ENTIRE SPACELIKE RADIAL GRAPHS IN THE MINKOWSKI SPACE, ASYMPTOTIC TO THE LIGHT-CONE, WITH PRESCRIBED SCALAR CURVATURE

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Abstract. We prove the existence and uniqueness in \( \mathbb{R}^{n,1} \) of entire spacelike hypersurfaces contained in the future of the origin \( O \) and asymptotic to the light-cone, with scalar curvature prescribed at their generic point \( M \) as a negative function of the unit vector pointing in the direction of \( \vec{OM} \), divided by the square of the norm of \( \vec{OM} \) (a dilation invariant problem). The solutions are sought as graphs over the future unit-hyperboloid emanating from \( O \) (the hyperbolic space); radial upper and lower solutions are constructed which, relying on a previous result in the Cartesian setting, imply their existence.

Résume. On prouve l’existence et l’unicité dans \( \mathbb{R}^{n,1} \) d’hypersurfaces entières de genre espace contenues dans le futur de l’origine \( O \) et asymptotes au cône de lumière, dont la courbure scalaire est prescrite au point générique \( M \) comme fonction négative du vecteur unité pointant en direction de \( \vec{OM} \), divisé par le carré de la norme du vecteur \( \vec{OM} \) (un problème invariant par homothétie). Les solutions sont cherchées comme graphes sur l’hyperboïde-unité futur émanant de \( O \) (l’espace hyperbolique); des solutions supérieure et inférieure radiales sont construites qui, d’après un résultat antérieur en cartésien, impliquent l’existence de telles solutions.

Introduction

The Minkowski space \( \mathbb{R}^{n,1} \) is the affine Lorentzian manifold \( \mathbb{R}^n \times \mathbb{R} \) endowed with the metric
\[
ds^2 = dX'^2 - dX_{n+1}^2, \quad \text{where } dX'^2 = dx_1^2 + \ldots + dx_n^2,
\]
setting \( X = (X', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \), and time-oriented by \( dX_{n+1} > 0 \). Distinguishing the origin \( O \) of \( \mathbb{R}^{n,1} \), let
\[
\mathbb{H} = \{ x \in \mathbb{R}^{n,1}, |\vec{Ox}|^2 = |x'|^2 - x_{n+1}^2 = -1, \ x_{n+1} > 0 \},
\]
be the future unit-hyperboloid, model of the hyperbolic space in \( \mathbb{R}^{n,1} \). If \( \varphi \) is a real function defined on \( \mathbb{H} \), we define the radial graph of \( \varphi \) by
\[
\text{graph}_\mathbb{H} \varphi = \{ X \in \mathbb{R}^{n,1}, \vec{OX} = e^{\varphi(x)} \vec{Ox}, \ x \in \mathbb{H} \}.
\]
This is a hypersurface contained in the future open solid cone
\[
C^+ = \{ X \in \mathbb{R}^{n,1}, \ x_{n+1} > |X'| \}.
\]
We say that \( \varphi \) is spacelike if its graph is a spacelike hypersurface, which means that the metric induced on it is Riemannian. Conversely, a spacelike and connected hypersurface in \( C^+ \) is the radial graph of a uniquely determined function \( \varphi : \mathbb{H} \to \mathbb{R} \). Of course, such a graph may also be considered as the Cartesian graph of some function \( u : \mathbb{R}^n \to \mathbb{R} \)
\[
\text{graph}_\mathbb{R}^n u = \{ (x', u(x')) \}, \ x' \in \mathbb{R}^n.
\]

The first author was supported by the project DGAPA-UNAM IN 101507; the second author is supported by the CNRS.

and the correspondence between the two representations is bijective passing from the Cartesian chart \( X = (X', X_{n+1}) \) restricted to \( C^+ \), to the polar chart \((x, \rho) \in \mathbb{H} \times (0, \infty) \) of \( C^+ \) defined by:

\[
\rho = \sqrt{-|\overrightarrow{OX}|^2}, \quad \overrightarrow{x} = \frac{1}{\rho} \overrightarrow{OX}.
\]

Recall that the principal curvatures \((\kappa_1, \ldots, \kappa_n)\) at a point of a spacelike hypersurface are the eigenvalues of its curvature endomorphism \( dN \), where \( N \) is the future oriented unit normal field, and the \( m \)th mean curvature (denoted by \( H_m \)) is the \( m \)th elementary symmetric function of its principal curvatures: \( H_m = \sigma_m(\kappa_1, \ldots, \kappa_n) \).

For each real \( \lambda > 0 \), the cone \( C^+ \) is globally invariant under the ambient dilation \( X \mapsto \lambda X \) of \( \mathbb{R}^{n+1} \) and the above \( m \)-th mean curvature is \((-m)\)-homogeneous; specifically, it transforms like \( H_m(\lambda X) = \lambda^{-m}H_m(X) \). It is thus natural to pose, as in [6, Theorem 1], the following inverse problem for \( H_m \): given a positive function \( h > 0 \) on \( \mathbb{H} \) tending to 1 at infinity, find a spacelike hypersurface \( \Sigma \) in \( C^+ \), asymptotic to \( \partial C^+ \) at infinity, such that, for each point \( X \in \Sigma \), the \( m \)-th mean curvature of \( \Sigma \) at \( X \) is given by:

\[
(1) \quad \overline{H}_m := \frac{1}{(m)_m} H_m(X) = \frac{1}{(\sqrt{-|\overrightarrow{OX}|^2})^m} \left[ h(x) \right]^m, \text{ with } \overrightarrow{x} = \frac{\overrightarrow{OX}}{\sqrt{-|\overrightarrow{OX}|^2}}.
\]

By construction, this problem is dilation invariant; moreover, as explained below, the positivity of \( h \) makes it elliptic. Actually, introducing the positivity cone [9] of \( \sigma_m \):

\[
\Gamma_m = \{ \kappa \in \mathbb{R}^n, \forall i = 1, \ldots, m, \ \sigma_i(\kappa) > 0 \},
\]

and recalling McLaurin’s inequalities (satisfied on \( \Gamma_m \)):

\[
0 < (\overline{H}_m)_{\frac{1}{m}} \leq (\overline{H}_{m-1})_{\frac{1}{m-1}} \leq \cdots \leq \overline{H}_2 \leq \overline{H}_1,
\]

we note that, if a hypersurface \( \Sigma = \text{graph}_{\mathbb{H}} u \) solves (1) with the asymptotic condition, then the time-function \( u \) must assume a minimum on \( \Sigma \) and, as readily checked (using e.g. [3, p.245]), the principal curvatures of \( \Sigma \) at such a minimum point of \( u \) must lie in \( \Gamma_m \). Now equation (1) combined with McLaurin’s inequalities forces the principal curvatures of \( \Sigma \) to stay in \( \Gamma_m \) everywhere. Let us call any spacelike hypersurface of \( C^+ \) having this property, \( m \)-admissible; accordingly, a function \( \varphi : \mathbb{H} \to \mathbb{R} \) (resp. \( \varphi : \mathbb{H}^* \to \mathbb{R} \)) is called \( m \)-admissible, provided \( \text{graph}_{\mathbb{H}} \varphi \) (resp. \( \text{graph}_{\mathbb{H}^*} \varphi \)) is so. The condition of \( m \)-admissibility is local (and open); one may thus speak of a function \( \varphi : \mathbb{H} \to \mathbb{R} \) being \( m \)-admissible at a point (hence nearby) whenever \( \text{graph}_{\mathbb{H}} \varphi \) is so at that point. We will seek the solution hypersurface \( \Sigma \) as the radial graph of some \( m \)-admissible function \( \varphi : \mathbb{H} \to \mathbb{R} \) vanishing at infinity (to comply with the asymptotic condition). Equation (1) then reads

\[
(2) \quad F_m(\varphi) = h,
\]

with the radial operator \( F_m \) defined by:

\[
F_m(\varphi) = e^\varphi \left[ \overline{H}_m(X) \right]_{\frac{m}{m}}, \quad X \in \text{graph}_{\mathbb{H}} \varphi.
\]

For briefness, we will not compute here explictely the general expression of the operator \( F_m \) (keeping it for a further study) – its restriction to radial functions will suffice (see section 3.3 below). We will rely instead on the well-known corresponding Cartesian expression (see e.g. [2]) combined with a few basic properties of \( F_m \) recorded in the next section (and proved with elementary arguments).

Furthermore, we will essentially restrict to the case \( m = 2 \) (and freely say ‘admissible’, for short, instead of ‘\( 2 \)-admissible’). Since \( H_2 \) is related to the scalar curvature \( S \) by \( S = -2H_2 \), our present study is really about the prescription of the scalar
Entire spacelike radial graphs with prescribed scalar curvature curvature, at a generic point $X$ of a radial graph, as a negative function of $x \in H$ (with $x$ given as in (1)) divided by the square of the norm of $\overrightarrow{OX}$. Aside from the origin $O$ of the ambient space $\mathbb{R}^{n,1}$, we will distinguish a point $o$ in $H$ and set $r = r(x)$ for the hyperbolic distance from $o$ to $x \in H$; accordingly, a function on $H$ will be called radial whenever it factors through a function of $r$ only. Our main result is the following:

**Theorem 1.** For $\alpha \in (0,1)$, let $h : H \to (0, \infty)$ be a function of class $C^{2,\alpha}$ with 
\[
\lim_{r(x) \to +\infty} h(x) = 1.
\]
Assume that the functions $h^-$ and $h^+$ defined on $\mathbb{R}^+$ by
\[
h^-(r) = \sup_{r(x) = r} h(x) \quad \text{and} \quad h^+(r) = \inf_{r(x) = r} h(x)
\]
satisfy
\[
\int_0^{+\infty} (h^- - 1)_+ dr < +\infty, \quad \int_0^{+\infty} (1 - h^+)_+ dr < +\infty,
\]
where $(h^- - 1)_+$ (resp. $(1 - h^+)_+$) means the positive part of $h^- - 1$ (resp. $1 - h^+$). Then the equation
\[
F_2(\varphi) = h
\]
has a unique admissible solution of class $C^{4,\alpha}$ such that \(\lim_{r(x) \to +\infty} \varphi(x) = 0\).

**Remark 1.** From Lemma 4 below, anytime the function $h$ is radial, the integral convergence conditions of Theorem 1 appears necessary for the existence of bounded solutions.

An analogous problem in the Euclidean setting is solved for the Gauss curvature in \cite[Théorème 1]{6}, and in \cite[5]{13} some related problems are studied. In the Lorentzian setting, the prescription of the mean curvature for entire graphs is studied in \cite{1} and that of the Gauss curvature in \cite[8, 4]{11}. In \cite{3}, the scalar curvature is prescribed in Cartesian coordinates $x_{n+1} = u(x_1, \ldots, x_n)$.

The outline of the paper is as follows. In section 1, we prove that there exists at most one solution vanishing at infinity for equation (2) with $m \in \{1, \ldots, n\}$. In section 2, relying on [3], we prove the existence of a solution when $m = 2$, provided upper and lower barriers are known. The latter are constructed, as radial functions, in section 3.

### 1. Uniqueness

We first require a few basic properties of the operator $F_m$. It is a nonlinear second order scalar differential operator defined on $m$-admissible real functions on $H$. The dilation invariance of (1) implies the identity:

\[
F_m(\psi + c) \equiv F_m(\psi),
\]
for every $m$-admissible function $\psi : H \to \mathbb{R}$ and constant $c$; linearizing at $\psi$ yields
\[
dF_m(\psi)(1) \equiv 0.
\]
Furthermore, we have:

**Lemma 1.** For each $m$-admissible function $\psi$, the linear differential operator $dF_m(\psi)$ is elliptic everywhere on $H$, with positive-definite symbol.
Summarizing for later use, the expression of \( dF_m(\psi) \), in the chart \( x' \in \mathbb{R}^n \) of \( \mathbb{H} \), at a fixed \( m \)-admissible function \( \psi \) reads like:

\[
\delta \psi \mapsto dF_m(\psi)(\delta \psi) = \sum_{1 \leq i,j \leq n} B_{ij} \frac{\partial^2}{\partial x'_i \partial x'_j}(\delta \psi) + \sum_{i=1}^n B_i \frac{\partial}{\partial x'_i}(\delta \psi),
\]

with the \( n \times n \) matrix \( (B_{ij}) \) symmetric positive definite (depending on \( \psi \), of course, like the \( B_i \)'s). We now proceed to proving Lemma 1.

**Proof:** We require the Cartesian operator \( v \mapsto G_m(v) := F_m(\psi) \) defined on \( m \)-admissible functions \( v : \mathbb{R}^n \to \mathbb{R} \) by:

\[
dG_m(v)(\delta v) = \sum_{1 \leq i,j \leq n} A_{ij} \frac{\partial^2}{\partial X'_i \partial X'_j}(\delta v) + \text{lower order terms},
\]

with the matrix \( (A_{ij}) \) symmetric positive definite. The \( m \)-admissible function \( \psi \) on \( \mathbb{H} \) such that (6) holds, is related to \( H \) [10, 14, 2]. Its expression thus starts out like

\[
v(X') = \sqrt{1 + |x'|^2} \exp \left[ \psi(x') \right], \quad \text{with } OX' = e^{\psi(x')} \overrightarrow{Ox'}.
\]

Varying \( \psi \) by \( \delta \psi \) thus yields for the corresponding variation \( \delta v \) of \( v \) the following expression:

\[
\delta v(X') = w(X') \delta \psi(x'),
\]

with

\[
w(X') = \left[ v - \sum_{i=1}^n X'_i \frac{\partial v}{\partial X'_i} \right](X').
\]

Since the graph lies in \( C^+ \) and it is spacelike, we have \( v(X') > |X'| \) and (using Schwarz inequality) \( \sum_{i=1}^n X'_i \frac{\partial v}{\partial X'_i} < |X'| \), therefore \( w > 0 \). Moreover, up to lower order terms, we have:

\[
\frac{\partial^2}{\partial X'_i \partial X'_j}(\delta v)(X') = w(X') \sum_{1 \leq i,j \leq n} \frac{\partial^2}{\partial x'_k \partial x'_l}(\delta \psi)(x') \frac{\partial x'_k}{\partial X'_i} \frac{\partial x'_l}{\partial X'_j},
\]

with \( x'_k = \frac{X'_k}{\sqrt{w(X')} - |X'|} \). We thus find in (5):

\[
B_{kl} = w(X') \sum_{1 \leq i,j \leq n} A_{ij} \frac{\partial x'_k}{\partial X'_i} \frac{\partial x'_l}{\partial X'_j},
\]

and the ellipticity of \( \delta \psi \mapsto dF_m(\psi)(\delta \psi) \) follows. \( \Box \)

We need also a more specific (ellipticity) property of the operator \( F_m \), namely:

**Lemma 2.** For each couple \( (\varphi_0, \varphi_1) \) of \( m \)-admissible real functions on \( \mathbb{H} \) and each point \( x_0 \in \mathbb{H} \) where \( \varphi = \varphi_1 - \varphi_0 \) assumes a local extremum, the whole segment \( t \in [0,1] \to \varphi_t = \varphi_0 + t \varphi \) consists of \( m \)-admissible functions at the point \( x_0 \).

**Proof:** The analogue of Lemma 2 is fairly standard in the Cartesian setting, using the expression of the operator \( G_m \) introduced in the proof of Lemma 1 (see [2]) together with the well-known fact: \( \forall \kappa \in \Gamma_m, \forall i \in \{1, \ldots, n\}, \frac{\partial \sigma_m}{\partial x_i}(\kappa) > 0 \). Here, we will simply reduce the proof to that setting (and let the reader complete the argument). Let us first normalize the situation at an extremum point \( x_0 \in \mathbb{H} \) of \( \varphi \). From (4), we may assume \( \varphi(x_0) = 0 \). Moreover, we may assume that \( \varphi \) has a local minimum at \( x_0 \) (if not, exchange \( \varphi_0 \) and \( \varphi_1 \)). Finally, setting \( \text{graph}_\mathbb{H} \varphi_a = \text{graph}_{\mathbb{R}^n} u_a \) for \( a = 0, 1 \), and performing if necessary a suitable Lorentz transform (hyperbolic rotation), we may take \( x_0 = (0,1) \in \mathbb{R}^n \times \mathbb{R} \) thus with \( u_a(0) = 1 \). For
Let \( t \in [0, 1] \) and near \( x_0 \), set \( \Sigma_t = \text{graph}_{\mathbb{R}_+} u_t \) for the hypersurface \( \text{graph}_{\mathbb{H}} \varphi_t \). We must prove that \( \Sigma_t \) is \( m \)-admissible at \( x_0 \). For \( X_t \in \mathbb{R}^{n,1} \) lying in \( \Sigma_t \), we have:

\[
OX_t = e^{t \varphi(x)} OX_0 \quad \text{with} \quad \dot{O}x = \frac{OX_0}{\sqrt{|OX_0|^2}}.
\]

In the Cartesian setting, we thus have (sticking to the \( \mathbb{R}^n \)-valued charts used in the preceding proof):

\[
\varphi_t = e^{t \varphi(x)} u[0,t] \quad \text{on} \quad X_t' = e^{t \varphi(x')} X_0',
\]

where \( \varphi_t \) is a nonzero local extremum (a maximum, say, with no loss of generality) at some \( C \in \Sigma_t \) has a unique admissible solution of class \( \mathcal{C}^2 \). We readily infer [2] that, for each \( t \in [0, 1] \), the principal curvatures \( \kappa_1 \leq \ldots \leq \kappa_n \) of the hypersurface \( \Sigma_t \) at \( x_0 \) (each repeated according to its multiplicity) satisfy:

\[\forall i \in \{1, \ldots, n\}, \kappa_i \geq \kappa_{i+1}.\]

The latter implies that the \( n \)-tuple \( (\kappa_1, \ldots, \kappa_n) \) lies in the cone \( \Gamma_m \), since \((\kappa_{n-1}, \ldots, \kappa_{n-1}) \in \Gamma_m \). \( \square \)

Theorem 2. The operator \( F_m \) is one-to-one on \( m \)-admissible functions of class \( C^2 \) vanishing at infinity.

\textbf{Proof} : Let us argue by contradiction. Let \( \varphi_0, \varphi_1 \) be two \( m \)-admissible \( C^2 \) functions vanishing at infinity and having the same image by \( F_m \). For \( t \in [0,1] \), set \( \varphi_t = \varphi_0 + t \varphi_1 \). Since \( \varphi \) vanishes at infinity, if \( \varphi \equiv 0 \), it assumes a nonzero local extremum (a maximum, say, with no loss of generality) at some point \( x_0 \in \mathbb{H} \). By Lemma 2, the whole segment \( t \in [0,1] \) is \( m \)-admissible in a neighborhood \( \Omega \) of \( x_0 \) where \( \varphi \) thus satisfies the second order linear equation \( L \varphi = 0 \) with \( L1 = 0 \) and the operator \( L \) given by \( L = \int_0^1 dF_m(\varphi_t)dt \). Combining Lemma 1 above with Hopf’s strong Maximum Principle (see [7]), we get \( \varphi \equiv \varphi(x_0) \) throughout \( \Omega \). By connectedness, we infer \( \varphi \equiv \varphi(x_0) \neq 0 \) on the whole of \( \mathbb{H} \), contradicting \( \lim_{r(x) \to +\infty} \varphi = 0 \). So, indeed, we must have \( \varphi \equiv 0 \), in other words \( F_m \) is one-to-one. \( \square \)

2. Existence of a solution reduced to that of upper and lower solutions

Theorem 3. Let \( h : \mathbb{H} \to \mathbb{R} \) be a function of class \( C^{2,\alpha} \), for some \( \alpha \in (0,1) \), such that there exists \( \varphi^- \in C^{4,\alpha}(\mathbb{H}) \) with \( \text{graph}_{\mathbb{H}} \varphi^- \) strictly convex and spacelike, and \( \varphi^+ \in C^2(\mathbb{H}) \) with \( \text{graph}_{\mathbb{H}} \varphi^+ \) spacelike, satisfying

\[
F_2(\varphi^-) \geq h, \quad F_2(\varphi^+) \leq h \quad \text{and} \quad \lim_{r(x) \to +\infty} \varphi^+ = 0.
\]

Then the equation

\[
F_2(\varphi) = h
\]

has a unique admissible solution of class \( C^{4,\alpha} \) such that \( \lim_{r(x) \to +\infty} \varphi(x) = 0 \). Moreover \( \varphi \) satisfies the pinching:

\[
\varphi^- \leq \varphi \leq \varphi^+.
\]
Remark 2. Since $\varphi$ is a bounded function, the hypersurface $M = \text{graph}_H(\varphi)$ is entire. More precisely, denoting by $\varphi_{\text{min}}$ and $\varphi_{\text{max}}$ two constants such that $\varphi_{\text{min}} \leq \varphi \leq \varphi_{\text{max}}$, the function $u : \mathbb{R}^n \to \mathbb{R}$ such that $\text{graph}_{\mathbb{R}^n}(u) = \text{graph}_H(\varphi)$ satisfies $u_{\text{min}} \leq u \leq u_{\text{max}}$ where $u_{\text{min}}$ (resp. $u_{\text{max}}$) is such that $\text{graph}_{\mathbb{R}^n}(u_{\text{min}}) = \text{graph}_H(\varphi_{\text{min}})$ (resp. $\text{graph}_{\mathbb{R}^n}(u_{\text{max}}) = \text{graph}_H(\varphi_{\text{max}})$). Noting that the graphs of $u_{\text{min}}$ and $u_{\text{max}}$ are hyperboloids, we see that the inequality $u \geq u_{\text{min}}$ implies that $M$ is entire, and the inequality $u \leq u_{\text{max}}$ implies that $M$ is asymptotic to the lightcone.

Proof: The asserted uniqueness follows from Theorem 2; so let us focus on the existence part. A straightforward comparison principle, using (5) and Lemma 2, implies $\varphi^- \leq \varphi^+$ on $\mathbb{H}$. Let $u^-, u^+ : \mathbb{R}^n \to \mathbb{R}$ be such that $\text{graph}_{\mathbb{R}^n}(u^\pm) = \text{graph}_H(\varphi^\pm)$. Set $H$ for the function on $\mathbb{R}^{n,1}$ defined by:

$$H(X) = \frac{(\frac{2}{n})}{|X_{n+1}|^2 - |X|^2} \left[ h \left( \frac{X}{\sqrt{|X_{n+1}|^2 - |X|^2}} \right) \right]^2.$$

The spacelike functions $u^-$ and $u^+$ satisfy:

$$H_2[u^-] \geq H(., u^-), \ H_2[u^+] \leq H(., u^+), \ u^- \leq u \leq u^+ \text{ and } \lim_{|x'| \to \infty} [u^+(x') - |x'|] = 0,$$

where $H_2[u^\pm]$ stands for the second mean curvature of the graph of $u^\pm$. Theorem 1.1 in [3] asserts the existence of a function $u : \mathbb{R}^n \to \mathbb{R}$, belonging to $C^{1,\alpha}$, spacelike, such that $H_2[u] = H(., u)$ in $\mathbb{R}^n$, $\lim_{|x'| \to \infty} u(x') - |x'| = 0$, and $u^- \leq u \leq u^+$. The function $\varphi : \mathbb{H} \to \mathbb{R}$ such that $\text{graph}_H(\varphi) = \text{graph}_{\mathbb{R}^n}(u)$ is a solution of our original problem. \hfill $\square$

3. Construction of radial upper and lower solutions

In the sequel of the paper, we first solve the Dirichlet problem on a bounded set in $\mathbb{H}$ (section 3.1) then proceed to proving the existence and uniqueness of an entire solution in the radial case and study its properties (sections 3.2 and 3.3); finally, we construct the required radial barriers (section 3.4).

3.1. The Dirichlet problem.

Theorem 4. Given $\alpha \in (0,1)$, let $\Omega$ be a uniformly convex bounded open subset of $\mathbb{H}$ with $C^{2,\alpha}$ boundary, $h : \Omega \to \mathbb{R}$ be a positive function of class $C^{2,\alpha}$, and $\varphi_0 : \mathbb{H} \to \mathbb{R}$ be a spacelike function of class $C^{2,\alpha}$ whose radial graph is strictly convex. Then the Dirichlet problem

$$F_2(\varphi) = h \text{ in } \Omega, \ \varphi = \varphi_0 \text{ on } \partial \Omega,$$

has a unique admissible solution of class $C^{1,\alpha}$.

Proof: We first prove uniqueness, by contradiction: let $\varphi_0, \varphi_1$ be two admissible solutions of (8), and, for $t \in [0,1]$, set $\varphi_t = t\varphi_0 + (1-t)\varphi$ with $\varphi = \varphi_1 - \varphi_0$. Since $\varphi$ vanishes on $\partial \Omega$, if $\varphi \not\equiv 0$, it assumes a nonzero local extremum. Following the arguments of the proof of Theorem 2 we obtain a contradiction with the Hopf’s strong Maximum Principle.

Let us focus now on the existence part. Setting $x = (x', \sqrt{1 + |x'|^2}) \in \mathbb{R}^n \times \mathbb{R}$, and

$$\Omega' = \{e^{\varphi_0}(x)x', \ x \in \Omega\}, \ u_0(e^{\varphi_0}(x)x') = e^{\varphi_0}(x) \sqrt{1 + |x'|^2},$$

problem (8) is equivalent to the Dirichlet problem:

$$H_2[u] = H(., u) \text{ in } \Omega', \ u = u_0 \text{ on } \partial \Omega'.$$
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where $H_2$ is the scalar curvature operator acting on spacelike graphs defined on $\Omega' \subset \mathbb{R}^n$, and $H$ is defined on $\Omega' \times \mathbb{R}$ by (7).

Let us consider the Banach space

$$E = \{ \overline{v} \in C^{2,\alpha}(\overline{\Omega'}) \mid \overline{v} = 0 \text{ on } \partial\Omega' \},$$

and the open convex subset of $E$

$$U = \{ \overline{v} \in E \mid \sup_{\overline{\Omega'}} |D(\overline{v} + u_0)| < 1 \}.$$

We first note that for every $\overline{v} \in U$, $\text{graph}_{\mathbb{R}^n}(\overline{v} + u_0)$ belongs to the dependence set $K$ of $\text{graph}_{\mathbb{R}^n} u_0$. Here, by definition, $X \in \mathbb{R}^{n,1}$ belongs to $K$ if for every $\xi \in \mathbb{R}^{n,1}$ with $\langle \xi, \xi \rangle \leq 0$ and $\xi \neq 0$, the ray $X + \mathbb{R}\xi$ meets $\text{graph}_{\mathbb{R}^n} u_0$. The set $K$ is a compact subset of the open cone $C^+$.

For each $(\overline{v}, t) \in U \times [0, 1]$, we know from [2, 15] that the Dirichlet problem

$$H_2[u] = tH(., \overline{v} + u_0) + (1 - t)H_2[u_0] \quad \text{in } \Omega', \; u = u_0 \text{ on } \partial\Omega'$$

has a unique admissible solution (belonging to $C^{4,\alpha}$). We define the map

$$T : [0, 1] \times U \to E \quad (t, \overline{v}) \mapsto \overline{u}$$

where $\overline{u}$ is such that $u = \overline{v} + u_0$ is the admissible solution of (10).

For each $t \in [0, 1]$ the fixed points of $T(t, \cdot)$ are under control: indeed, suppose $T(t, \overline{u}) = \overline{u}$, then the function $u = \overline{v} + u_0$ solves the Dirichlet problem

$$H_2[u] = \tilde{H}(., u) \quad \text{in } \Omega', \; u = u_0 \text{ on } \partial\Omega'$$

where

$$\tilde{H}(., u) = tH(., u) + (1 - t)H_2[u_0].$$

The following \textit{a priori} estimates are carried out in [3, p.251]: there exist $\vartheta \in (0, 1)$ and $C > 0$ such that

$$\sup_{\overline{\Omega'}} |Du| < 1 - \vartheta \quad \text{and} \quad \|u\|_{2,\alpha,\overline{\Omega'}} < C.$$ 

The constants $\vartheta, C$ only depend on $\text{diam}(\Omega')$, $\inf_K \tilde{H}$, $\|\tilde{H}\|_{2,K}$, $\|u_0\|_{4,\overline{\Omega'}}$, and on a positive lower bound on the minimum eigenvalue of $D^2u_0$ on $\overline{\Omega'}$. The expression of $\tilde{H}$ implies that they are independent of the parameter $t \in [0, 1]$.

In order to prove that $T(1, \cdot)$ has a fixed point, we now consider the (nonempty) convex subset of the Banach space $E$:

$$U_{\vartheta, C} = \{ \overline{v} \in U \mid |D(\overline{v} + u_0)| < 1 - \vartheta \quad \text{and} \quad \|\overline{v} + u_0\|_{2,\alpha,\overline{\Omega'}} < C \},$$

and the map $T : [0, 1] \times U_{\vartheta, C} \to E$. Then the following properties hold:

(i) $T$ is continuous with compact image due to the above estimates on the solutions of the Dirichlet problem (10);

(ii) $T(0, \cdot) \equiv 0$ by definition;

(iii) for every $t \in [0, 1]$, $T(t, \cdot)$ does not have any fixed point on $\partial U_{\vartheta, C}$, since each fixed point of $T(t, \cdot)$ belongs to $U_{\vartheta, C}$ by the definitions of $\vartheta$ and $C$.

An elementary version of the Leray-Schauder theorem (due to Browder and Potter [12]) implies that $T(1, \cdot)$ has a fixed point, which proves that (8) has a solution. □
3.2. Existence and uniqueness of entire radial solutions. The aim of this section is to prove the following result:

**Theorem 5.** For $\alpha \in (0, 1)$, let $h : \mathbb{R}^+ \to \mathbb{R}$ be a positive function of class $C^{2, \alpha}$ constant on some neighborhood of 0 and let $\varphi_0$ be a real number. Recall $r = r(x)$ denotes the hyperbolic distance of $x \in \mathbb{H}$ from a fixed origin $o \in \mathbb{H}$. The problem:

\begin{equation}
F_2(\varphi)(x) = h(r) \text{ for all } x \in \mathbb{H}, \quad \varphi(o) = \varphi_0,
\end{equation}

admits a unique admissible radial solution $\varphi : \mathbb{H} \to \mathbb{R}$ of class $C^{4, \alpha}$.

**Proof:** Existence: let $B_i$ denote the ball in $\mathbb{H}$ with center $o$ and radius $i \in \mathbb{N}^*$, and $\varphi_i$ be the admissible solution of the Dirichlet problem:

\begin{equation}
F_2(\varphi) = h, \quad \varphi|_{\partial B_i} = 0,
\end{equation}

given by Theorem 4. By radial symmetry and uniqueness, $\varphi_i$ is a radial function: $\varphi_i(x) = f_i(r)$ for some function $f_i : [0, i] \to \mathbb{R}$. By uniqueness again, for $j > i$, the function $\varphi_j - \varphi_i$ must be constant on $B_i$. Therefore $f_j'(r) \equiv f_i'(r) \text{ for } r \in [0, i]$. We may thus define $g$ on $\mathbb{R}^+$ by $g = f_i'$ on each $[0, i]$. Now the function $\varphi$ defined by

$$\varphi(x) = \varphi_0 + \int_0^r g(u) du$$

is a radial solution of (14).

**Uniqueness:** assume that $\varphi_1$ and $\varphi_2$ are admissible radial solutions of (14): $\varphi_1(x) = f_1(r), \varphi_2(x) = f_2(r)$ where $f_1, f_2$ are functions $\mathbb{R}^+ \to \mathbb{R}$. For each real $R > 0$, set

$$\varphi_{1,R}(x) = - \int_{r}^{R} f_1'(u) \, du \quad \text{and} \quad \varphi_{2,R}(x) = - \int_{r}^{R} f_2'(u) \, du.$$ 

The functions $\varphi_{1,R}$ and $\varphi_{2,R}$ are both admissible solutions of the Dirichlet problem (15) on $B_R$. As such, they must coincide on $B_R$, hence $f_1' = f_2'$ on $[0, R]$, which implies the desired result. \(\square\)

3.3. Properties of the radial solutions. The following lemma describes the monotonicity of a solution $\varphi$ of equation (14) depending on the sign of $h - 1$:

**Lemma 3.** Let $h : \mathbb{R}^+ \to \mathbb{R}$ and $\varphi : \mathbb{H} \to \mathbb{R}$ be as in Theorem 5, and let $f : \mathbb{R}^+ \to \mathbb{R}$ be such that $\varphi(x) = f[r(x)], \forall x \in \mathbb{H}$.

(i) If $h \leq 1$, then $f$ is non-increasing; in particular, if $\varphi_0 = 0$, the function $\varphi$ is non-positive.

(ii) If $h \geq 1$, then $f$ is non-decreasing; in particular, if $\varphi_0 = 0$, the function $\varphi$ is non-negative.

**Proof:** Here, we need to calculate explicitly the expression of equation (14) in the radial case. Set $e_1, \ldots, e_{n+1}$, for the standard orthonormal basis of the vector space $\mathbb{R}^{n,1}$. Fix $x \in \mathbb{H}$ and take, with no loss of generality, $o = e_{n+1} = (0, \ldots, 0, 1)$, $x = (\sinh r, 0, \ldots, 0, \cosh r)$ with $r$, the hyperbolic distance between $o$ and $x$. Consider the orthonormal basis of $T_x \mathbb{H}$ defined by:

$$\partial_r = \cosh r \, e_1 + \sinh r \, e_{n+1}, \quad \partial_\theta = e_\theta, \quad \theta = 2, \ldots, n,$$

and the vectors, tangent to $M = \text{graph}_{\mathbb{H}} \varphi$ at $e^{\varphi(x)} x$, induced by the embedding $x \in \mathbb{H} \to e^{\varphi(x)} x \in M$, given by:

$$u_r = e^f(f \, x + \partial_r), \quad u_\theta = e^f \partial_\theta, \quad \theta = 2, \ldots, n.$$
The future oriented unit normal to $M$ at $e^x(x)$ is the vector:

$$N(r) = \frac{f'}{\sqrt{1 - f'^2}} \partial_r + \frac{1}{\sqrt{1 - f'^2}} x.$$  

Let $S$ be the shape endomorphism of $M$ at $e^x(x)$, with respect to the future unit normal $N(r)$. Using the formulas

$$D_\partial, \partial_r(x) = x, \quad D_\partial, \partial_r(x) = \frac{1}{\tanh r} \partial_\partial$$

where $D$ denotes the canonical flat connection of $\mathbb{R}^{n,1}$ and $\partial_r$ the unit radial vector field of $H$ with respect to the point $o$, we readily get:

$$S(u_\partial) = dN(\partial_\partial) = \frac{e^{-f}}{\sqrt{1 - f'^2}} \left( \frac{f''}{1 - f'^2} + 1 \right) u_\partial,$$

and, for $\partial = 2, \ldots, n,$

$$S(u_\partial) = dN(\partial_\partial) = \frac{e^{-f}}{\sqrt{1 - f'^2}} \left( \frac{f'}{\tanh r} + 1 \right) u_\partial.$$

The principal curvatures of $M$ at $r > 0$ are thus equal to:

$$\frac{e^{-f}}{\sqrt{1 - f'^2}} \left( \frac{f''}{1 - f'^2} + 1 \right) \quad \text{(simple),} \quad \frac{e^{-f}}{\sqrt{1 - f'^2}} \left( \frac{f'}{\tanh r} + 1 \right) \quad \text{(multiplicity } n - 1).$$

Setting $s = s(r)$ for the hyperbolic distance from $o$ to $N(r)$, we infer from (16):

$$s(r) = r + \text{Argth}(f').$$

In terms of the new radial unknown $s(r)$, for $r > 0$, the principal curvatures read

$$\left( e^{-f} \cosh(r - s)s', e^{-f} \sinh s \sinh r, \ldots, e^{-f} \sinh s \sinh r \right),$$

and the equation $F_2(\varphi) = h$ reads

$$2s' \cosh(r - s) \sinh r \sinh s = nh^2 \sinh^2 r - (n - 2) \sinh^2 s.$$  

We now prove the first statement of the lemma. Since $f' = \tanh(s - r)$, we must prove: $s \leq r$ on $[0, +\infty)$. Suppose first $h < 1$. Since $s(0) = 0$ and $s'(0) = h(0) < 1$ (from (19)), there exists $r_0 > 0$ such that $s \leq r$ on $[0, r_0]$. Moreover, we get from (19):

$$s' \leq \frac{1}{2 \cosh(r - s)} \left( n \frac{\sinh r}{\sinh s} - (n - 2) \frac{\sinh s}{\sinh r} \right).$$

We observe that the function $s(r) = r$ is a solution of the ODE:

$$s' = \frac{1}{2 \cosh(r - s)} \left( n \frac{\sinh r}{\sinh s} - (n - 2) \frac{\sinh s}{\sinh r} \right)$$

on $[r_0, +\infty)$. So the comparison theorem for solutions of ordinary differential equations implies $s \leq r$ on $[r_0, +\infty)$. Suppose only $h \leq 1$, fix $A > 0$ and consider $h_\delta = h - \delta$, where $\delta$ is some small positive constant such that $h_\delta > 0$ on $[0, A]$. Denoting by $s_\delta$ and $s_\delta$ the corresponding solutions of (14) and (19) on the ball of radius $A$, the function $s_\delta - r$ is non-positive; we now prove that $s_\delta - r$ converges uniformly to $s - r$ as $\delta$ tends to zero, which will yield the desired result. Set $B_A$ for the ball of radius $A$ in $\mathbb{H}$ and $U = \{ \psi \in C^{2,\alpha}(\overline{B_A}), \ \psi + \varphi \text{ is admissible in } \overline{B_A}, \ \psi|_{\partial B_A} = 0 \}$; consider the auxiliary map:

$$\Phi : \psi \in U \rightarrow \Phi(\psi) := F_2(\psi + \varphi) \in C^{\alpha}(\overline{B_A}).$$

Since $\Phi(0) = h$ and since, classically [7] (recalling (5)), the linearized map $d\Phi(0)$ is an isomorphism from $\{ \xi \in C^{2,\alpha}(\overline{B_A}), \ \xi|_{\partial B_A} = 0 \}$ to $C^{\alpha}(\overline{B_A})$, the inverse function theorem implies: $\forall \epsilon > 0, \exists h_0 > 0, \forall \delta \in (0, h_0)$, the solution $\psi_\delta \in U$ of
that the following vector field, associated to the differential equation (19):

\[ F_\delta(\psi_\delta + \varphi) = h_\delta \] satisfies \(|\psi_\delta|_2,\alpha \leq \varepsilon\). Since \(\varphi_\delta = \psi_\delta + \varphi - \psi_\delta(0)\), we obtain \(|\varphi_\delta - \varphi|_2,\alpha \leq 2\varepsilon\), which implies the convergence of \(\varphi_\delta\) to \(\varphi\) in \(C^1\) and thus the uniform convergence of \(s_\delta\) to \(s\).

The proof of statement (ii) is analogous and thus omitted \(\square\)

Our next lemma provides a simple necessary and sufficient condition for an entire radial solution to be bounded.

**Lemma 4.** Let \(h : \mathbb{R}^+ \to \mathbb{R}\) and \(\varphi : \mathbb{R} \to \mathbb{R}\) be as in Theorem 5.

(i) Assume \(h \leq 1\), and \(\lim_{r \to \infty} h = 1\). Then

\[
\lim_{r(\varphi) \to +\infty} \varphi(x) > -\infty \text{ if and only if } \int_0^{+\infty} (1-h)dr \text{ converges.}
\]

(ii) Assume \(h \geq 1\), and \(\lim_{r \to \infty} h = 1\). Then

\[
\lim_{r(\varphi) \to +\infty} \varphi(x) < +\infty \text{ if and only if } \int_0^{+\infty} (h-1)dr \text{ converges.}
\]

**Proof:** Let us prove statement (i), thus assuming \(h \leq 1\), with \(\lim_{r \to \infty} h = 1\).

We stick to the notations used in the proof of Lemma 3. From (17), we get at once:

\[
\varphi(x) = \varphi_0 - \int_0^{r(x)} \tanh(u-s(u))du.
\]

Statement (i) amounts to proving that \(\int_0^{+\infty} \tanh(u-s(u))du\) converges if and only if \(\int_0^{+\infty} (1-h)dr\). We split the proof of this fact into five steps.

**Step 1:** the solution \(s\) of (19) is an increasing function.

Let us consider in the \((r, s)\) plane the curve \(C\) with equation:

\[ nh^2 \sinh^2 r = (n-2) \sinh^2 s, \ r, s \geq 0.\]

The slope of its tangent at \((0, 0)\) is \(\sqrt{\frac{n}{n-2}} h(0)\). Since the solution \(s\) satisfies \(s(0) = 0\) and \(s'(0) = h(0)\), we infer that the graph of \(s\) stays under the curve \(C\) near 0. Noting that the following vector field, associated to the differential equation (19):

\[(r, s) \mapsto (2 \cosh(r-s) \sinh r \sinh s, nh^2 \sinh^2 r - (n-2) \sinh^2 s),\]

is horizontal on \(C\), and that the height \(s\) of the curve \(C\) is increasing with \(r\), we conclude that the solution \(s\) of (19) remains trapped below \(C\). In other words \(nh^2 \sinh^2 r \geq (n-2) \sinh^2 s\) for all \(r\), and (19) implies: \(s' \geq 0\).

**Step 2:** \(r - s\) has a limit at \(+\infty\).

By contradiction, assume \(\liminf(r-s) < \limsup(r-s) = \delta\). Thus there exists a sequence \(r_k \to +\infty\) such that \(r_k - s(r_k) \to \delta\) and \(s'(r_k) = 1\). Denoting \(s(r_k)\) by \(s_k\), we get from equation (19):

\[
1 = \frac{1}{2 \cosh(r_k - s_k)} \left[ nh^2(r_k) \frac{\sinh r_k}{\sinh s_k} - (n-2) \frac{\sinh s_k}{\sinh r_k} \right].
\]

We distinguish two cases:

**First case:** \(\delta < +\infty\). We then have \(s_k \to +\infty\), \(\frac{\sinh r_k}{\sinh s_k} \sim e^{r_k-s_k} \sim e^{\delta}\) and \(\frac{\sinh s_k}{\sinh r_k} \sim e^{s_k-r_k} \sim e^{-\delta}\) as \(k\) tends to infinity (here and below, the equivalence \(\sim\) between two quantities means that their quotient has limit 1). So (21) yields

\[
1 = \frac{1}{2 \cosh \delta} \left[ ne^{\delta} - (n-2) e^{-\delta} \right].
\]
Using \( e^\delta \geq e^{-\delta} \) we get \( 1 \geq \frac{e^\delta}{\cosh \delta} \), which is absurd.

**Second case** \( \delta = +\infty \). First assuming that \( s_k \) is not bounded, and since \( s \) is an increasing function (Step 1), we have: \( s_k \to +\infty \) and \( \frac{\sinh s_k}{\sinh r_k} \sim e^{r_k-s_k} \to 0 \) as \( k \) tends to infinity. Equation (21) yields

\[
1 \sim \frac{n}{2\cosh(r_k - s_k)} e^{r_k - s_k},
\]

which is absurd since \( \cosh(r_k - s_k) \sim e^{r_k-s_k}/2 \). If we now assume \( s_k \) bounded, since \( s \) is an increasing function with \( s'(0) > 0 \), we get that \( s_k \) converges to \( l > 0 \), and, since \( \frac{\sinh s_k}{\sinh r_k} \to 0 \), we obtain from (21):

\[
1 \sim \frac{n}{2\cosh(r_k - s_k)} \sinh l,
\]

with \( \sinh r_k \sim e^{r_k}/2 \), \( \cosh(r_k - s_k) \sim e^{r_k-s_k}/2 \sim e^{-l}e^{r_k} \); so \( 1 \geq \frac{n}{2} e^\delta \), which is absurd.

**Step 3:** \( r - s \) tends to 0 at infinity.

Having proved that \( r - s \) converges, let us set \( \delta = \lim_{r \to +\infty} r - s \) and prove by contradiction that \( \delta = 0 \). There are two cases:

**First case:** \( 0 < \delta < +\infty \). We get \( s \to +\infty \), hence \( \frac{\sinh r}{\sinh s} \sim e^{-s} \sim e^\delta \), \( \frac{\sinh s}{\sinh r} \sim e^{s-r} \sim e^{-\delta} \) as \( r \) tends to infinity, and thus, from (19):

\[
s' \sim \frac{1}{2\cosh \delta} \left[ ne^\delta - (n - 2)e^{-\delta} \right].
\]

The latter expression is larger than 1, which contradicts \( r \geq s \).

**Second case:** \( \delta = +\infty \). We first note that \( \frac{\sinh s}{\sinh r} \to 0 \) (if \( s \) is bounded this is trivial; if \( s \) is not bounded, \( s \to +\infty \) since \( s \) is increasing, and we have \( \frac{\sinh s}{\sinh r} \sim e^{s-r} \to 0 \) since \( r - s \to +\infty \)). Moreover we have \( \liminf h^2 \frac{\sinh r}{\sinh s} \geq n \) since \( r \geq s \). We thus infer from equation (19):

\[
s' \sim \frac{n}{2\cosh(r - s)} \frac{\sinh r}{\sinh s}.
\]

Assuming \( s \to +\infty \), we get \( \frac{\sinh r}{\sinh s} \sim e^{r-s} \) and \( \cosh(r - s) \sim e^{r-s} \), hence \( s' \sim n \), which is impossible since \( s \leq r \).

Finally, assuming \( s \) bounded yields \( s \to l > 0 \); since \( r - s \to +\infty \), we infer \( \cosh(r - s) \sim e^{r-s}/2 \) and \( \sinh r \sim e^r/2 \), hence from (19), \( e^{r-s} s' \sim n e^\delta /2 \sinh l \) and thus \( s' \sim n e^\delta /2 \sinh l \), which contradicts the boundedness assumption on \( s \).

**Step 4:** \( \lim_{\varphi(x) \to +\infty} \varphi(x) = -\infty \) if and only if \( \psi(r) := r - s \) is integrable on \([0, +\infty)\).

This is straightforward from (20) combined with \( \tanh(u - s(u)) \sim \varepsilon(u) \) which holds as \( u \to +\infty \) due to Step 3.

**Step 5:** \( \varepsilon \) is integrable on \([0, +\infty)\) if and only if \( \beta := 1 - h^2 \) is integrable on \([0, +\infty)\).

First observation: \( \lim_{r \to +\infty} s' = 1 \). Indeed, at infinity, we have \( r - s \to 0 \), so \( s \to +\infty \), hence:

\[
\frac{\sinh r}{\sinh s} \sim e^{r-s} \sim 1, \quad \frac{\sinh s}{\sinh r} \sim e^{s-r} \sim 1,
\]

and (19) yields \( s' \to 1 \).
Using Step 3, the assumptions on \( h \) and the preceding observation, we get
\[
\varepsilon(r) \to 0, \beta(r) \to 0, \text{ and } \varepsilon'(r) = 1 - s'(r) \to 0
\]
as \( r \) tends to infinity. Plugging the definitions of \( \varepsilon \) and \( \beta \) in (19) and using the expansions
\[
cosh \varepsilon = 1 + o(\varepsilon), \sinh(r - \varepsilon) = \sinh r (1 - \varepsilon + o(\varepsilon)),
\]
yields
\[
(n - 1)\varepsilon + \varepsilon' + o(\varepsilon) = \frac{n}{2} \beta.
\]
Fixing a real \( \delta > 0 \), there readily exists \( r_\delta > 0 \) such that, for all \( r \geq r_\delta \),
\[
\varepsilon' + (n - 1 - \delta)\varepsilon \leq \frac{n}{2} \beta,
\]
and
\[
\varepsilon' + (n - 1 + \delta)\varepsilon \geq \frac{n}{2} \beta.
\]
Integrating (23), we get, for \( r \geq r_\delta \),
\[
\varepsilon(r) \leq e^{-(n-1-\delta)r} \left[ C(r_\delta) + \frac{n}{2} \int_{r_\delta}^{r} \beta(u)e^{(n-1-\delta)u} du \right].
\]
Integrating again and using Fubini Theorem yields, with \( \delta \) such that \( n - 1 - \delta > 0 \),
\[
\int_{r_\delta}^{+\infty} \varepsilon(r) dr \leq C'(r_\delta) + \frac{n}{2} \int_{r_\delta}^{+\infty} \beta(u)e^{(n-1-\delta)u} \left( \int_{u}^{+\infty} e^{-(n-1-\delta)r} dr \right) du,
\]
\[
\leq C'(r_\delta) + \frac{n}{2(n-1-\delta)} \int_{r_\delta}^{+\infty} \beta(u) du.
\]
We conclude that \( \varepsilon \) is integrable provided \( \beta = 1 - h^2 \) is integrable. Analogously, using (24), we get
\[
\varepsilon(r) \geq e^{-(n-1+\delta)r} \left[ C(r_\delta) + \frac{n}{2} \int_{r_\delta}^{r} \beta(u)e^{(n-1+\delta)u} du \right],
\]
and
\[
\int_{r_\delta}^{+\infty} \varepsilon(r) dr \geq C'(r_\delta) + \frac{n}{2} \int_{r_\delta}^{+\infty} \beta(u)e^{(n-1+\delta)u} \left( \int_{u}^{+\infty} e^{-(n-1+\delta)r} dr \right) du,
\]
\[
\geq C'(r_\delta) + \frac{n}{2(n-1+\delta)} \int_{r_\delta}^{+\infty} \beta(u) du.
\]
Taking \( \delta > 0 \) arbitrary, we find that \( \beta \) is integrable if \( \varepsilon \) is integrable.
The proof of statement (iii) is analogous and thus omitted \( \square \)

3.4. Construction of radial barriers.

**Lemma 5.** Let \( h : \mathbb{H} \to \mathbb{R} \) be a positive and continuous function on the hyperbolic space such that
\[
\lim_{r(x) \to +\infty} h(x) = 1
\]
and such that the functions \( h^- \) and \( h^+ \) defined on \( \mathbb{R}^+ \) by
\[
h^-(r) = \sup_{r(x) = r} h(x) \text{ and } h^+(r) = \inf_{r(x) = r} h(x)
\]
satisfy
\[
\int_{0}^{+\infty} (h^- - 1)_+ dr < +\infty, \quad \int_{0}^{+\infty} (1 - h^+)_+ dr < +\infty,
\]
where $(h^- - 1)_+$ (resp. $(1 - h^+)_+$) means the positive part of $h^- - 1$ (resp. $1 - h^+$). Then there exist $\varphi^-, \varphi^+ \in C^\infty(\mathbb{H})$, with strictly convex spacelike graphs, satisfying:

$$F_2(\varphi^-) \geq h, \quad F_2(\varphi^+) \leq h \quad \text{and} \quad \lim_{r \to +\infty} \varphi^\pm = 0.$$ 

**Proof**: First, considering $1 + (h^- - 1)_+$ instead of $h^-$ and $1 - (1 - h^+)_+$ instead of $h^+$, we may suppose without loss of generality that $h^-$ and $h^+$ are two continuous functions such that: $\forall x \in \mathbb{H}$, with $r = r(x)$,

$$h^-(r) \geq h(x) \geq h^+(r) > 0,$$

(25)

$$h^- \geq 1 \geq h^+, \quad \lim_{r \to +\infty} h^-(r) = \lim_{r \to +\infty} h^+(r) = 1,$$

(26)

and

$$\int_0^{+\infty} (h^--1)dr < +\infty, \quad \int_0^{+\infty} (1-h^+)dr < +\infty.$$

(27)

If we now consider

$$h^- + \frac{\varepsilon_0}{r^2} \text{ if } r \geq 1, \quad h^- + \varepsilon_0 \text{ if } r \leq 1$$

instead of $h^-$, and

$$h^+ - \frac{\varepsilon_0}{r^2} \text{ if } r \geq 1, \quad h^+ - \varepsilon_0 \text{ if } r \leq 1$$

instead of $h^+$, where $\varepsilon_0$ is chosen sufficiently small such that $\inf h^+ > \varepsilon_0$, we may moreover assume the following:

$$h^- \geq \max(1, h) + \frac{\varepsilon_0}{r^2} \text{ and } h^+ \leq \min(1, h) - \frac{\varepsilon_0}{r^2} \text{ if } r \geq 1.$$ 

We now prove that we can approximate $h^\pm$ by smooth functions $g^\pm$ such that

$$|h^\pm - g^\pm| \leq \min\left(\frac{\varepsilon_0}{r^2}, \varepsilon_0\right).$$

(28)

For each $i \in \mathbb{N}$, let us denote by $g^-_i$ a smooth function on $[0, i + 1)$ such that $|h^- - g^-_i| \leq \frac{\varepsilon_0}{(i+1)^2}$ on $[0, i + 1]$. Let $\vartheta \in C^\infty_c(\mathbb{R})$ such that $0 \leq \vartheta \leq 1$, $\vartheta(x) = 1$ if $|x| \leq \frac{1}{4}$ and $\vartheta(x) = 0$ if $|x| \geq \frac{3}{4}$. We define $g^-$ on $[i, i + 1]$ by

$$g^- = \vartheta_i g^-_i + (1 - \vartheta_i) g^-_{i+1},$$

where $\vartheta_i = \vartheta(-i)$. By construction, we have $g^- = g^-_i$ on a neighborhood of $i$. The function $g^-$ is thus smooth on $[0, +\infty)$, and satisfies on $[i, i + 1]$

$$|g^- - h^-| \leq \vartheta_i |g^- - h^-| + (1 - \vartheta_i) (g^-_{i+1} - h^-) \leq \frac{\varepsilon_0}{(i+1)^2},$$

which implies the estimate (28). We may thus assume that (25), (26) and (27) hold, where $h^\pm$ are two smooth functions on $\mathbb{R}^+$. Considering $\vartheta \sup_{\mathbb{R}} h^- + (1 - \vartheta) h^-$ instead of $h^-$, and $\vartheta \inf_{\mathbb{R}} h^+ + (1 - \vartheta) h^+$ instead of $h^+$, we may also assume that the functions $h^\pm$ are constant on some neighborhood of 0. Let $\varphi^-$ and $\varphi^+$ be smooth radial functions given by Theorem 5 (with some arbitrary initial condition $\varphi_0$) such that $F_2(\varphi^\pm) = h^\pm$. From Lemma 4, subtracting constants if necessary, we obtain

$$\lim_{r \to +\infty} \varphi^\pm(r) = 0 \quad \Box$$

Now, we can complete the proof of Theorem 1 as follows. Lemma 5 provides two barriers which tend to 0 at infinity; by Theorem 3, we get an entire solution of equation (3) pinched between these barriers, and thus tending to 0 at infinity, so the existence part of Theorem 1 is proved. Uniqueness was proved in Theorem 2.
References


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