Additive and unstable algebra structures for the MU-cohomology of profinite spaces, resolutions and application to the cohomology of mapping spaces from $\mathbb{C}\mathrm{P}^\infty$

A survey (March 2003)

1. Introduction

We consider division functors for MU-cohomology related to the MU-cohomology of mapping spaces, in the spirit of Lannes' T-functor for mod p-cohomology.

Let $(\mathrm{MU}_n)_n$ denote the Ω-spectra representing the cobordism cohomology theory. The coefficient ring MU^* is the polynomial algebra over $\mathbb{Z}$ generated by elements x_k of degree $2k$, $k \geq 1$. As we will consider spaces like $\mathbf{map}(\mathbb{C}\mathrm{P}^\infty, \mathrm{MU}_n)$ which already has infinite mod p cohomology group in each degree, we will use profinite completion and continuous cohomology. So we fix a prime number p and let $\hat{\mathcal{S}}$, respectively $\mathrm{h}\hat{\mathcal{S}}$, denote the Quillen model category of profinite spaces, respectively the associated homotopy category, where the weak equivalences are the maps inducing an isomorphism in continuous mod p cohomology ([Mo]).

For W a finite simplicial set, the mapping space $\mathbf{map}(W, Y) \in \hat{\mathcal{S}}$ is defined by the adjonction

$$\mathrm{Hom}_{\hat{\mathcal{S}}}(X, \mathbf{map}(W, Y)) \simeq \mathrm{Hom}_{\hat{\mathcal{S}}}(W \times X, Y) \ .$$

This extends to general simplicial set W by defining the external cartesian product $W \hat{\times} X$ as the colimit in $\hat{\mathcal{S}}$ of the $W_\alpha \times X$ and $\mathbf{map}(W, Y)$ as the limit in $\hat{\mathcal{S}}$ of the $\mathbf{map}(W_\alpha, Y)$, W_α spanning the simplicial finite subsets of W. We thus get a counit $W \hat{\times} \mathbf{map}(W, Y) \to Y$ in $\hat{\mathcal{S}}$. We still have a bijection when changing $\hat{\mathcal{S}}$ to the homotopy category $\mathrm{h}\hat{\mathcal{S}}$ if Y is fibrant in $\hat{\mathcal{S}}$ and in particular a bijection $\pi_0\mathbf{map}(W, Y) \simeq [W, Y]$, where $[W, Y]$ denotes the (profinite) set of homotopy classes of maps from W to the underlying space of Y.

We define the continuous MU-cohomology of a profinite space X in degree $n \in \mathbb{Z}$ as the set $\mathrm{Hom}_{\mathrm{h}\hat{\mathcal{S}}}(X, \hat{\mathrm{MU}}_n)$, where $\hat{\mathrm{MU}}_n$ denotes the profinite completion of MU_n if $n \geq 1$ and the profinite space $\Omega^{1-n}\hat{\mathrm{MU}}_1$ if $n \leq 0$; we denote it by $\hat{\mathrm{MU}}^n X$. The ring structure on MU induces an MU^*-algebra structure on $\hat{\mathrm{MU}}^* X$ for every profinite space X. Note that if $\hat{X}$ is the profinite completion of some space X then the continuous mod p-cohomology of $\hat{X}$ identifies with the ordinary mod p cohomology of X and the continuous MU-cohomology of $\hat{X}$ identifies with the p-completed MU-cohomology of X.

We start by pointed out some facts concerning the continuous mod p cohomology of profinite spaces, which we denote by $\mathrm{H}^*(-)$. We let $\mathcal{E}$ denote the category of $\mathbb{Z}$-graded $\mathbb{F}_p$-vector spaces.

1) Let E be a non negatively graded $\mathbb{F}_p$-vector space, then there exists a profinite space $\mathrm{K}(E)$ and a map $E \to \mathrm{H}^*\mathrm{K}(E)$ inducing a bijection $\mathrm{Hom}_{\mathrm{h}\hat{\mathcal{S}}}(X, \mathrm{K}(E)) \to \mathrm{Hom}_{\mathcal{E}}(E, \mathrm{H}^*X)$ for any profinite space X.

The category $\mathcal{K}_\mathrm{H}$ of unstable algebras over the mod p Steenrod algebra is exactly such that the natural map $\mathrm{Hom}_{\mathrm{h}\hat{\mathcal{S}}}(X, \mathrm{K}(E)) \to \mathrm{Hom}_{\mathcal{K}_\mathrm{H}}(\mathrm{H}^*\mathrm{K}(E), \mathrm{H}^*X)$ is a bijection for all X, so $\mathrm{H}^*\mathrm{K}(E)$ is the "free unstable algebra on $E \in \mathcal{E}$".

2) We have a Künneth formula $\mathrm{H}^*X \otimes \mathrm{H}^*Y \simeq \mathrm{H}^*X \times Y$ for all X and Y in $\hat{\mathcal{S}}$. Let W be a simplicial set whose mod p cohomology is degree-wise finite and let X be in $\hat{\mathcal{S}}$, then the map $\mathrm{H}^*W \otimes \mathrm{H}^*X \to \mathrm{H}^*(W \hat{\times} X)$ is an isomorphism. In other words, the map $W \hat{\times} X \to \hat{W} \times X$ is a weak equivalence in $\hat{\mathcal{S}}$. So the set $\mathrm{Hom}_{\mathrm{h}\hat{\mathcal{S}}}(W \hat{\times} X, \mathrm{K}(E))$ is functorial in H^*X; more over it is a representable functor in $\mathrm{H}^*X \in \mathcal{K}_\mathrm{H}$. Let then $(- : \mathrm{H}^*W)$ denote the left adjoint of the functor $\mathrm{H}^*W \otimes -$ in $\mathcal{K}_\mathrm{H}$, then the canonical map $(\mathrm{H}^*\mathrm{K}(E) : \mathrm{H}^*W) \to \mathrm{H}^*\mathbf{map}(W, \mathrm{K}(E))$ is an isomorphism.

3) If W is the classifying space of $\mathbb{Z}/p$ then the isomorphisms in 1) and 2) generalize to isomorphisms

$$\mathrm{Hom}_{\mathrm{h}\hat{\mathcal{S}}}(\mathrm{B}\mathbb{Z}/p, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}_\mathrm{H}}(\mathrm{H}^*X, \mathrm{H}^*\mathrm{B}\mathbb{Z}/p)$$

and

$$(\mathrm{H}^*X : \mathrm{H}^*\mathrm{B}\mathbb{Z}/p) \xrightarrow{\sim} \mathrm{H}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p, X)$$

for all (fibrant) profinite space Y (cf [La], [Mo]).

The purpose of this work is to provide analogues of the statements 1) and 2) for the continuous MU-cohomology. Motivation for this comes from some analogue of point 3) :

3') Let X be a 1-connected space whose ordinary homology is a free abelian group of finite type in each dimension then the map $[\mathrm{BS}^1, \hat{X}] \to \mathrm{Hom}_{\mathcal{K}_\mathrm{K}}(\mathrm{K}^*X, \mathrm{K}^*\mathrm{BS}^1)$ is a bijection, where $\hat{X}$ denotes the p-completion of X, $\mathcal{K}_\mathrm{K}$ the category of p-adic lambda-rings and $\mathrm{K}^*(-)$ the p-completed K-theory. The proof uses an MU-unstable resolution of the space X ([DL]).

Note that this map is also a bijection when X is the classifying space of a connected compact Lie group, even if its ordinary homology has torsion ([NS]).

We end this introduction by pointing out some difficulties that arise when using MU-cohomology instead of mod p cohomology.

Let $\mathcal{M}$ denote the category of MU^*-modules and let M be in $\mathcal{M}$. We should not expect to find a space $\mathrm{K}(M)$ and a bijection $\mathrm{Hom}_{\mathrm{h}\hat{\mathcal{S}}}(X, \mathrm{K}(M)) \simeq \mathrm{Hom}_{\mathcal{M}}(M, \mathrm{MU}^*X)$ if the functor $X \mapsto \mathrm{Hom}_{\mathcal{M}}(M, \mathrm{MU}^*X)$ does not satisfy the exactness requirement for the Brown representability theorem. But if M is a free MU^*-module on a graded set S, we can define $\mathrm{K}(M)$ as the product $\prod_{s\in S} \hat{\mathrm{MU}}_{|s|}$ where $|s|$ refers to the degree of $s \in S$.

For the same reason that MU^*-modules may have torsion, the continuous MU-cohomology of a product $X \times Y$ is not in general functorial in the continuous MU^*-cohomology of Y. We will have instead a spectral sequence whose E^2-term is functorial in MU^*X and MU^*Y and which is strongly convergeant if X and Y are finite dimensional.

Finally if X and Y are ordinary spaces whose ordinary homology is a free abelian group of finite type in each dimension then the ordinary MU-cohomology of the product $X \times Y$ is the completion of the tensor product of the MU-cohomologies of X and Y with respect to the skeleton filtration, so some filtration on the MU-cohomology of a space is needed when the space is not finite dimensional.

2. Generalized Eilenberg - Mac Lane spaces and unstable algebra structure

We state the analogue of 1) for MU by defining the free unstable algebra on a graded set instead of an MU^*-module. Let gr-Set denote the category of $\mathbb{Z}$-graded sets.

For S in gr-Set, we define $\mathrm{K}(S)$ as the product $\prod_{s\in S} \hat{\mathrm{MU}}_{|s|}$ (where $|s|$ is the degree of s). The profinite space $\mathrm{K}(S)$ comes with a natural map $S \to \hat{\mathrm{MU}}^*\mathrm{K}(S)$, which assigns to an element $s \in S$ the projection on the factor indexed by $s : \prod_{t\in S} \mathrm{MU}_{|t|} \to \mathrm{MU}_{|s|}$. This map induces for any profinite space X a bijection $\mathrm{Hom}_{\mathrm{h}\hat{\mathcal{S}}}(X, \mathrm{K}(S)) \to \mathrm{Hom}_{\text{gr-Set}}(S, \hat{\mathrm{MU}}^*X)$. The functor $\mathrm{G} : S \mapsto \hat{\mathrm{MU}}^*\mathrm{K}(S)$ gets from this adjointness the structure of a monad on gr-Set. We denote by $\mathcal{K}_{\mathrm{MU}}$ the category of G-algebras of gr-Set and call its elements the MU-unstable algebras.

Examples.

- Let S be a graded set; the natural transformation $\mathrm{G} \circ \mathrm{G} \to \mathrm{G}$ makes $\mathrm{G}(S)$ a G-algebra. The map from S to $\mathrm{G}(S)$ induces a bijection $\mathrm{Hom}_{\mathrm{K}_{\mathrm{MU}}}(\mathrm{G}(S), N) \to \mathrm{Hom}_{\text{gr-Set}}(S, N)$ for any G-algebra N so that $\mathrm{G}(S)$ is the free unstable algebra on S.

- Let X be a profinite space, then the map $X \to \mathrm{K}(\hat{\mathrm{MU}}^*X)$, adjoint to the identity of $\hat{\mathrm{MU}}^*X$, induces a map $\mathrm{G}(\hat{\mathrm{MU}}^*X) \to \hat{\mathrm{MU}}^*X$ which makes $\hat{\mathrm{MU}}^*X$ an MU-unstable algebra.

Let M be a G-algebra. From the monad structure on G we get a simplicial G-algebra

$$\cdots \mathrm{G}^2(M) \leftleftarrows \mathrm{G}(M) \ ,$$

image in MU-cohomology of a cosimplicial diagram in $\mathrm{h}\hat{\mathcal{S}}$, and a morphism from it to the constant simplicial G-algebra M which is canonically a homotopy equivalence between simplicial graded sets. In particular the diagram $\mathrm{G}^2(M) \leftleftarrows \mathrm{G}(M) \to M$ is canonically a split coequalizer in gr-Set. Any natural structure on free G-algebras coming from a monad then induces a similar structure on M.

Our aim is to describe such a structure which is abelian and which gives the link between the continuous MU-cohomology of a profinite space and its continuous mod p cohomology. This is made possible by the fact that any free MU-unstable algebra is the continuous MU-cohomology of some "torsion free" profinite space for which we have a simple universal coefficient formula.

3. Additive structure on the continuous MU-cohomology of a torsion free profinite space

We choose a decreasing filtration $(\mathrm{f}^n\mathrm{MU}^*)$ of the ring MU^* such that $\mathrm{f}^1\mathrm{MU}^*$ is the kernel of the map $\mathrm{MU}^* \to \mathbb{Z}/p$ and $\mathrm{f}^n\mathrm{MU}^*/\mathrm{f}^{n+1}\mathrm{MU}^*$ is a finite graded $\mathbb{F}_p$-vector space. It induces a decreasing filtration on any MU^*-module M which we call the coefficient filtration. We let $\mathcal{M}_{\mathrm{f}}$ denote the category of MU^*-modules M with a decreasing filtration $(\mathrm{f}^n M)_{n\in\mathbb{N}}$ less thin than the coefficient filtration, and $\hat{\mathcal{L}}$ the full subcategory of $\mathcal{M}_{\mathrm{f}}$ whose objects are those isomorphic to the completion for the coefficient filtration of some free MU^*-module, given with the limit filtration.

We will say that a space is torsion free if its continuous p-adic cohomology is torsion free.

Proposition 3.1. *Let X be a profinite space of finite dimension (i.e. equal to some of its skeleton), then X is torsion free if and only if the continuous* MU*-cohomology of X with the coefficient filtration is in $\hat{\mathcal{L}}$. If so, the map $\hat{\mathrm{MU}}^*X/\mathrm{f}^1 \to \mathrm{H}^*X$, induced by the orientation $\mathrm{MU} \to \mathrm{H}\mathbb{Z}/p$, is an isomorphism.*

The proof uses an *ad hoc* Atyiah Hirzebruch spectral sequence.

Let X be a profinite space and let $\mathrm{Sk}_s X$ denote its s-skeleton for any $s \geq 0$. We define the skeleton filtration $\mathrm{F}^s_X\hat{\mathrm{MU}}^*X$ of $\hat{\mathrm{MU}}^*X$ has the kernel of the map $\hat{\mathrm{MU}}^*X \to \hat{\mathrm{MU}}^*\mathrm{Sk}_s X$ and the skeleton closure of the coefficient filtration of $\hat{\mathrm{MU}}^*X$ as the filtration given by $\mathrm{f}^n\hat{\mathrm{MU}}^*X = \cap_s(\mathrm{f}^n\hat{\mathrm{MU}}^*X + \mathrm{F}^s_X\hat{\mathrm{MU}}^*X)$. We obtain the following proposition by a limit argument :

Proposition 3.2. *Let X be a torsion free profinite space, then the continuous* MU*-cohomology of X with the skeleton closure of the coefficient filtration is in $\hat{\mathcal{L}}$ and the map $\hat{\mathrm{MU}}^*X/\mathrm{f}^1 \to \mathrm{H}^*X$ is an isomorphism.*

For M in $\hat{\mathcal{L}}$ and n an integer $\mathrm{F}^n M$ denotes the sub-object of M generated by the elements of degree greater or equal to n, then the decreasing sequence $(\mathrm{F}^n M)$ is a complete filtration of M in $\hat{\mathcal{L}}$. Let M and N be in $\hat{\mathcal{L}}$; we define the tensor product $M\hat{\otimes}N$ of M and N as the complete filtered MU^*-module given by $(M\hat{\otimes}N)/\mathrm{f}^n = \lim_s((M/\mathrm{F}^s)/\mathrm{f}^n \otimes_{\mathrm{MU}^*} (N/\mathrm{F}^s)/\mathrm{f}^n)$, then $M\hat{\otimes}N$ is in $\hat{\mathcal{L}}$. For torsion free profinite spaces X and Y, the map $\hat{\mathrm{MU}}^*X \otimes_{\mathrm{MU}^*} \hat{\mathrm{MU}}^*Y \to \hat{\mathrm{MU}}^*X \times Y$ induces a map $\hat{\mathrm{MU}}^*X\hat{\otimes}\hat{\mathrm{MU}}^*Y \to \hat{\mathrm{MU}}^*X \times Y$ which is an isomorphism if X and Y are torsion free.

Example. For any graded set S the profinite space $\mathrm{K}(S)$ is torsion free by results of Wilson ([Wi]) so the map $\mathrm{G}(S)/\mathrm{f}^1 \to \mathrm{H}^*\mathrm{K}(S)$ is an isomorphism.

4. Additive structure in the general case, free resolutions

For any graded set S, we define $\hat{\mathrm{L}}(S)$ as the completion for the coefficient filtration of the free MU^*-module with basis S, equipped with the limit filtration. The filtered MU^*-module $\hat{\mathrm{L}}(S)$ comes with a map $S \to \hat{\mathrm{L}}(S)$ which induces a bijection $\mathrm{Hom}_{\mathcal{M}_\mathrm{f}}(\hat{\mathrm{L}}(S), N) \to \mathrm{Hom}_{\text{gr-Set}}(S, N)$ for any complete object N of $\mathcal{M}_\mathrm{f}$; so we again obtain a monad structure on $\hat{\mathrm{L}} : \text{gr-Set} \to \text{gr-Set}$. We let $\hat{\mathcal{M}}$ denote the category of $\hat{\mathrm{L}}$-algebras of gr-Set.

Example. Let M be a G-algebra, then the natural free $\hat{\mathrm{L}}$-algebra structure on $\mathrm{G}^2(M)$ and $\mathrm{G}(M)$ induces a natural $\hat{\mathrm{L}}$-algebra structure on M.

It turns out that the category $\hat{\mathcal{M}}$ is an abelian category. The objects of $\hat{\mathcal{L}}$ are exactly the projective objects of $\hat{\mathcal{M}}$. From the monad structure on $\hat{\mathrm{L}}$, every $\hat{\mathrm{L}}$-algebra M has a canonical free resolution $\hat{\mathrm{L}}_*(M) \to M$.

For M and N in $\hat{\mathcal{M}}$, we define their tensor product as the coequalizer of the diagram $\hat{\mathrm{L}}^2(M)\hat{\otimes}\hat{\mathrm{L}}^2(N) \rightrightarrows \hat{\mathrm{L}}(M)\hat{\otimes}\hat{\mathrm{L}}(N)$, where $\hat{\mathrm{L}}^2(M) \rightrightarrows \hat{\mathrm{L}}(M) \to M$ and $\hat{\mathrm{L}}^2(N) \rightrightarrows \hat{\mathrm{L}}(N) \to N$ are the canonical free presentations of M and N. Note that if M and N are unstable algebras then $M\hat{\otimes}N$ is naturally an unstable algebra and as such the sum of M and N in $\mathcal{K}_{\mathrm{MU}}$. We define the graded $\hat{\mathrm{L}}$-algebra $\mathrm{Tor}_*^{\hat{\mathcal{L}}}(M, N)$ as the homology of the complex $\hat{\mathrm{L}}_*(M)\hat{\otimes}N$, where $\hat{\mathrm{L}}_*(M)$ is the canonical free resolution of M.

Exemple. For any $\hat{\mathrm{L}}$-algebra M and any integer n, the quotient M/f^n inherits from M an $\hat{\mathrm{L}}$-algebra structure. The natural map $M\hat{\otimes}(\mathrm{MU}^*/\mathrm{f}^n) \to M/\mathrm{f}^n$ is an isomorphism in $\hat{\mathcal{M}}$.

Proposition 4.1.

(a) *Let M be an $\hat{\mathrm{L}}$-algebra, $M_* \to M$ a free resolution of M and n a non negative integer, then the cokernel of the morphism $M_{n+1} \to M_n$ is in $\hat{\mathcal{L}}$ if and only if the $\hat{\mathrm{L}}$-algebra $\mathrm{Tor}_{n+1}^{\hat{\mathcal{L}}}(M, \mathrm{MU}^*/\mathrm{f}^1)$ is null.*

(b) *Let M_* be a non negatively graded complex of $\hat{\mathcal{L}}$ and let M be the cokernel of the morphism $M_1 \to M_0$ in $\hat{\mathcal{M}}$. Then M is in $\hat{\mathcal{L}}$ and $M_* \to M$ is a free resolution of M if and only if the homology of the complex M_*/f^1 is concentrated in degree 0.*

By definition the $\hat{\mathrm{L}}$-algebra $\mathrm{Tor}_1^{\hat{\mathcal{L}}}(M, N)$ is null if M is in $\hat{\mathcal{L}}$. We need some finiteness assumptions to prove that it is also null if N si in $\hat{\mathcal{L}}$, that is, to prove that the tensor product by N is exact.

Proposition 4.2. *Let M and N be $\hat{\mathrm{L}}$-algebras null in high degree and such that N is in $\hat{\mathcal{L}}$; then the $\hat{\mathrm{L}}$-algebra $\mathrm{Tor}_1^{\hat{\mathcal{L}}}(M, N)$ is null.*

The proof uses some change of ring formula related to the sub polynomial rings of $\hat{\mathrm{MU}}^*$ generated by $x_1, \ldots, x_n$, $n \in \mathbb{N}$. We also obtain from a syzygy theorem :

Proposition 4.3. *Let M be an $\hat{\mathrm{L}}$-algebra and $M_1 \to M_0 \to M$ a free presentation of M. Suppose that M/f^1 and M/f^0 are concentrated in a finite range of degrees then M admits a free resolution of finite length.*

Let X be a profinite space; a free resolution of X is a sequence of maps $C_n \to X_n \to C_{n+1}$, $n \geq 0$, between profinite spaces such that C_0 is X, the profinite space X_n is torsion free and the sequence $C_n \to X_n \to C_{n+1}$ is a cofiber sequence inducing an exact sequence $0 \to \tilde{\hat{\mathrm{MU}}}{}^*C_{n+1} \to \hat{\mathrm{MU}}^*X_n \to \hat{\mathrm{MU}}^*C_n \to 0$ for all n. Such a resolution of X gives rise to a free resolution $\cdots \to \tilde{\hat{\mathrm{MU}}}{}^*X_1 \to \hat{\mathrm{MU}}^*X_0 \to \hat{\mathrm{MU}}^*X$ of $\hat{\mathrm{MU}}^*X$. Every profinite space X admits a canonical free resolution defined inductively by $C_0 = X$, $X_n = \mathrm{K}(\hat{\mathrm{MU}}^*C_n)$ and $C_{n+1} = \mathrm{Cofiber}(C_n \to \mathrm{K}(\hat{\mathrm{MU}}^*C_n))$. Moreover if X if finite dimensional then X admits a free resolution by finite dimensional profinite spaces : it suffices to replace inductively $\mathrm{K}(\hat{\mathrm{MU}}^*C_n)$ by its k-th skeleton in the construction above, where k is the dimension of C_n. We say that a free resolution (C_n, X_n) of X is of finite length if C_n is the point for some n. From the proposition above and the proposition 3.1 we deduce (cf [Ad] or [CS]) :

Proposition 4.4. *Let X be a profinite space of finite dimension, then X admits a free resolution of finite length.*

Proposition 4.5. *Let X and Y be finite dimensional profinite spaces, them there exists a spectral sequence whose* E^2 *term is given by* $\mathrm{E}^2_{s,*} = \mathrm{Tor}^{\hat{\mathcal{L}}}_s(\hat{\mathrm{MU}}^*X, \hat{\mathrm{MU}}^*Y)$ *and which strongly converges to* $\hat{\mathrm{MU}}^*X \times Y$.

Proposition 4.6. *Let X be a finite dimensional profinite space, then there exists a spectral sequence whose* E^2 *term is given by* $\mathrm{E}^2_{s,*} = \mathrm{Tor}^{\hat{\mathcal{L}}}_s(\hat{\mathrm{MU}}^*X, \mathrm{MU}^*/\mathrm{f}^1)$ *and which strongly converges to* H^*X

Both spectral sequence are formed by using a free resolution of X and by considering the resulting exact couple $\tilde{\hat{\mathrm{MU}}}{}^*C_{n+1} \times Y \to \tilde{\hat{\mathrm{MU}}}{}^*X_n \times Y \to \tilde{\hat{\mathrm{MU}}}{}^*C_n \times Y$ for the first one, $\tilde{\mathrm{H}}^*C_{n+1} \to \tilde{\mathrm{H}}^*X_n \to \tilde{\mathrm{H}}^*C_n$ for the second one (cf [Ad] for the case of finite CW-complexes or spectra).

We use some limit argument to approach the $\hat{\mathrm{L}}$-algebra $\mathrm{Tor}^{\hat{\mathcal{L}}}(M, N)$ when M or N are $\hat{\mathrm{L}}$-algebras not necessarily bounded above. We obtain :

Proposition 4.7. *Let L be in $\hat{\mathcal{L}}$ such that L/f^1 is degree-wise finite, then :*

(a) *The $\hat{\mathrm{L}}$-algebra $\mathrm{Tor}^{\hat{\mathcal{L}}}_1(M, L)$ is null for all $M \in \hat{\mathcal{M}}$.*

(b) *Let M_s be a tower of $\hat{\mathrm{L}}$-algebras then the map $M_\infty \hat{\otimes} L \to \lim_s(M_s \hat{\otimes} L)$ and $(\lim^1_s M_s)\hat{\otimes} L \to \lim^1_s(M_s \hat{\otimes} L)$ are isomorphisms.*

We can generalize the previous proposition, replacing L by an $\hat{\mathrm{L}}$-algebra M such that M admits a free resolution $M_* \to M$ with M_k/f^1 degree-wise finite for all k.

Applications.

(a) Let X and Y be profinite spaces such that X is torsion free and H^*X is degree-wise finite, then the map $\hat{\mathrm{MU}}^*X \hat{\otimes} \hat{\mathrm{MU}}^*Y \to \hat{\mathrm{MU}}^*X \times Y$ is an isomorphism.

(b) Let n be an integer and X a profinite space. Then

$$\mathrm{Tor}^{\hat{\mathcal{L}}}_k(\hat{\mathrm{MU}}^*X, \mathrm{MU}^*/\mathrm{f}^n) \to \lim_s \mathrm{Tor}^{\hat{\mathcal{L}}}_k(\hat{\mathrm{MU}}^*X/\mathrm{F}^s_X, \mathrm{MU}^*/\mathrm{f}^n)$$

is an isomorphism for all k if $\lim^1_s \mathrm{Tor}^{\hat{\mathcal{L}}}_k(\hat{\mathrm{MU}}^*X/\mathrm{F}^s_X, \mathrm{MU}^*/\mathrm{f}^n)$ is null for all k, in particular if X is the profinite completion of some space.

(c) We take $M = \hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n$ for some n. The $\hat{\mathrm{L}}$-algebra $\hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n$ admits a free resolution of length 1. We let N_s be a tower of $\hat{\mathrm{L}}$-algebras such that each N_s has a free presentation $N_{s,1} \to N_{s,0} \to N_s$ with $N_{s,1}/\mathrm{f}^1$ and $N_{s,0}/\mathrm{f}^1$ concentrated in a finite range of degrees. We can prove that $\mathrm{Tor}^{\hat{\mathcal{L}}}_1(\hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n, N_s)$ is null so we obtain an exact sequence $0 \to \hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n \hat{\otimes} N_\infty \to \lim_s(\hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n \hat{\otimes} N_s) \to \mathrm{Tor}^{\hat{\mathcal{L}}}_1(\hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n, \lim^1_s N_s) \to 0$. If X is the profinite completion of some space then taking $N_s = \hat{\mathrm{MU}}^*X_s$ where X_s is the s-th skeleton of X, we obtain an isomorphism $\hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n \hat{\otimes} \hat{\mathrm{MU}}^*X \to \hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n \times X$. In general we are unable to prove such a formula (see also [RWY]).

(d) Let X be a space such that the $\hat{\mathrm{L}}$-algebra $\lim^1_s \mathrm{Tor}^{\hat{\mathcal{L}}}_k(\hat{\mathrm{MU}}^*X/\mathrm{F}^s_X, \mathrm{MU}^*/\mathrm{f}^1)$ is null for all k then the filtration of $\hat{\mathrm{MU}}^*X$ coming from a free resolution of X coincides with the skeleton closure of the coefficient filtration.

5. Back to MU-unstable algebra

From the isomorphism $\mathrm{G}(S)/\mathrm{f}^1 \simeq \mathrm{H}^*\mathrm{K}(S)$ we get an unstable algebra structure over the mod p Steenrod algebra on $\mathrm{G}(S)/\mathrm{f}^1$. As any MU-unstable algebra M is the split coequaliser in gr-Set of the diagram $\mathrm{G}^2(M) \rightleftarrows \mathrm{G}(M)$, we obtain an unstable algebra structure on M/f^1 for all M.

Proposition 5.1.

(a) *For all* $M \in \mathcal{K}_{\mathrm{MU}}$*, the morphism* $\mathrm{Hom}_{\mathcal{K}_{\mathrm{MU}}}(M, \hat{\mathrm{MU}}^*) \to \mathrm{Hom}_{\mathcal{K}_{\mathrm{H}}}(M/\mathrm{f}^1, \mathbb{Z}/p)$ *is a bijection.*

(b) *For all profinite space* X*, the morphism* $\mathrm{Hom}_{\mathcal{K}_{\mathrm{MU}}}(\hat{\mathrm{MU}}^*X, \hat{\mathrm{MU}}^*) \to \mathrm{Hom}_{\mathcal{K}_{\mathrm{H}}}(\mathrm{H}^*X, \mathbb{Z}/p) \simeq \pi_0 X$ *is a bijection.*

Let P be a free $\hat{\mathrm{L}}$-algebra such that P/f^1 is degree-wise finite, then the functor $\hat{\mathcal{M}} \to \hat{\mathcal{M}}$, $M \mapsto P\hat{\otimes}M$ has a left adjoint $(- : P)_{\hat{\mathcal{M}}}$. If S and S' are graded sets with S' concentrated in a finite number of degrees and finite in each degree, then $(\hat{\mathrm{L}}(S) : \hat{\mathrm{L}}(S'))_{\hat{\mathcal{M}}}$ is isomorphic to $\hat{\mathrm{L}}(S \times S'^-)$, where S'^- is the graded set obtained from S' by reversing the sign of the degrees.
By adjonction we obtain :

Proposition 5.2.

(a) *Let* P *be in* $\mathcal{K}_{\mathrm{MU}} \cap \hat{\mathcal{L}}$ *such that* P/f^1 *is degree-wise finite, then the funtor* $\mathcal{K}_{\mathrm{MU}} \to \mathcal{K}_{\mathrm{MU}}$*,* $M \mapsto P\hat{\otimes}M$ *has a left adjoint* $(- : P)_{\mathcal{K}_{\mathrm{MU}}}$*.*

(b) *Let* S *be a graded set and* W *a simplicial space whose* p*-adic cohomology is torsion free and finite dimensional in each degree, then the counit* $W\hat{\times}\mathbf{map}(W, \mathrm{K}(S)) \to \mathrm{K}(S)$ *induces an isomorphism*
$$(\mathrm{G}(S) : \hat{\mathrm{MU}}^*W)_{\mathcal{K}_{\mathrm{MU}}} \to \hat{\mathrm{MU}}^*\mathbf{map}(W, \mathrm{K}(S)) \ .$$

This is the analogue of statement (2) for continuous MU-cohomology.

6. Application to the continuous MU-cohomology of mapping spaces from the classifying space of $\mathbb{Z}/p^n$ or S^1.

Recall that the map $\mathrm{colim}_n \mathrm{B}\mathbb{Z}/p^n \to \mathrm{BS}^1$ is a mod p homology equivalence. For M in $\mathcal{K}_{\mathrm{MU}}$ and n in $\mathbb{N} \cup \{\infty\}$, we define the unstable algebra $\mathrm{T}_n(M)$ as the coequalizer of the diagram
$$\hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, \mathrm{K}(\mathrm{G}(M))) \rightleftarrows \hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, \mathrm{K}(M))$$
induced by the diagram $\mathrm{G}^2(M) \rightleftarrows \mathrm{G}(M)$ in $\mathcal{K}_{\mathrm{MU}}$. The last proposition states that T_∞ is left adjoint to $\hat{\mathrm{MU}}^*\mathrm{BS}^1\hat{\otimes}-$ in $\mathcal{K}_{\mathrm{MU}}$. A weaker form of this adjonction can be shown for T_n for finite n. At least we still have a map $\mathrm{T}_n\hat{\mathrm{MU}}^*X \to \hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, X)$ for any profinite space X and a bijection $\mathrm{Hom}_{\mathcal{K}_{\mathrm{MU}}}(\mathrm{T}_n M, \hat{\mathrm{MU}}^*) \simeq \mathrm{Hom}_{\mathcal{K}_{\mathrm{MU}}}(M, \hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n)$.

Let X be a torsion free profinite space. The augmented simplicial G-algebra $\mathrm{G}_\bullet(\hat{\mathrm{MU}}^*X) \to \hat{\mathrm{MU}}^*X$ is the image in continuous MU-cohomology of a coaugmented cosimplical profinite space $X \to \mathrm{R}^\bullet X$. As $\hat{\mathrm{MU}}^*X$ is in $\hat{\mathcal{L}}$ and $\mathrm{G}_\bullet(\hat{\mathrm{MU}}^*X) \to \hat{\mathrm{MU}}^*X$ is acyclic in $\hat{\mathcal{M}}$, the augmented simplicial $\mathbb{F}_p$-vector space $\mathrm{H}^*\mathrm{R}^\bullet X \to \mathrm{H}^*X$ is acyclic. We use the following key-input from [DL] :

Proposition 6.1. *The augmented simplicial* $\mathbb{F}_p$*-vector space*
$$\mathrm{H}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, \mathrm{R}^\bullet X) \to \mathrm{H}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, X)$$
is acyclic

As the morphism $\mathrm{T}_n\hat{\mathrm{MU}}^*\mathrm{R}^kX \to \hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, \mathrm{R}^kX)$ is iso for all k, we obtain

Theorem 6.2. *Let* X *be a torsion free profinite space, then the morphism*
$$\mathrm{T}_n\hat{\mathrm{MU}}^*X \to \hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, X)$$

is an isomorphism.

For general X we have the following proposition :

PROPOSITION 6.3. *Let $X \to X_0 \to C$ be a cofiber sequence in $\hat{\mathcal{S}}$, then the induced sequence*

$$\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, X) \to \mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, X_0) \to \mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, C)$$

is a cofiber sequence.

We can apply this to the begining of a free resolution $X = C_0 \to X_0 \to C_1$ of X. As the associated long exact sequence in MU-cohomology is short, we obtain a coequalizer diagram $\hat{\mathrm{MU}}^*C_1 \hat{\otimes} \hat{\mathrm{MU}}^*X_0 \rightrightarrows \hat{\mathrm{MU}}^*X_0 \to \hat{\mathrm{MU}}^*X$. Using the fact that T_n commutes by construction to reflexive coequalizers, we obtain :

PROPOSITION 6.4. *Suppose that the morphism $(\mathrm{T}_n\hat{\mathrm{MU}}^*Y)/\mathrm{f}^1 \to (\hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, Y))/\mathrm{f}^1$ is epi for $Y = X$ and $Y = C_1$ then $\mathrm{T}_n\hat{\mathrm{MU}}^*X \to \hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, X)$ is an isomorphism*

Example. Let X be a profinite space whose continuous MU-cohomology is null in odd degree, then we can choose X_0 having the same property. If $M \in \mathcal{K}_{\mathrm{MU}}$ is null in odd degree then the same is true for T_nM. Suppose that $\hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, Y)$ is null in odd degree for $Y = X$ and $Y = C_1$, then the proposition applies and $\mathrm{T}_n\hat{\mathrm{MU}}^*X \to \hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, X)$ is an isomorphism. This applies to the case of a product of Eilenberg - Mac Lane spaces for mod p-cohomology : their MU-cohomology are null in odd degrees by a result of Ravenel-Wilson-Yagita ([RWY]).

To conclude, recall that we have a bijection $\mathrm{Hom}_{\mathcal{K}_{\mathrm{MU}}}(\mathrm{T}_nM, \hat{\mathrm{MU}}^*) \simeq \mathrm{Hom}_{\mathcal{K}_{\mathrm{MU}}}(M, \hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n)$. So, letting $[\mathrm{B}\mathbb{Z}/p^n, X]$ denote the homotopy classes of maps from $\mathrm{B}\mathbb{Z}/p^n$ to the underlying space of a profinite space X, we have :

PROPOSITION 6.5. *Let X be a profinite space such that $\mathrm{T}_n\hat{\mathrm{MU}}^*X \to \hat{\mathrm{MU}}^*\mathbf{map}(\mathrm{B}\mathbb{Z}/p^n, X)$ is an isomorphism, then the map*

$$[\mathrm{B}\mathbb{Z}/p^n, X] \to \mathrm{Hom}_{\mathcal{K}_{\mathrm{MU}}}(\hat{\mathrm{MU}}^*X, \hat{\mathrm{MU}}^*\mathrm{B}\mathbb{Z}/p^n)$$

is a one to one correspondance.

References

[AD] J. F. ADAMS, *Lectures on Generalized Cohomology Theories, Springer L. N. M.*, **99**, 1969.

[CS] P .E. CONNER and L. SMITH, On the complex bordism of finite complexes, *Publ. Math. I. H. E. S.*, **37** (1969), 117-221.

[DE] F.-X. DEHON, Cobordisme complexe des espaces profinis et foncteur T de Lannes, preprint (2001-03).

[DL] F.-X. DEHON and J. LANNES, Sur les espaces fonctionnels dont la source est le classifiant d'un groupe de Lie compact commutatif, *Publ. Math. I. H. E. S.*, **89** (1999).

[LA] J. LANNES, Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire, *Publ. Math. I. H. E. S.*, **75** (1992), 135-244.

[MO] F. MOREL, Ensembles profinis simpliciaux et interprétation géométrique du foncteur T, *Bull. Soc. Math. France*, **124** (1996), 347-373.

[NS] D. NOTBOHM and L. SMITH, Fake Lie groups and maximal tori I, *Math. Ann.*, **288** (1990), 637-661.

[RWY] D. C. RAVENEL, W. S. WILSON and N. YAGITA, Brown-Peterson cohomology from Morava K-theory, *K-Theory*, **15** (1998), 147-199.

[WI] W. S. WILSON, The Ω-spectrum for Brown-Peterson cohomology, Part I, *Comm. Math. Helv.*, **48** (1973), 45-55.

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