Mean-Field Games

Lectures at the Imperial College London

4th Lecture: Master Equation and Convergence

François Delarue (Nice – J.-A. Dieudonné)

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Joint works with R. Carmona; J.F. Chassagneux and D. Crisan; P. Cardaliaguet

Part I. Master Equation

Part I. Master Equation

a. Revisiting the PDE interpretation

Reminder

- Recall MFG when $\sigma^0 \equiv 0$

- Define the asymptotic equilibrium state of the population as the solution of a fixed point problem

(1) fix a flow of probability measures $(\mu_t)_{0 \le t \le T}$ (with values in $\mathcal{P}_2(\mathbb{R}^d)$)

(2) solve the stochastic optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t)dW_t$$

- with $X_0 = \xi$ being fixed on some set-up $(\Omega, \mathbb{F}, \mathbb{P})$ with a d-dimensional B.M.
- with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$

(3) let $(X_t^{\star,\mu})_{0 \le t \le T}$ be the unique optimizer (under nice assumptions)

$\rightsquigarrow$ find $(\mu_t)_{0 \le t \le T}$ such that

$$\mu_t = \mathcal{L}(X_t^{\star,\mu}), \quad t \in [0, T]$$

PDE point of view: HJB

- PDE characterization of the optimal control problem when σ is the identity
- Value function in environment $(\mu_t)_{0 \le t \le T}$

$$U(t,x) = \inf_{\alpha \text{ processes}} \mathbb{E}\Big[g(X_T, \mu_T) + \int_t^T f(X_s, \mu_s, \alpha_s) ds \big| X_t = x\Big]$$

- U solution Backward HJB

$$\Big(\partial_t U + \frac{\partial^2_{xx} U}{2}\Big)(t,x) + \underbrace{\inf_{\alpha \text{ scalar}} \big[b(x, \mu_t, \alpha) \cdot \partial_x U(t,x) + f(x, \mu_t, \alpha)\big]}_{\text{standard Hamiltonian in HJB}} = 0$$

  - $\alpha \rightsquigarrow \alpha = \alpha^\star(x, \mu_t, \partial_x U(t,x))$
- Terminal boundary condition: $U(T, \cdot) = g(\cdot, \mu_T)$
- Pay attention that U depends on $(\mu_t)_t$!

Fokker-Planck

- Need for a PDE characterization of $(\mathcal{L}(X_t^{\star,\mu}))_t$
- Dynamics of $X^{\star,\mu}$ at equilibrium

$$dX_t^{\star,\mu} = b(X_t^{\star,\mu}, \mu_t, \alpha^\star(X_t^{\star,\mu}, \mu_t, \partial_x U(t, X_t^{\star,\mu})))dt + dW_t$$

- Law $(X_t^{\star,\mu})_{0\leq t\leq T}$ satisfies Fokker-Planck (FP) equation

$$d_t\mu_t = -\mathrm{div}(\underbrace{b(x, \mu_t, \alpha^\star(x, \mu_t, \partial_x U(t,x))}_{b^\star(t,x)}\mu_t)dt + \frac{1}{2}\partial_{xx}^2\mu_t dt$$

- MFG equilibrium described by forward-backward in ∞ dimension
    - ∞ dimensional analogue of

$$\dot{x}_t = b(x_t, y_t)dt, \quad x_0 = x^0$$
$$\dot{y}_t = -f(x_t, y_t)dt, \quad y_T = g(x_T)$$

    - $\sigma^0 \equiv 0 \rightsquigarrow$ deterministic FB system
    - if $\sigma^0 \not\equiv 0 \rightsquigarrow$ stochastic FB system

MFG as characteristics of a PDE

- Find the decoupling field of the ∞ dimensional FB system
- Find a function $\mathcal{U}$ such that

$$\underbrace{U}_{\text{HJB}}(t,\ \cdot\ ) = \mathcal{U}(t,\ \cdot\ ,\ \underbrace{\mu_t}_{\text{FP}}\ )$$

  - $\mathcal{U} : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to C(\mathbb{R}^d, \mathbb{R})$
  - $\mathcal{U}(T, \cdot, \mu_T) = g(\cdot, \mu_T)$
- Write (master?) PDE for $\mathcal{U}$
- Procedure for the formal identification of the PDE
  - martingale increment

$$d\mathcal{U}(t, X_t^\star, \mu_t) + f(X_t^\star, \mu_t, \alpha^\star(X_t^\star, \mu_t, \partial_x \mathcal{U}(t, X_t^\star, \mu_t)))dt$$

  - compare with Itô's formula
  - requires a chain rule on $\mathcal{P}_2(\mathbb{R}^d)$

MFG as characteristics of a PDE

- Find the decoupling field of the ∞ dimensional FB system
- Find a function $\mathcal{U}$ such that

$$\underbrace{U}_{\text{HJB}}(t,x) = \mathcal{U}(t,x,\underbrace{\mu_t}_{\text{FP}})$$

  - $\mathcal{U} : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
  - $\mathcal{U}(T,x,\mu_T) = g(x,\mu_T)$
- Write (master?) PDE for $\mathcal{U}$
- Procedure for the formal identification of the PDE
  - martingale increment

$$d\mathcal{U}(t,X_t^\star,\mu_t) + f(X_t^\star,\mu_t,\alpha^\star(X_t^\star,\mu_t,\partial_x\mathcal{U}(t,X_t^\star,\mu_t)))dt$$

  - compare with Itô's formula
  - requires a chain rule on $\mathcal{P}_2(\mathbb{R}^d)$

Part I. Master Equation

b. Deriving the master equation

Reminder

- Recall Lions differentiation on $\mathcal{P}_2(\mathbb{R}^d)$
- Consider $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
- Lifted-version of $\mathcal{U}$

$$\hat{\mathcal{U}} : L^2(\Omega, \mathbb{P}) \ni X \mapsto \mathcal{U}(\mathrm{Law}(X))$$

  - $\mathcal{U}$ differentiable if $\hat{\mathcal{U}}$ Fréchet differentiable (Lions)
- Differential of $\mathcal{U}$
  - Fréchet derivative of $\hat{\mathcal{U}}$

$$D\hat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \partial_\mu \mathcal{U}(\mu) : \mathbb{R} \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \quad \mu = \mathcal{L}(X)$$

  - derivative of $\mathcal{U}$ at $\mu \rightsquigarrow \partial_\mu \mathcal{U}(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$
- Finite dimensional projection

$$\partial_{x_i}\Big[\mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)\Big] = \frac{1}{N}\partial_\mu \mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)(x_i), \quad x_1, \ldots, x_N \in \mathbb{R}^d$$

Chain rule on $\mathcal{P}_2(\mathbb{R}^d)$

- Itô process $dX_t = b_t dt + \sigma_t dW_t, \quad \int_0^T \mathbb{E}[|b_t|^2 + |\sigma_t|^4]dt < \infty$
  - μ_t = law of X_t
- $\hat{\mathcal{U}}$ twice Fréchet differentiable
  - chain rule for $(\mathcal{U}(\mu_t))_{t \geq 0}$?
- Approximate μ_t by particle system

$$\mu_t \sim \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^j} \quad \text{and} \quad d_t \Big[\mathcal{U}\Big(\frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^j} \Big) \Big]$$

  - expand the right-hand side and pass to the limit
- Chain rule
  - need $\mathbb{R}^d \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \in \mathbb{R}^d$ differentiable

$$\frac{d}{dt}\mathcal{U}(\mu_t) = \mathbb{E}[\langle b_t, \partial_\mu \mathcal{U}(\mu_t)(X_t)\rangle] + \frac{1}{2}\mathbb{E}[\text{Trace}(\sigma_t \sigma_t^\dagger \partial_x(\partial_\mu \mathcal{U}(\mu_t))(X_t))]$$

Shape of the master equation

- Formal identification of zero dt term in expansion of

$$d\mathcal{U}(t,X_t^\star,\mu_t) + f(X_t^\star,\mu_t,\alpha^\star(X_t^\star,\mu_t,\partial_x\mathcal{U}(t,X_t^\star,\mu_t)))dt$$

∘ requires an extension of Itô's formula to handle all the coordinates $\rightsquigarrow$ no bracket!

- Formal derivation $\rightsquigarrow$ first-order master equation:

$$\partial_t\mathcal{U}(t,x,\mu) + \underbrace{\int_{\mathbb{R}^d}\langle b^\star(t,v),\partial_\mu\mathcal{U}(t,x,\mu)(v)\rangle d\mu(v)}_{\text{transport in } \mu}$$

$$+ \underbrace{\langle b^\star(t,x),\partial_x\mathcal{U}(t,x,\mu)\rangle + f(x,\mu,\alpha^\star(t,x,\partial_x\mathcal{U}(t,x,\mu),\mu))}_{\text{standard Hamiltonian}}$$

$$+ \frac{1}{2}\text{Trace}\Big(\underbrace{\partial_x^2\mathcal{U}(t,x,\mu)}_{\text{standard diffusion}}\Big) + \underbrace{\int_{\mathbb{R}^d}\text{Trace}\Big(\partial_v\partial_\mu\mathcal{U}(t,x,\mu)(v)\Big)d\mu(v)}_{\text{bracket}} = 0$$

∘ Not a HJB! (MFG $\neq$ optimization)

Part I. Master Equation

c. Sketch of the proof

Program

- Prove existence of a classical solution
  - holds in small time if smooth coefficients
- Long time $\rightsquigarrow$ emergence of singularities
  - no singularity in x $\leftrightsquigarrow$ Laplace ∂_x^2
  - if ~~Laplace~~ $\rightsquigarrow$ use convexity in x in cost functional
  - regularity in μ $\rightsquigarrow$ Laplace does not help need monotonicity condition

main issue is to control $\partial_\mu \mathcal{U}$!

- Lasry Lions monotonicity condition
  - b doesn't depend on μ
  - $f(x,\mu,\alpha) = f_0(x,\mu) + f_1(x,\alpha)$ (μ and α are separated)
  - monotonicity property for f_0 and g w.r.t. μ

$$\int_{\mathbb{R}^d} (h(x,\mu) - h(x,\mu'))d(\mu-\mu')(x) \geq 0, \quad h = f_0, g$$

Master equation in linear case

- Forget forward-backward and consider the decoupled case

$$dX_t^\star = b(X_t^\star, \mathcal{L}(X_t^\star))dt + dW_t, \quad X_0^\star = X_0$$

  - choose $\sigma = \mathrm{Id}$ for simplicity

- Analogue with the master equation?
  - notice that $\mathcal{L}(X_t^\star)$ only depends on $\mathcal{L}(X_0)$
  - define the semi-group

$$(P_t\phi)(\mathcal{L}(X_0)) = \phi(\mathcal{L}(X_t^\star)), \quad t \in [0,T], \quad \phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$$

  - dynamics of $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto P_t\phi(\mu)$?

- Shape of the master equation

$$\partial_t(P_t\phi)(\mu) - \int_{\mathbb{R}^d} b(v,\mu) \cdot \partial_\mu(P_t\phi)(\mu)(v)d\mu(v)$$
$$- \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_v\partial_\mu(P_t\phi)(\mu)(v)\Big)d\mu(v) = 0, \quad (P_0\phi)(\mu) = \phi(\mu)$$

Derivative of the semi-group of a MKV SDE

- Regularity of $P_t\phi$ when ϕ is smooth $\rightsquigarrow$ investigate smoothness of the flow of the MKV SDE
- Lift of $\phi \rightsquigarrow \hat{\phi} : L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto \hat{\phi}(X) = \phi(\mathcal{L}(X))$
  - $P_t\phi(\mathcal{L}(X_0)) = \hat{\phi}(X_t^\star)$
- Perturbation of X_0 in direction $\zeta \in L^2$
  - $X_0^\varepsilon = X_0 + \varepsilon\zeta \rightsquigarrow (X_t^{\star,\varepsilon})_{0\le t\le T} \Rightarrow \partial_\zeta X_t^\star = \frac{dX_t^{\star,\varepsilon}}{d\varepsilon}\big|_{\varepsilon=0}$
- Derivative of $P_t\phi$ reads

$$\mathbb{E}\Big[\langle \partial_\mu (P_t\phi)(\mathcal{L}(X_0))(X_0), \zeta\rangle\Big] = \mathbb{E}\Big[\langle \partial_\mu \phi(\mathcal{L}(X_t^\star))(X_t^\star), \partial_\zeta X_t^\star\rangle\Big]$$

  - get the estimate

$$\underbrace{\mathbb{E}\Big[\big|\partial_\mu (P_t\phi)(\mathcal{L}(X_0))(X_0)\big|^2\Big]^{1/2}}_{\text{derivative of semigroup at } \mathcal{L}(X_0)}$$

$$\le \underbrace{\mathbb{E}\Big[\big|\partial_\mu \phi(\mathcal{L}(X_t^\star))(X_t^\star)\big|^2\Big]^{1/2}}_{\text{derivative of } \phi \text{ along SDE}} \underbrace{\sup_{\zeta:\mathbb{E}[|\zeta|^2]\le 1} \mathbb{E}\Big[\big|\partial_\zeta X_t^\star\big|^2\Big]^{1/2}}_{L^2 \text{ flow of SDE}}$$

Derivative of the flow of MKV SDE

- Recall MKV dynamics

$$dX_t^\star = b(X_t^\star, \mathcal{L}(X_t^\star))dt + dW_t, \quad X_0^\star = X_0$$

- Dynamics of $\partial_\zeta X^\star$

$$\begin{aligned} d\partial_\zeta X_t^\star &= \partial_x b(X_t^\star, \mathcal{L}(X_t^\star))\partial_\zeta X_t^\star dt \\ &\quad + \hat{\mathbb{E}}[\partial_\mu b(X_t^\star, \mathcal{L}(X_t^\star))(\hat{X}_t^\star)\partial_\zeta \hat{X}_t^\star]dt, \quad \partial_\zeta X_0^\star = \zeta \end{aligned}$$

  ◦ $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$ auxiliary space with copies of the r.v. $\rightsquigarrow$ McKean-Vlasov derivative system

- L^2 estimate of $\mathbb{E}[|\partial_\zeta X_t^\star|^2]$

$$\begin{aligned} d\mathbb{E}[|\partial_\zeta X_t^\star|^2] &= 2\mathbb{E}[\langle \partial_\zeta X_t^\star, \partial_x b(X_t^\star, \mathcal{L}(X_t^\star))\partial_\zeta X_t^\star \rangle]dt \\ &\quad + \mathbb{E}\hat{\mathbb{E}}[\langle \partial_\zeta X_t^\star, \partial_\mu b(X_t^\star, \mathcal{L}(X_t^\star))(\hat{X}_t^\star)\partial_\zeta \hat{X}_t^\star \rangle]dt \end{aligned}$$

  ◦ deduce $\mathbb{E}[|\partial_\zeta X_t^\star|^2] \le C\mathbb{E}[|\zeta|^2]$ with

$$C = C\Big(T, \sup_{x,\mu}\big|\partial_x b(x,\mu)\big|^2, \sup_{x,\mu} \int |\partial_\mu b(x,\mu)(v)|^2 d\mu(v)\Big)$$

Higher-order derivatives

- Master equation $\rightsquigarrow$ differentiate once again w.r.t. v

$$(\mu, v) \mapsto \partial_\mu P_t \phi(\mu)(v)$$

- Derivatives in the direction v/X_0
  - freeze ζ and consider new perturbation $X_0 \rightsquigarrow X_0^\varepsilon$

$$\mathcal{L}(X_0^\varepsilon) \quad \text{independent of} \quad \varepsilon \Rightarrow \mathcal{L}(X_t^{\star,\varepsilon}) \quad \text{independent of} \quad \varepsilon$$

- differentiate the formula for the derivative

$$\mathbb{E}\Big[\Big\langle \partial_v \partial_\mu \Big(P_t \phi(\mathcal{L}(X_0^0))\Big)(X_0^0), \zeta \otimes \frac{dX_0^\varepsilon}{d\varepsilon}_{|\varepsilon=0} \Big\rangle\Big]$$

$$= \mathbb{E}\Big[\Big\langle \partial_v \partial_\mu \phi(\mathcal{L}(X_t^{0,\star}))(X_t^{0,\star}), \partial_\zeta X_t^{0,\star} \otimes \frac{d}{d\varepsilon}_{|\varepsilon=0} X_t^{\varepsilon,\star} \Big\rangle\Big]$$

$$+ \mathbb{E}\Big[\Big\langle \partial_\mu \phi(\mathcal{L}(X_t^{0,\star}))(X_t^{0,\star}), \frac{d}{d\varepsilon}_{|\varepsilon=0} \partial_\zeta X_t^{\varepsilon,\star} \Big\rangle\Big]$$

  - example $X_0^\varepsilon = X_0 + \delta\Big(\cos(\varepsilon) Z + \sin(\varepsilon) Z'\Big)$
  - with $(Z, Z') \sim \mathcal{N}(0,1)^{\otimes 2}$ and (Z, Z') independent of X_0

Example in coupled case

- Linear-quadratic cost in $d = 1$
  - $b(x, \mu, \alpha) = \alpha, \quad f_1(x, \alpha) = \alpha^2/2$
  - g, f_0 bounded, smooth and Lasry-Lions

$$dX_t^\star = -\partial_x \mathcal{U}(t, X_t^\star, \mathcal{L}(X_t^\star))dt + dB_t$$

- Dynamics of $\partial_\zeta X^\star$

$$\begin{aligned} d\partial_\zeta X_t^\star &= -\partial_{xx}^2 \mathcal{U}(X_t^\star, \mathcal{L}(X_t^\star), \cdot)\partial_\zeta X_t^\star dt \\ &\quad - \hat{\mathbb{E}}[\partial_\mu(\partial_x \mathcal{U})(t, X_t^\star, \mathcal{L}(X_t^\star))(\hat{X}_t^\star)\partial_\zeta \hat{X}_t^\star]dt \end{aligned}$$

  - $\partial_{xx}^2 \mathcal{U}$ already estimated! (thanks to Laplace)
- Propagation of monotonicity

$$\mathbb{E}\hat{\mathbb{E}}[\partial_x(\partial_\mu \mathcal{U})(t, X_t^\star, \mathcal{L}(X_t^\star))(\hat{X}_t^\star)\partial_\zeta \hat{X}_t^\star \partial_\zeta X_t^\star] \geq 0.$$

- Conclusion $\rightsquigarrow \mathbb{E}[|\partial_\zeta X_t^\star|^2] \leq C\mathbb{E}[|\zeta|^2]$
  - gives a way to control derivative in $\mu \rightsquigarrow$ avoid any blow-up

Checking the monotonicity condition

- Lasry-Lions monotonicity condition (choose $d = 1$)

$$\int_{\mathbb{R}} (h(x,\mu') - h(x,\mu)) d(\mu' - \mu)(x) \geq 0$$

  - $X \sim \mu$ and $X' \sim \mu'$

$$\mathbb{E}\Big[h(X', \mathcal{L}(X')) - h(X', \mathcal{L}(X)) - \big(h(X, \mathcal{L}(X')) - h(X, \mathcal{L}(X))\big)\Big] \geq 0$$

- Make a perturbation $X' = X + \varepsilon Y$
  - first step

$$\mathbb{E}\hat{\mathbb{E}}\Big[\partial_\mu h(X', \mathcal{L}(X))(\hat{X})\hat{Y} - \partial_\mu h(X, \mathcal{L}(X))(\hat{X})\hat{Y}\Big] + o(\varepsilon) \geq 0$$

  - need copies $\hat{X}$ and $\hat{Y}$ on another space
  - second step

$$\mathbb{E}\hat{\mathbb{E}}\Big[\partial_x \partial_\mu h(X, \mathcal{L}(X))(\hat{X})\hat{Y} Y\Big] \geq 0$$

Notes and complements

- Case with a common noise

  ∘ HJB and FP become stochastic PDEs

  ∘ but $\mathcal{U}$ remains deterministic! decoupling field of a stochastic FBSDE in ∞ dimension

- Master equation with a common noise $\rightsquigarrow$ involves second-order derivatives in the direction of the measure $\rightsquigarrow$ example

  ∘ $b(x,\mu,\alpha) = -x + b(m) + \alpha$, $m = \int x' d\mu(x')$

  ∘ $f(x,\mu,\alpha) = \frac{1}{2}[(x+f(m))^2 + \alpha^2]$, $g(x,\mu) = \frac{1}{2}(x+g(m))^2$

- Stochastic Pontryagin $\rightsquigarrow$ strong solution if $Y_t = X_t + \chi_t$

$$
\begin{aligned}
dm_t &= (b(m_t) - 2m_t - \chi_t)dt + dW_t^0, \\
d\chi_t &= -(f+b)(m_t)dt + \zeta_t dW_t^0, \quad \chi_T = g(m_T)
\end{aligned}
$$

  ∘ $\partial_x \mathcal{U}(t,x,\mu_t) = x + v(t,m_t)$ with m_t mean of μ_t

$$
\partial_t v(t,m) + \frac{1}{2}\partial^2_{mm} v(t,m) + \partial_m v(t,m)(b(m) - 2m - v(t,m)) + (f+b)(m) = 0
$$

Part II. The convergence problem

Part II. The convergence problem

a. General prospect

Revisiting the N-player game

- Controlled dynamics (1d to simplify)

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i$$

  - independent Brownian motion $\boldsymbol{W}^1, \ldots, \boldsymbol{W}^N$, progressively-measurable controls $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^N$
  - mean-field interaction $\rightsquigarrow \bar{\mu}_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}$

- Cost functionals to player i

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_s^i, \bar{\mu}_s^N, \alpha_s^i)ds\Big]$$

  - try to minimize $\rightsquigarrow$ Nash equilibrium?

- Rigorous connection between Nash equilibria with N players and MFG?

Two roads for making the connection

- Prove the convergence of the Nash equilibria as N tends to ∞

∘ difficulty $\rightsquigarrow$ no uniform smoothness on the optimal feedback function $\alpha^{\star,N}$ w.r.t to N

$$\underbrace{\alpha_t^{\star,i,N}}_{\text{optimal control to player } i} = \alpha^{\star,N}(X_t^i; \underbrace{X^1, \dots, X^{i-1}, X^{i+1}, \dots, X^N}_{\text{states of the others}})$$

$\rightsquigarrow$ no compactness on the feedback functions

∘ several attempts $\rightsquigarrow$ weak compactness arguments on the control (notion of relaxed controls) for equilibria over open loop controls

∘ below $\rightsquigarrow$ use the master equation

- Implement feedback function for MFG into finite player game

∘ limit setting $\rightsquigarrow$ optimal control has the form

$$\alpha_t^\star = \alpha^\star(X_t, \underbrace{\mu_t}_{\text{population at equilibrium}})$$

∘ use $\alpha_t^N = \alpha^\star(X_t^i, \mu_t) \rightsquigarrow$ what about Nash?

Part II. The convergence problem

b. Convergence of the equilibria

Reminder

- Recall FBSDE associated with Markov loop

$$dX_t^i = b\Big(X_t^i, \bar{\mu}_t^N, \alpha^\star(X_t^i, \bar{\mu}_t^N, Z_t^{i,i}\sigma^{-1}(X_t^i, \bar{\mu}_t^N))\Big)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i$$

$$dY_t^i = -f\Big(X_t^i, \bar{\mu}_t^N, \alpha^\star(X_t^i, \bar{\mu}_t^N, Z_t^{i,i}\sigma^{-1}(X_t^i, \bar{\mu}_t^N))\Big)dt + \sum_{j=1}^N Z_t^{i,j} dW_t^j$$

with $Y_T^i = g(X_T^i, \mu_T^N)$ as terminal condition

  - $\alpha^\star$ is the minimizing function of the Hamiltonian

$$\alpha^\star(x, \mu, z) = \inf_{\alpha \in A} H(x, \mu, \alpha, z) \quad H(x, \mu, \alpha, z) = b(x, \mu, \alpha) \cdot z + f(x, \mu, z)$$

  - difficulty $Z_t^{i,i} = \underbrace{\partial_{x_i} u^{i,N}}_{\text{derivative of } x_i \text{ if } i\text{th value function}} (t, X_t^1, \dots, X_t^N)$

- Same assumption as for optimal control under non-degenerate σ (with A bounded) in 1st Lecture $\rightsquigarrow$ existence and uniqueness
  - again $\rightsquigarrow$ no uniform control of $\partial_{x_i} u^{i,N}$

N-player game as a perturbation

- Idea is to use the master equation (if smooth solution)
  - recall the meaning of Y and Z in the MFG

$$Y_t = \mathcal{U}(t, X_t, \mathcal{L}(X_t)) \quad Z_t = \partial_x \mathcal{U}(t, X_t, \mathcal{L}(X_t)) \underbrace{\sigma(X_t, \mathcal{L}(X_t))}_{\text{choose } \sigma = \mathrm{Id}}$$

- Perturbed version $\rightsquigarrow$ go back to N-player game equilibrium
  - FBSDE for $\begin{array}{l} \mathcal{Y}_t^i = \mathcal{U}(t, X_t^i, \bar{\mu}_t^N) \\ \mathcal{Z}_t^i = \partial_x \mathcal{U}(t, X_t^i, \bar{\mu}_t^N) \end{array}$?

- Get it by applying Itô's formula to $\mathcal{Y}_t^{\star,i}$ $(d = 1)$

$$\partial_{x_i}[\mathcal{U}(t, x_j, \mu^{N,x})] = \partial_x \mathcal{U}(t, x_i, \mu^{N,x})\delta_i^j + \frac{1}{N}\partial_\mu \mathcal{U}(t, x_j, \mu^{N,x})(x_i)$$

$$\partial_{x_i}^2[\mathcal{U}(t, x, \mu^{N,x})] = \partial_x^2 \mathcal{U}(t, x_i, \mu^{N,x})\delta_i^j + \frac{1}{N}\partial_v \partial_\mu \mathcal{U}(t, x_j, \mu^{N,x})(x_i) + O(N^{-1})\delta_i^j + O(N^{-2})$$

  - $\mu^{N,x} = N^{-1} \sum_{\ell=1}^N \delta_{x_\ell}$

Perturbed FBSDE

- Let $\alpha_t^{\star,i,\infty} = \alpha^\star(X_t^i, \bar\mu_t^N, \mathcal{Z}_t^i)$ artificial control
  $\alpha_t^{\star,i,N} = \alpha^\star(X_t^i, \bar\mu_t^N, Z_t^{i,i})$ true control
- Itô expansion yields

$$
\begin{aligned}
& d\mathcal{Y}_t^i \\
& = \Big[b(X_t^i, \bar\mu_t^N, \alpha_t^{\star,i,N}) - b(X_t^i, \bar\mu_t^N, \alpha_t^{\star,i,\infty}) \Big] \cdot \partial_x \mathcal{U}(t, X_t^i, \bar\mu^N) dt \\
& + \frac{1}{N} \sum_{j=1}^N \Big[b(X_t^j, \bar\mu_t^N, \alpha_t^{\star,j,N}) - b(X_t^j, \bar\mu_t^N, \alpha_t^{\star,j,\infty}) \Big] \cdot \partial_\mu \mathcal{U}(t, X_t^i, \bar\mu_t^N)(X_t^j) dt \\
& - f(X_t^i, \bar\mu_t^N, \alpha_t^{\star,i,\infty}) dt + O(N^{-1}) dt \\
& + \mathcal{Z}_t^{\star,i} dW_t^i + \underbrace{\frac{1}{N} \sum_{j=1}^N \mathcal{Z}_t^{\star,i,j} dW_t^j}_{\text{bracket} \sim N^{-1}}
\end{aligned}
$$

◦ reminiscent of the expansion of $(Y_t^i)_{0 \le t \le T} \rightsquigarrow$ make the difference between both

Stability argument

- Difference between the two FBSDEs

$$
\begin{aligned}
& d(\mathcal{Y}_t^i - Y_t^i) \\
& = \Big[b(X_t^i, \bar{\mu}_t^N, \alpha_t^{\star,i,N}) - b(X_t^i, \bar{\mu}_t^N, \alpha_t^{\star,i,\infty})\Big]\partial_x \mathcal{U}(t, X_t^i, \bar{\mu}^N)dt \\
& + \frac{1}{N}\sum_{j=1}^N \Big[b(X_t^j, \bar{\mu}_t^N, \alpha_t^{\star,j,N}) - b(X_t^j, \bar{\mu}_t^N, \alpha_t^{\star,j,\infty})\Big]\partial_\mu \mathcal{U}(t, X_t^i, \bar{\mu}_t^N)(X_t^j)dt \\
& - \Big[f(X_t^i, \bar{\mu}_t^N, \alpha_t^{\star,i,\infty} - f(X_t^i, \bar{\mu}_t^N, \alpha_t^{\star,i,N})\Big]dt + O(N^{-1})dt \\
& + \big(\mathcal{Z}_t^i - Z_t^{i,i}\big)dW_t^i + \frac{1}{N}\sum_{j=1}^N (\mathcal{Z}_t^{i,j} - Z_t^{i,j})dW_t^j
\end{aligned}
$$

- Lipschitz differences!
  - recall $|\alpha^{\star,i,\infty} - \alpha^{\star,i,N}| \leq C|\mathcal{Z}_t^{\star,i} - Z_t^{\star,i,i}|$
  - if $|\partial_x \mathcal{U}(t,x,\mu)| \leq C$ and $\hat{\mathbb{E}}[|\partial_\mu \mathcal{U}(t,x,\mu)(\hat{X})|^2]^{1/2} \leq C$ for $\hat{X} \sim \mu$
  - use variation of Cauchy-Lipschitz $\rightsquigarrow$ stability!

Conclusion

- Stability yields and symmetry

$$\mathbb{E}[\sup_{0\le t\le T} |\mathcal{Y}_t^i - Y_t^i|^2] + \mathbb{E}\int_0^T |\mathcal{Z}_t^i - Z_t^{i,i}|^2 dt \underset{N\to\infty}{\longrightarrow} 0$$

- Plug into the forward equation

$$\begin{aligned} dX_t^i &= b\Big(X_t^i, \bar{\mu}_t^N, \alpha^\star(X_t^i, \bar{\mu}_t^N, Z_t^{i,i})\Big)dt + dW_t^i \\ &\approx b\Big(X_t^i, \bar{\mu}_t^N, \alpha^\star(X_t^i, \bar{\mu}_t^N, \partial_x \mathcal{U}(t, X_t^i, \bar{\mu}_t^N))\Big)dt + dW_t^i \end{aligned}$$

- Recover the standard MKV setting!

  ◦ require $\partial_x \mathcal{U}$ to be Lipschitz

  ◦ then particles get independent in the limit with following dynamics

$$dX_t = b\Big(X_t, \mathcal{L}(X_t), \alpha^\star(X_t, \mathcal{L}(X_t), \partial_x \mathcal{U}(t, X_t, \mathcal{L}(X_t)))\Big)dt + dW_t$$

  ◦ recover the dynamics of the MFG equilbrium

Part II. The convergence problem

c. Construction of quasi-equilibria

Implementing the limit optimal feedback

- Shape of the optimal feed back in the limit MFG problem

$$\alpha^\star(x, \mu_t^\star, \partial_x \mathcal{U}(t, x, \mu_t^\star))$$

  - $\alpha^\star$ minimizes the Hamiltonian
  - $\mu_t^\star$ is the law of the population at time when in equilibrium
  - $\partial_x \mathcal{U}(t, x, \mu_t^\star)$ matches $\partial_x U^{\boldsymbol{\mu}^\star}(t, x)$ where $U^{\boldsymbol{\mu}^\star}$ is the value function in environment $\boldsymbol{\mu}^\star$
  - under same assumptions as in Lecture 1 $\rightsquigarrow$ $\partial_x U^{\boldsymbol{\mu}^\star}(t, \cdot)$ is Lipschitz continuous in x

- Go back to the dynamics of the finite player system
  - assume that σ is identity (for simplicity)

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, \alpha^\star(t, X_t^i, \mu_t^\star, \partial_x U^{\boldsymbol{\mu}^\star}(t, X_t^i)))dt + dW_t^i$$

  - compute first $\partial_x U^{\boldsymbol{\mu}^\star}$ and $\boldsymbol{\mu}^\star$ numerically and plug them!

Propagation of chaos

- N-player system

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, \alpha^\star(t, X_t^i, \partial_x U^{\boldsymbol{\mu}^\star}(t, X_t^i), \mu_t^\star))dt + dW_t^i$$

  - fits the framework of MKV SDE

- As N tends to ∞
  - for k fixed

$$(X_t^1, \ldots, X_t^k)_{0 \le t \le T} \underset{\mathcal{L}}{\longrightarrow} \mathcal{L}((X_t^\star)_{0 \le t \le T})^{\otimes k}$$

  - where $(X_t^\star)_{0 \le t \le T}$ optimal dynamics in the limit

$$dX_t^\star = b(X_t^\star, \mu_t^\star, \alpha^\star(X_t^\star, \mu_t, \partial_x U^{\boldsymbol{\mu}^\star}(t, X_t^\star)))dt + dW_t$$

  - moreover, for each $t \in [0, T]$, $\bar{\mu}_t^N \underset{\mathcal{L}}{\longrightarrow} \mu_t^\star$

Quasi-Nash property

- Notations

  ∘ $\alpha_t^i = \alpha^\star(t, X_t^i, \partial_x U^{\boldsymbol{\mu}^\star}(t, X_t^i), \mu_t^\star)$ controls taken from the limit feedback function

  ∘ call $J^\star$ the optimal cost in the MFG setting

  ∘ under assumptions used throughout the lectures

$$J(\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^N) \underset{N\to\infty}{\longrightarrow} J^\star$$

- Check that $(\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^N)$ forms a quasi-Nash equilibrium

  ∘ change $\boldsymbol{\alpha}^1$ into $\boldsymbol{\beta}^1$ and freeze the others (Nash over open loop)

  ∘ $\exists N_0$ s.t. for $N \geq N_0$, $A > 0$, $\exists C$ s.t.

$$\mathbb{E}\int_0^T |\beta_t^1|^2 dt \geq C \Rightarrow J^1(\beta^1, \alpha^2, \dots, \alpha^N) \geq J^\star + A$$

  ∘ for $A > 0$, $\exists (\varepsilon_N)_{N\geq 1} \downarrow 0$ s.t. 0, such that

$$\mathbb{E}\int_0^T |\beta_t^1|^2 dt \leq A \Rightarrow \begin{array}{l} J^1(\beta^1, \alpha^2, \dots, \alpha^N) \geq J^\star - \varepsilon_N \\ J^i(\beta^1, \alpha^2, \dots, \alpha^N) \geq J^\star - \varepsilon_N, \quad 2 \leq i \leq N \end{array}$$