Rough paths and 1d SDEs driven by a distributional drift

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(Work in collaboration with R. Diel)

Background

- 1d SDE with a rough drift

$$dX_t = dB_t + \partial_x Y_t(X_t)\, dt$$

- $(Y_t(x))_{t\geq 0, x\in\mathbb{R}} \rightsquigarrow$ environment
  - α-Hölder continuous in space: $\alpha < 1 \Rightarrow$ not differentiable
  - time dependent
- Challenges
  - notion of solutions (existence and uniqueness)
  - infinitesimal dynamics (differential calculus)
- Time independent setting
  - Flandoli-Russo-Wolf (2003-2004), Bass-Chen (2001)

Motivation

- KPZ Equation $t \geq 0$, $x \in \mathbb{R}$

$$\partial_t u_t(x) = \frac{1}{2}\partial^2_{xx} u_t(x) + \frac{1}{2}|\partial_x u_t(x)|^2 + \dot{W}(t,x)$$

- Growth of a random surface
- Originally introduced by Kardar-Parisi-Zhang (86)
  - Ill-posed because of the non-linearity
  - notion of solutions? Bertini-Giacomin (96), Hairer (2013)
  - long-time behavior? Amir-Corwin-Quastel (2011)
- HJB equation in random environment
  - dynamics of the optimal path?

Polymer measure

- 1d Brownian path in random potential $\dot{W}$

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z_T^{-1} \exp\left(\int_0^T \dot{W}(t, B_t)\, \mathrm{d}t \right)$$

- Dynamics of $(B_t)_{0 \le t \le T}$ under $\mathbb{Q}$?
  - If solution u to KPZ smooth enough... $(u(0, \cdot) \equiv 0)$

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^T \left[\partial_x u_{T-t}(B_t)\, \mathrm{d}B_t - \tfrac{1}{2} |\partial_x u_{T-t}(B_t)|^2\, \mathrm{d}t \right] \right)$$

  - Girsanov theorem $\Rightarrow$ under $\mathbb{Q}$

$$\mathrm{d}B_t = \partial_x u_{T-t}(B_t)\, \mathrm{d}t + \mathrm{d}B_t^{\mathbb{Q}}$$

- But u not smooth! ($\alpha < 1/2$-Hölder in space)
  - expect same equation (don't care about renormalization)

Related PDE

- Recall the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t + \partial_x Y_t(X_t)\,\mathrm{d}t$$

∘ $Y_t(\cdot)$ α-Hölder $\rightsquigarrow$ $\alpha > 1/3$ but choose close to $1/2$ to simplify

- Weak solvability ⇔ solvability of related PDEs?

$$\partial_t u_t(x) + \frac{1}{2}\partial_x^2 u_t(x) + \partial_x Y_t(x)\partial_x u_t(x) + f_t(x) = 0$$

∘ Solvability on the same model as Hairer's method for KPZ

∘ Expansion of solutions with respect to heat kernels $\rightsquigarrow$ rough integrals (L. Zambotti's talk)

∘ Extension to more general operators (with smooth coefficients)

Expansion of the solution

- Central problem, $0 \le t \le T$, $x \in \mathbb{R}$,

$$\partial_t u_t(x) + \frac{1}{2}\partial^2_{xx} u_t(x) + \partial_x Y_t(x) \partial_x u_t(x) = 0$$

  - with $u(T, x) = x$ (ideal case)
  - perturbation of the identity by dynamics of X

- Mild representation of $v_t(x) = \partial_x u_t(x)$

$$v_t(x) = 1 + \int_t^T \int_{\mathbb{R}} \partial_x p_{s-t}(x-y) v_s(y) \partial_y Y_s(y) \, dy \, ds$$

  - First order expansion $\rightsquigarrow$ replace $v_s(y)$ by 1

$$v_t(x) = 1 + \underbrace{\int_t^T \int_{\mathbb{R}} \partial_x p_{s-t}(x-y) \partial_y Y_s(y) \, dy \, ds}_{Z_t(x)} + \ldots$$

The co-environment

- $Z_t(x) \rightsquigarrow$ central role

$$Z_t(x) = \int_t^T \int_{\mathbb{R}} \partial_x p_{s-t}(x-y) \partial_y Y_s(y) \, \mathrm{d}y \, \mathrm{d}s$$

  - IBP $\rightsquigarrow$ solves the PDE, $0 \leq t \leq T$, $x \in \mathbb{R}$

$$\partial_t Z_t(x) + \frac{1}{2} \partial_x^2 Z_t(x) = \partial_x^2 Y_t(x)$$

- $Y_s(\cdot)$ is α Hölder continuous + IBP

$$Z_t(x) = \int_t^T \underbrace{\int_{\mathbb{R}} \partial_x^2 p_{s-t}(x-y) \left[Y_s(y) - Y_s(x) \right] \mathrm{d}y}_{\sim (s-t)^{-1+\alpha/2}} \mathrm{d}s$$

  - well-defined! $\alpha - \epsilon$ Hölder continuous

Higher order expansion

- Recall

$$v_t(x) = 1 + \int_t^T \int_{\mathbb{R}} \partial_x p_{s-t}(x-y) v_s(y) \partial_y Y_s(y) \, dy \, ds$$

  - $v_s(y) \approx 1 + Z_s(y)$

- Iteration of the expansion

$$v_t(x) = 1 + Z_t(x) + \int_t^T \underbrace{\underbrace{\int_{\mathbb{R}} \partial_x p_{s-t}(x-y) Z_s(y) \partial_y Y_s(y) \, dy}_{\int_{\mathbb{R}} \partial_x p_{s-t}(x-y) Z_s(y) \, d Y_s(y)}}_{\text{No longer way to make an IBP!}} ds + \dots$$

Higher order expansion

- Recall

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Cross-integral

- Need $\int^x Z_s(y)\,\mathrm{d}Y_s(y)$ to make sense for any s
  - as a Hölder continuous function
- Example 1: if $\alpha > 1/2 \Rightarrow$ well-defined as Young integral
  - integral is limit of Riemann sums

$$\sum Z_s(y_i)(Y_s(y_{i+1}) - Y_s(y_i))$$

- Example 2: if Y is constant in time
  - the cross integral match

$$\underbrace{\int^x \left(\int p_{T-t}(y-z)Y(z)\,\mathrm{d}z\right)\mathrm{d}Y(y)}_{\text{IBP applies}} - \int^x \underbrace{Y(y)\,\mathrm{d}Y(y)}_{\frac{1}{2}\mathrm{d}[Y(y)]^2}$$

- Example 3: statistical decorrelation

Rough paths

- Assume: $\int Z_t(y)\,\mathrm{d}Y_t(y)$ exists
  - defined as limit of mollified $\int Z_t^n(y)\,\mathrm{d}Y_t^n(y)$
- Objective: prove that it is sufficient to solve

$$v_t(x) = 1 + \int_t^T \underbrace{\int_{\mathbb{R}} \partial_x p_{s-t}(x-y)\, v_s(y) \partial_y Y_s(y)\,\mathrm{d}y}_{\int_{\mathbb{R}} \partial_x^2 p_{s-t}(x-y) \int_x^y v_s(z)\,\mathrm{d}Y_s(z)\,\mathrm{d}y}\,\mathrm{d}s$$

- Scheme: prove that $v_s(\cdot)$ is controlled by $Z_s(\cdot)$ (Gubinelli 04)

$$v_s(z) = v_s(x) + v_s'(x)\big(Z_s(z) - Z_s(x)\big) + O(|x-y|^\beta)$$

  - $\beta \gg 0 \;\Rightarrow\; \int_x^y v_s(z)\,\mathrm{d}Y_s(z)$ limit of Riemann sums

$$\sum \left[v_s(y_i)\big(Y_s(y_{i+1}) - Y_s(y_i)\big) + v_s'(y_i) \int_{y_i}^{y_{i+1}} \big(Z_s(z) - Z_s(y_i)\big)\,\mathrm{d}Y_s(z) \right]$$

Rough paths

- Assume: $\int Z_t(y)\,\mathrm{d}Y_t(y)$ exists
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- Objective: prove that it is sufficient to solve

$$v_t(x) = 1 + \int_t^T \underbrace{\int_{\mathbb{R}} \partial_x p_{s-t}(x-y) v_s(y) \partial_y Y_s(y)\,\mathrm{d}y}_{\int_{\mathbb{R}} \partial_x^2 p_{s-t}(x-y) \int_x^y v_s(z)\,\mathrm{d}Y_s(z)\,\mathrm{d}y} \,\mathrm{d}s$$

- Scheme: prove that $v_s(\cdot)$ is controlled by $Z_s(\cdot)$ (Gubinelli 04)

$$v_s(z) = v_s(x) + v_s'(x)\big(Z_s(z) - Z_s(x)\big) + O(|x-y|^\beta)$$

  - $\beta \sim 2\alpha \Rightarrow \int_x^y v_s(z)\,\mathrm{d}Y_s(z)$ limit of Riemann sums

$$\sum \Big[v_s(y_i)\big(Y_s(y_{i+1}) - Y_s(y_i)\big) + v_s'(y_i) \int_{y_i}^{y_{i+1}} \big(Z_s(z) - Z_s(y_i)\big)\,\mathrm{d}Y_s(z) \Big]$$

Solvability result

- Computation of v'_t

$$v_t(x') - v_t(x) = \underbrace{\int_t^T \int_{\mathbb{R}} \left(\partial_x^2 p_{s-t}(x'-y) - \partial_x^2 p_{s-t}(x-y)\right) \int_x^y v_s(z)\,\mathrm{d}Y_s(z)\,\mathrm{d}y}_{\approx v_t(x)(Z_t(x') - Z_t(x))}$$

- If $\alpha \leq 1/2 \Rightarrow$ need existence and stability: $\exists \beta > 1-\alpha$ s.t.

$$\left| \int_x^y (Z_t(u) - Z_t(x))\,\mathrm{d}Y_t(u) \right| = O(|y-x|^{\beta})$$

  - In practice $\beta \sim 2\alpha$ so that $\alpha > 1/3$
- v exists and is $((\alpha-\varepsilon)/2, \alpha-\varepsilon)$ Holder in (t,x)

Connection with SDE

- Discuss the solvability of the SDE

$$\mathrm{d}X_t = \partial_x Y_t(X_t)\,\mathrm{d}t + \mathrm{d}B_t, \quad t \in [0, T]$$

  - if needed, localize Y (cut-off...)

- Zvonkin method

$$\partial_t u_t(x) + \frac{1}{2}\partial_x^2 u_t(x) + \partial_x Y_t(x)\partial_x u_t(x) = 0$$

  - If Itô's rule applies ($v = \partial_x u$)

$$\mathrm{d}u_t(X_t) = v_t(X_t)\,\mathrm{d}B_t = v_t(\Phi_t(u_t(X_t)))\,\mathrm{d}B_t$$

  - $\Phi_t(\cdot) = u_t^{\circ -1}(\cdot)$, well-defined since $v \sim 1 > 0$ for $T \ll 1$
  - Solve $\mathrm{d}M_t = v_t(\Phi_t(M_t))\,\mathrm{d}B_t$ and set $X_t = \Phi_t(M_t)$

- Strong solvability if $\alpha > 1/2$ and weak if $\alpha \leq 1/2$

Structure of the increments

- Expansion of $X_{t+h} - X_t$ for h small? (α close to but $< 1/2$)
  - Expand $u_{t+h}(X_{t+h}) - u_t(X_t)$

$$= u_{t+h}(X_{t+h}) - u_t(X_{t+h}) + u_t(X_{t+h}) - u_t(X_t)$$
$$= O\big(h^{3/4-\varepsilon}\big) + \partial_x u_t(X_t)(X_{t+h} - X_t) + \underbrace{|X_{t+h} - X_t|^{3/2-\varepsilon})}$$

  - Martingale $M_t = u_t(X_t)$

$$\underbrace{M_{t+h} - M_t} = \int_t^{t+h} \partial_x u_s(X_s) dB_s$$

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Structure of the increments

- Expansion of $X_{t+h} - X_t$ for h small? (α close to but $< 1/2$)
    - Expand $u_{t+h}(X_{t+h}) - u_t(X_t)$

$$= u_{t+h}(X_{t+h}) - u_t(X_{t+h}) + u_t(X_{t+h}) - u_t(X_t)$$

$$= O\big(h^{3/4-\varepsilon}\big) + \partial_x u_t(X_t)\big(X_{t+h} - X_t\big) + \big(\underbrace{|X_{t+h} - X_t|^{3/2-\varepsilon}}_{h^{3/4-\varepsilon}}\big)$$

    - Martingale $M_t = u_t(X_t)$

$$\underbrace{M_{t+h} - M_t}_{h^{1/2-\varepsilon}} = \int_t^{t+h} \partial_x u_s(X_s) dB_s$$

$$= \partial_x u_t(X_t)\big(B_{t+h} - B_t\big) + O(h^{3/4-\varepsilon})$$

- Dirichlet dynamics $X_{t+h} - X_t = B_{t+h} - B_t + O\big(h^{3/4-\varepsilon}\big)$

Analysis of the remainder

- Conditioning of the remainder:

$$X_{t+h} - X_t = B_{t+h} - B_t + \mathbb{E}\big[O\big(h^{3/4-\varepsilon}\big)|\mathcal{F}_t\big] + \underbrace{O\big(h^{3/4-\varepsilon}\big) - \mathbb{E}\big[O\big(h^{3/4-\varepsilon}\big)|\mathcal{F}_t\big]}_{\text{martingale increment}}$$

  - Sum of martingale increments of size $h^{3/4-\varepsilon}$

$$\underbrace{h^{-1}}_{\sharp \text{ increments}} \times \underbrace{O(h^{3/2-\varepsilon})}_{\text{local variance}} = \underbrace{O(h^{1/2-\varepsilon})}_{\text{global variance}}$$

  - No role when integrated w.r.t. prog. meas. processes!

- Almost-explicit function b (in terms of Y and Z):

$$\mathbb{E}\big[O\big(h^{3/4-\varepsilon}\big)|\mathcal{F}_t\big] = b(t, X_t, h) \sim h^{3/4-\varepsilon}$$

Notion of drift

- Locally $\mathrm{d}X_t = \mathrm{d}B_t + b(t, X_t, \mathrm{d}t) = \mathrm{d}B_t + O(\mathrm{d}t^{3/4-\varepsilon})$
- Integration w.r.t. processes that are progressively meas. and $1/4+\varepsilon$ Hölder $\rightsquigarrow$ Young's integral

$$\int_0^t f(X_s) b(s, X_s, \mathrm{d}s) \approx \sum_i f(X_{s_i}) b(s_i, X_{s_i}, s_{i+1} - s_i)$$

- Shape of the drift: with $Z_{t+h}^{(t+h)} \equiv 0$

$$b(t,x,h) = \underbrace{\int_t^{t+h} \int_{\mathbb{R}} \partial_x p_{r-t}(x-y) \big(Y_r(y) - Y_r(x) \big) \, \mathrm{d}y \, \mathrm{d}r}_{\sim h^{(1+\alpha)/2} = h^{3/4-\varepsilon} (\text{mollification of } \partial_x Y)}$$

$$+ \underbrace{\int_t^{t+h} \int_{\mathbb{R}} \partial_x p_{r-t}(x-y) \int_x^y Z_r^{(t+h)}(z) \, \mathrm{d}Y_r(z) \, \mathrm{d}y \, \mathrm{d}r}_{\sim h^{1/2+\alpha-\varepsilon} = h^{1-2\varepsilon} (\text{corrector term})}$$

KPZ

- Hairer's result (on torus)

$$\underbrace{Y_t(x)}_{\text{Solution to KPZ}} = \underbrace{\text{Stationary SHE}}_{\alpha < 1/2 \text{ Holder}} + \underbrace{\text{Random function}}_{1-\varepsilon \text{ Holder}}$$

- Dynamics of the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t + \underbrace{b(t, X_t, \mathrm{d}t)}_{O(\mathrm{d}t^{3/4-\varepsilon})}$$

  - only need to define the co-environment w.r.t. to SHE
  - other terms do not count in the correction of the drift
  - cross-integral makes sense by Gaussian arguments