On a characterization of an invariant Gaussian measure for linear semigroups

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Preliminaries

\[ X_0 \sim \mu \Rightarrow X_t = T_t(X_0) \sim \mu \circ T_t^{-1}. \]

\( \mu \) is \(( T_t )\)-invariant if:

\[ \mu(T_t^{-1}(E)) = \mu(E), \ \forall t > 0, \ \forall E \in \mathcal{B}(X) \]

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and its covariance operator \( R_\mu : X^* \to X \)

\[ \langle y^*, R_\mu x^* \rangle = \int_X \langle x^*, z \rangle \langle y^*, z \rangle d\mu(z), \quad \text{for every } x^*, \ y^* \text{ in } X^* \]
\[ X_0 \sim \mu \Rightarrow X_t = T_t(X_0) \sim \mu \circ T_t^{-1}. \]

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Theorem

(EL M 2015) A centred Gaussian measure $\mu$, on $X$ with covariance operator $R_\mu$, is invariant if and only if $AR_\mu + R_\mu A^* = 0$ holds on $D(A^*)$.

Proof.

Assume $\mu$ invariant, $R_\mu = T(t)R_\mu T^*(t)$, for all $t \geq 0$, and let $x^* \in D(A^*)$.

$$\lim_{t \to 0} \frac{1}{t} \langle (T^*(t) - I)x^*, x \rangle = \langle A^*x^*, x \rangle, \text{ for all } x \in X.$$ 

By uniform boundednes principle $(\frac{1}{t}(T^*(t) - I)x^*)_{0 < t < 1}$ is bounded in $X^*$.

$$(\forall (t_n) \to 0) \quad \frac{1}{t_n} (T(t_n)R_\mu x^* - R_\mu x^*) = \frac{1}{t_n} (T(t_n)R_\mu x^* - T(t_n)R_\mu T^*(t_n)x^*),$$

$$= \frac{1}{t_n} T(t_n)R_\mu (x^* - T^*(t_n)x^*).$$
Proof

\( R_\mu : X^* \to X \) compact:
\[ \exists (t_{n_k}) \left( R_\mu \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*) \right)_k \to w. \]

Claim: \( w = -R_\mu A^*x^* \),

For \( y^* \) arbitrarily in \( X^* \). Write
\[
\langle y^*, w \rangle = \lim_{k \to \infty} \langle y^*, R_\mu \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*) \rangle, \\
= \lim_{k \to \infty} \langle \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*), R_\mu y^* \rangle, \\
= \langle -A^*x^*, R_\mu y^* \rangle, \\
= \langle y^*, -R_\mu A^*x^* \rangle.
\]

We deduce that,
\[ \forall (t_n) \to 0, \exists (t_{n_k}) \lim_k \frac{1}{t_{n_k}}(T(t_{n_k})R_\mu x^* - R_\mu x^*) = -R_\mu A^*x^* \]

Finally, \( R_\mu x^* \in D(A) \) and \( AR_\mu x^* = -R_\mu A^*x^* \).
The Converse’s Proof

Let $t > 0$, $h \neq 0$ and $x^* \in D(A^*)$. We have

$$\frac{1}{h}(T_{t+h}R_\mu T_{t+h}^*x^* - T_tR_\mu T_t^*x^*) = T_{t+h}R_\mu \frac{1}{h}[T_{t+h}^* - T_t^*]x^*, \quad + \frac{1}{h}[T_{t+h} - T_t]R_\mu T_t^*x^*.$$ 

The first term converges to $T_tR_\mu A^* T_t^*x^*$ (Compactness again!). The second term converges to $T_tAR_\mu T_t^*x^*$. We obtain

$$\lim_{h \to 0} \frac{1}{h}(T(t+h)R_\mu T^*(t+h)x^* - T(t)R_\mu T^*(t)x^*) = T(t)[R_\mu A^* + AR_\mu] T^*x^* = 0.$$ 

Hence for all $t > 0$,

$$T(t)R_\mu T^*(t) = R_\mu,$$ on, $D(A^*)$. 

To conclude:
For all \( y^* \in X^* \),
\[
\langle \cdot, R_\mu y^* \rangle = \langle \cdot, T(t)R_\mu T^*(t)y^* \rangle,
\]
on the weak*-dense subspace \( D(A^*) \).
The two mapping are weak*-continuous
Thus,
\[
R_\mu y^* = T(t)R_\mu T^*(t)y^*
\]
holds for all \( y^* \in X^* \).
Mixing in $L^p(\Omega, d\sigma)$ (joint work with K.Latrach(DIE 2013))

$(\Omega, \mathcal{B}, \sigma)$ $\sigma$-finite measure space, and let $X := L^p(\Omega, d\sigma)$, $1 \leq p < +\infty$. Assume that $X$ is separable, and $A$ is the generator of a $C_0$-semigroup $T(\cdot)$ on $X$. Discrete case(F. Bayart and É. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Maths vol 179, (2009).)

**Theorem**

Assume $\sigma_p(A) \cap i\mathbb{R} \subset i(\omega_1, \omega_2)$ for some $\omega_1$ and $\omega_2$, and there is a measurable function $u : (\omega_1, \omega_2) \mapsto X$ satisfying the following conditions:

(i) $u_s := u(s) \in \ker(is - A)$ for a.e. $s \in (\omega_1, \omega_2)$,

(ii) $\left( \int_{\omega_1}^{\omega_2} |u_s(\cdot)|^2 \, ds \right)^{1/2} = v(\cdot) \in L^p(\Omega)$,

(iii) $\text{span}\{u_s, \, s \in (\omega_1, \omega_2) \setminus N\}$ is dense in $X$ for every subset $N$ with zero Lebesgue measure.

Then there exists an invariant Gaussian measure $\nu$, such that $\text{supp}(\nu) = X$ with respect to which $T(\cdot)$ is strong mixing.
Mixing Translation in $L^p(I, \rho(x)dx)$

\[ T(t)f(x) = f(x + t), \ x \in I, \ t \geq 0 \]

Proposition

If $\int_I \rho(x)dx < \infty$, then there exists an invariant Gaussian measure with full support with respect to which $T(\cdot)$ is strong mixing.
abnormal cell division model

\[
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= -\frac{\partial (xu(t,x))}{\partial x} + \gamma(x)u(t,x) - \beta(x)u(t,x) + 4\beta(2x)u(t,2x)\chi_{(0,\frac{1}{2})}(x), \\
\ u(0, \cdot) &= \phi \in L^1(0,1).
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial v(t,y)}{\partial t} &= e^y \frac{\partial (e^{-y}v(t,y))}{\partial y} + \gamma v(t,y) - \beta v(t,y) + 4\beta v(t,y - \ln 2)\chi_{(\ln 2,\infty)}(y), \\
\ v(0, \cdot) &= \psi \in L^1((0,\infty), e^{-y}dy),
\end{aligned}
\]

Under $\gamma - 3\beta > 0$, $0 \leq \beta \leq \frac{1}{2}$, The solution is chaotic + existence of a strongly mixing non degenerate Gaussian measure.
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\end{cases}
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Thank you for your attention