series for γ

Euler's conditionally convergent alternating series for γ is the sum

$$\gamma = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n}$$

(Euler, 1735)

Mascheroni's absolutely convergent series for γ is the sum

$$\gamma = \sum_{n=1}^{\infty} \frac{|G_n|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \cdots$$

(Mascheroni, 1790)

Gregory coefficients

The numbers G_n are called Gregory coefficients (or Bernoulli numbers of the second kind). The numbers $n!G_n$ are called Cauchy numbers. They can be computed recursively by $G_0 = 1$ and

$$\sum_{k=0}^{n} \frac{(-1)^k G_k}{n-k+1} = 0$$

Or explicitly by

$$G_n = \frac{1}{n!} \int_0^1 x(x-1)(x-2)\cdots(x-n+1)\,dx, \qquad n = 1, 2, 3, \ldots$$

Gregory coefficients

The Gregory coefficients alternate in sign. The first one are

$$G_1 = \frac{1}{2}\,,\, G_2 = \frac{-1}{12}\,,\, G_3 = \frac{1}{24}\,,\, G_4 = \frac{-19}{720}\,,\, G_5 = \frac{3}{160}\,,\, G_6 = \frac{-863}{60\,480}\,,\ldots$$

Asymptotically, they behave as

$$|G_n| \sim \frac{1}{n\,(\ln n)^2}\,, \qquad n \to +\infty$$

Gregory coefficients

Remark on the Mascheroni's series. If we remove the absolute values in this series, it may be shown that the sum

$$\sum_{n=1}^{\infty} \frac{G_n}{n} = \mathrm{li}(2) - \gamma = 0.46794811521\ldots$$

where $\mathrm{li}(x)$ is the logarithmic integral function.

The sequence (ν_k)

For each integer k with $k \geqslant -1$, we now consider the shifted series

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k}$$

They form a sequence of conditionally convergent series parametrized by k with $\nu_0 = \gamma$.

Suryanarayana's formula

A quite simple formula for ν_1 is

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1$$

(Suryanarayana, 1974)

A short proof of this formula is given by Singh and Verma (Yokohama Mathematical Journal, 31 (1983)).

The case $k \geq 1$

For any integer $k \geq 1$, we have shown that a general formula is

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

where C_k is a positive rational number whose explicit expression is given by

$$C_k = \frac{1}{k} + \sum_{n=2}^{k} \binom{k}{n} \frac{B_n H_{n-1}}{k+1-n}$$

In this expression, H_n stands for the n-th harmonic number,

$$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and B_n denotes the n-th Bernoulli number.

Bernoulli numbers

The Bernoulli numbers are defined by their generative function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| < 2\pi)$$

They can be computed recursively by $B_0 = 1$ and

$$\sum_{k=0}^{n} \frac{B_k}{k!(n-k+1)!} = 0$$

The first one are

$B_1 = \frac{-1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = \frac{-1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}$, etc.

Examples

The first terms of the sequence $(\nu_k)_{k\geqslant 1}$ are

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1$$
$$\nu_2 = \frac{\gamma}{3} - \frac{1}{2}\ln 2\pi - 2\zeta'(-1) + \frac{2}{3}$$
$$\nu_3 = \frac{\gamma}{4} - \frac{1}{2}\ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}$$
$$\nu_4 = \frac{\gamma}{5} - \frac{1}{2}\ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}$$
$$\nu_5 = \frac{\gamma}{6} - \frac{1}{2}\ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}$$

etc.

Link with the digamma function

The Taylor series expansion of the digamma function $\psi(x)$ (logarithmic derivative of the Γ-function) at $x = 1$ is

$$\psi(x+1) + \gamma = \sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1} \qquad (|x| < 1)$$

An integration between 0 and 1 gives

$$\nu_k = \frac{\gamma}{k+1} + \int_0^1 x^k \psi(x+1)\, dx \qquad (k \geq 1)$$

It follows that

$$\int_0^1 x^k \psi(x+1)\, dx = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

The case $k = -1$

The case $k = -1$ is of particular interest. By definition

$$\nu_{-1} := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n-1} = 1.2577468869\ldots$$

We have the following formulae:

- $\nu_{-1} = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n(n+1)}$
- $\nu_{-1} = -\sum_{n=2}^{\infty} \zeta'(n)$
- $\nu_{-1} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(\frac{1}{2} + ix)\cosh(\pi x)}\, dx$

This last formula (due to Blagouchine) may be easily proved by the residue theorem.

Higher Gregory coefficients

We consider now the Gregory coefficients of higher order defined by

$$G_n^{(k)} := \frac{1}{n!}\sum_{j=1}^{n}\frac{s(n,j)}{j+k} \qquad (k \geq 0, n \geq 1)$$

In this formula, $s(n,j)$ denotes the Stirling numbers of the first kind. The Gregory coefficients of higher order are representable by the integral

$$G_n^{(k)} = \frac{(-1)^{n+1}}{n!}\int_0^1 x^k(1-x)(2-x)\cdots(n-1-x)\,dx$$

In the specific case $k=1$, we recover the ordinary Gregory coefficients

$$G_n^{(1)} = G_n$$

Higher Gregory coefficients

As for the G_n, the Gregory coefficients $G_n^{(k)}$ alternate in sign.

$$G_n^{(k)} = (-1)^{n+1}|G_n^{(k)}|$$

We have shown that

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \qquad (k \geq 0)$$

In the specific case $k = 1$, we recover the Mascheroni series for γ.

Examples

$$\nu_{-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(0)}|}{n} = 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \cdots$$

$$\nu_{0} = \sum_{n=1}^{\infty} \frac{|G_n^{(1)}|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \cdots$$

$$\nu_{1} = \sum_{n=1}^{\infty} \frac{|G_n^{(2)}|}{n} = \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25\,200} + \frac{731}{725\,760} + \cdots$$

$$\nu_{2} = \sum_{n=1}^{\infty} \frac{|G_n^{(3)}|}{n} = \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100\,800} + \frac{5849}{10\,886\,400} + \cdots$$

Multizeta values

For integers $p \geq 0$ and $k \geq -1$, we consider now the more general alternating series

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n, \underbrace{1, \ldots, 1}_{p})$$

where

$$\zeta(s_1, s_2, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

In the specific case $p = 0$, we have

$$\nu_{k,0} = \nu_k$$

Multizeta values

We have shown that for all integers $p \geq 0$ and $k \geq -1$, we have

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}$$

In the specific case $k = 0$, we get

$$\nu_{0,p} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n, \underbrace{1, \dots, 1}_{p}) = \sum_{n=1}^{\infty} \frac{|G_n|}{n^{p+1}}$$

In particular, for $p = 1$,

$$\nu_{0,1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n, 1) = \sum_{n=1}^{\infty} \frac{|G_n|}{n^2} = 0.5290529699\ldots$$

We have shown the following beautiful relation

$$\nu_{0,1} = \nu_{-1} + \gamma_1 + \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2)$$

where γ_1 is the first Stieltjes constant.

Application to the harmonic zeta function

The Apostol-Vu harmonic zeta function ζ_H defined for $\Re(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s}$$

is an analytic function in the half-plane $\Re(s) > 1$ which can be extended meromorphically in the whole complex plane with a double pole at $s = 1$ and an infinity of simple poles at the integers $0, -1, -3, -5, -7, -9, \ldots$.

Application to the harmonic zeta function

The special values of the harmonic zeta function at negative even integers are $\zeta_H(-2k) = -B_{2k}/4k + B_{2k}/2$. The special values at positive integers are given by

$$2\zeta_H(n) = (n+2)\zeta(n+1) - \sum_{r=1}^{n-2} \zeta(r+1)\zeta(n-r) \qquad (n \geq 2).$$

This last formula was first obtained by Euler in a manuscript dated 1775 and several times rediscovered afterwards.

Application to the harmonic zeta function

We have shown the following nice identity

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = 0.916240149\ldots$$

where γ_1 is the first Stieltjes constant.

And we can also prove that in a neighborhood of $s = 1$,

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{(s-1)} + \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + O(s-1)$$