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Periodic geodesics on compact riemannian manifolds.
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The authors make an essential contribution to the problem of the existence of "many" prime closed geodesics on a compact Riemannian manifold M. For simplicity, let us assume that $\pi_1 M = 0$, which is the most difficult part anyhow. Associated to M we have the Hilbert manifold $\Lambda M = H^1(S^1, M)$ of H^1-maps of the circle into M, the so-called space of parametrized closed curves [cf. the reviewer, Ann. of Math. (2) **89** (1969), 69–91; MR0239624 (39 #981)]. The authors denote ΛM by ΩM. On ΛM there are naturally given a Riemannian metric and a differentiable function E, the energy integral, which satisfy the condition (C) of Palais and Smale and therefore permit one to develop the Morse-Ljusternik-Šnirel′man theory on ΛM with respect to E. The critical points of E are the closed geodesics. The main result of the paper is the following theorem: If $(*)$ the sequence $\{b_k(\Lambda M)\}$ of real Betti numbers of ΛM is not bounded, then there exists an infinite number of prime closed geodesics on M, i.e., closed geodesics on M that are not coverings of other closed geodesics. To understand the beautiful simple basic idea of the proof, the reader should first consider the case that all closed geodesics on M are non-degenerate. In that case, the Morse inequalities and Bott's formula [R. Bott, Comm. Pure Appl. Math. **9** (1956), 171–206; MR0090730 (19,859f)] for the index of a covering of a prime closed geodesic imply immediately: The existence of only a finite number of prime closed geodesics on M contradicts the hypothesis $(*)$. The bulk of the paper is devoted to the general case where one allows degenerate closed geodesics. Here the authors rely upon an earlier paper [Topology **8** (1969), 361–369; MR0246329 (39 #7633)] in which they define a local invariant of a degenerate closed geodesic.

It is easy to give examples of M for which $(*)$ is satisfied, but it is also clear that there are M where $(*)$ is not satisfied, e.g., M of the homotopy type of a symmetric space of rank 1. The results of P. Klein quoted in the paper are not yet proved in full generality. In any case it seems reasonable to say that the hypothesis $(*)$ means for M that its real cohomology ring must contain a subring isomorphic to the product of the cohomology of two manifolds.

Reviewed by *W. Klingenberg*