ON THE LOCAL NIRENBERG PROBLEM FOR THE $Q$-CURVATURES

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The local image of each conformal $Q$-curvature operator on the sphere admits no scalar constraint, although identities of Kazdan–Warner type hold for its graph.

1. Introduction

Let $(m, n)$ be positive integers such that $n > 1$, and $n \geq 2m$ in case $n$ is even. We work on the standard $n$-sphere $(\mathbb{S}^n, g_0)$, with pointwise conformal metric $g_u = e^{2u}g_0$. (All objects will be taken to be smooth.)

We are interested in the structure near $u = 0$ of the image of the conformal $2m$-th order $Q$-curvature increment operator $u \mapsto Q_{m,n}[u] = Q_{m,n}(g_u) - Q_{m,n}(g_0)$ (see Section 2), thus considering a local Nirenberg-type problem (Nirenberg’s problem was for $m = 1$; see, for example, [Moser 1973; Kazdan and Warner 1974; 1975; Aubin 1982, p. 122]). At the infinitesimal level, the situation looks as follows (dropping henceforth the subscript $(m, n)$):

Lemma 1.1. Let $L = d\mathcal{Q}[0]$ stand for the linearization at $u = 0$ of the conformal $Q$-curvature increment operator and $\Lambda_1$, for the $(n+1)$-space of first spherical harmonics on $(\mathbb{S}^n, g_0)$. Then $L$ is self-adjoint and $\text{Ker} L = \Lambda_1$.

Further, the graph $\Gamma(Q) := \{(u, Q[u]), u \in C^\infty(\mathbb{S}^n)\}$ of $Q$ in $C^\infty(\mathbb{S}^n) \times C^\infty(\mathbb{S}^n)$ admits scalar constraints which are the analogue for $Q$ of the so-called Kazdan–Warner identities for the conformal scalar curvature (i.e., the case $m = 1$); see [Kazdan and Warner 1974; 1975; Bourguignon and Ezin 1987]. Here, a scalar constraint means a real-valued submersion defined near $\Gamma(Q)$ in $C^\infty(\mathbb{S}^n) \times C^\infty(\mathbb{S}^n)$ and vanishing on $\Gamma(Q)$. Specifically:

Theorem 1.2. For each $(u, q) \in C^\infty(\mathbb{S}^n) \times C^\infty(\mathbb{S}^n)$ and each conformal Killing vector field $X$ on $(\mathbb{S}^n, g_0)$, the condition $(u, q) \in \Gamma(Q)$ implies the vanishing of the...
integral \( \int_{\mathbb{S}^n}(X \cdot q) \, d\mu_u \), where \( d\mu_u = e^{nu}d\mu_0 \) stands for the Lebesgue measure of the metric \( g_u \). In particular, there is no solution \( u \in C^\infty(\mathbb{S}^n) \) to the equation

\[
Q(g_u) = z + \text{constant} \quad \text{with} \quad z \in \Lambda_1.
\]

Due to the naturality of \( Q \) (Remark 3.1) and the self-adjointness of \( dQ[u] \) in \( L^2(M_n, d\mu_u) \) (Remarks 3.2 and 3.3), this theorem holds as a particular case of the more general Theorem 2.1 below.

Can one do better than Theorem 1.2 and drop the \( u \) variable occurring in the constraints and find constraints bearing on the sole image of the operator \( Q \)? Since \( L \) is self-adjoint in \( L^2(M_n, d\mu_u) \) (see [Graham and Zworski 2003]), Lemma 1.1 shows that the map \( u \mapsto Q[u] \) misses infinitesimally at \( u = 0 \) a vector space of dimension \( n + 1 \). How does this translate at the local level? Calling a real-valued map \( K \) a scalar constraint for the local image of \( Q \) near 0 if \( K \) is a submersion defined near 0 in \( C^\infty(\mathbb{S}^n) \) such that \( K \circ Q = 0 \) near 0 in \( C^\infty(\mathbb{S}^n) \), a spherical symmetry argument as in [Delanoë 2003, Corollary 5] shows that if the local image of \( Q \) admits a scalar constraint near 0, it must admit \( n + 1 \) independent such constraints, the maximal number to be expected. In this context, our main result is quite in contrast with Theorem 1.2:

**Theorem 1.3.** The local image of \( Q \) near 0 admits no scalar constraint.

The picture about the local image of the \( Q \)-curvature increment operator on \((\mathbb{S}^n, g_0)\) is completed with a remark:

**Remark 1.4.** The local Nirenberg problem for \( Q \) near 0 is governed by the nonlinear Fredholm formula (9) below. Thus, as in [Delanoë 2003, Corollary 5], a local result of Moser type [1973] holds: If \( f \in C^\infty(\mathbb{S}^n) \) is close enough to zero and invariant under a nontrivial group of isometries of \((\mathbb{S}^n, g_0)\) acting without fixed points,\(^1\) then \( \varrho(f) = 0 \) in (9), so \( f \) lies in the local image of \( Q \).

The outline of the paper is as follows. In Section 2 we present an independent account on general Kazdan–Warner type identities, implying Theorem 1.2. Then we focus on Theorem 1.3: we recall basic facts for the \( Q \)-curvature operators on spheres in Section 3 and sketch the proof of Theorem 1.3 in Section 4, relying on [Delanoë 2003] and reducing it to Lemma 1.1 and another key lemma. In the last two sections we carry out the proofs of these lemmas, deferring to an Appendix some eigenvalues calculations.

## 2. General identities of Kazdan–Warner type

The following statement is essentially due to Jean–Pierre Bourguignon [1986]:

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\(^1\)This condition is more general than a free action.
Theorem 2.1. Let $M_n$ be a compact $n$-manifold and $g \mapsto D(g) \in C^\infty(M)$ be a scalar differential operator defined on the open cone of Riemannian metrics on $M_n$, and natural in the sense of [Stredder 1975] (see (5) below). Given a conformal class $c$ and a Riemannian metric $g_0 \in c$, sticking to the notation $g_u = e^{2u} g_0$ for $u \in C^\infty(M)$, consider the operator $u \mapsto D[u] := D(g_u)$ and its linearization $L_u = dD[u]$ at $u$. Assume that, for each $u \in C^\infty(M)$, the linear differential operator $L_u$ is formally self-adjoint in $L^2(M, d\mu_u)$, where $d\mu_u = e^{nu} d\mu_0$ stands for the Lebesgue measure of $g_u$. Then, for any conformal Killing vector field $X$ on $(M_n, c)$ and any $u \in C^\infty(M)$, we have

$$\int_M X \cdot D[u] \, d\mu_u = 0.$$

In particular, if $(M_n, c)$ is equal to $\mathbb{S}^n$ equipped with its standard conformal class, there is no solution $u \in C^\infty(\mathbb{S}^n)$ to the equation

$$D[u] = z + \text{constant} \quad \text{with } z \in \Lambda_1$$

(a first spherical harmonic).

Proof. We rely on Bourguignon’s functional integral invariants approach and follow the proof of [Bourguignon 1986, Proposition 3] (using freely notations from p. 101 of the same paper), presenting its functional geometric framework with some care. We consider the affine Fréchet manifold $\Gamma$ whose generic point is the volume form (possibly of odd type in case $M$ is not orientable [de Rham 1960]) of a Riemannian metric $g \in c$; we denote by $\omega_g$ the volume form of a metric $g$ (recall the tensor $\omega_g$ is natural [Stredder 1975, Definition 2.1]). The metric $g_0 \in c$ yields a global chart of $\Gamma$ defined by

$$\omega_g \in \Gamma \mapsto u := \frac{1}{n} \log \frac{d\omega_g}{d\omega_{g_0}} \in C^\infty(M_n)$$

(viewing volume-forms like measures and using the Radon–Nikodym derivative) — in other words, such that $\omega_g = e^{nu} \omega_0$; changes of such charts are indeed affine (and pure translations). It will be easier, though, to avoid the use of charts on $\Gamma$, except for proving that a 1-form is closed (see below). The tangent bundle to $\Gamma$ is trivial, equal to $T\Gamma = \Gamma \times \Omega^1(M_n)$ (setting $\Omega^k(A)$ for the $k$-forms on a manifold $A$), and there is a canonical Riemannian metric on $\Gamma$ of Fischer type [Friedrich 1991], given at $\omega_g \in \Gamma$ by

$$\langle v, w \rangle := \int_M \frac{dv}{d\omega_g} \frac{dw}{d\omega_g} \omega_g \quad \text{for } (v, w) \in T_{\omega_g} \Gamma.$$

From Riesz’s theorem, a tangent covector $a \in T^*_{\omega_g} \Gamma$ may thus be identified with a tangent vector $a^2 \in \Omega^1(M_n)$ or else with the function $da^2/d\omega_g =: \rho_g(a) \in C^\infty(M_n)$...
such that

\[(1) \quad a(\omega) = \int_M \rho_g(a) \omega \quad \text{for } \omega \in T_{\omega_0} \Gamma. \]

We also consider the Lie group $G$ of conformal maps on $(M_n, c)$, acting on the manifold $\Gamma$ by

\[(\varphi, \omega_\gamma) \in G \times \Gamma \to \varphi^* \omega_\gamma \in \Gamma \]

(indeed, we have $\varphi^* \omega_\gamma = \omega_{\varphi^* \gamma}$ by naturality and $\varphi \in G \Rightarrow \varphi^* g \in c$). For each conformal Killing field $X$ on $(M_n, c)$, the flow of $X$ as a map $t \in \mathbb{R} \to \varphi_t \in G$ yields a vector field $\bar{X}$ on $\Gamma$ defined by

\[\omega_\gamma \mapsto \bar{X}(\omega_\gamma) := \frac{d}{dt} \left( \varphi_t^* \omega_\gamma \right)_{t=0} \equiv L_X \omega_\gamma \]

($L_X$ standing here for the Lie derivative on $M_n$). In this context, regardless of any Banach completion, one may define the (global) flow $t \in \mathbb{R} \to \bar{\varphi}_t \in \text{Diff}(\Gamma)$ of $\bar{X}$ on the Fréchet manifold $\Gamma$ by setting

\[\bar{\varphi}_t(\omega_\gamma) := \varphi_t^* \omega_\gamma \quad \text{for } \omega_\gamma \in \Gamma; \]

indeed, the latter satisfies

\[\frac{d}{dt} \left( \varphi_t^* \omega_\gamma \right) = \varphi_t^* (L_X \omega_\gamma) \equiv L_X (\varphi_t^* \omega_\gamma) = \bar{X} \left[ \bar{\varphi}_t(\omega_\gamma) \right] \]

(see [Kobayashi and Nomizu 1963, p. 33], for example). With the flow $(\bar{\varphi}_t)_{t \in \mathbb{R}}$ at hand, we can define the Lie derivative $L_{\bar{X}}$ of forms on $\Gamma$ as usual, by setting

\[L_{\bar{X}} a := (d/dt) \left( \bar{\varphi}_t^* a \right)_{t=0}. \]

Finally, one can check Cartan’s formula for $\bar{X}$, namely

\[(2) \quad L_{\bar{X}} = i_{\bar{X}} d + di_{\bar{X}}, \]

where $i_{\bar{X}}$ denotes the interior product with $\bar{X}$, by verifying it for a generic function $f$ on $\Gamma$ and for its exterior derivative $df$ (with $d$ defined as in [Lang 1962]).

Following [Bourguignon 1986], and using our global chart $\omega_\gamma \mapsto u$, we apply (2) to the 1-form $\sigma$ on $\Gamma$ defined at $\omega_\gamma$ by the function $\rho_G(\sigma) := D[u]$; see (1). Arguing as on p. 102 of the same reference, one readily verifies in the chart $u$ (and using constant local vector fields on $\Gamma$) that the 1-form $\sigma$ is closed due to the self-adjointness of the linearized operator $L_u$ in $L^2(M_n, d\mu_u)$; furthermore (dropping the chart $u$), one derives at once the $G$-invariance of $\sigma$ from the naturality of $g \mapsto D(g)$. We thus have $d\sigma = 0$ and $L_{\bar{X}} \sigma = 0$, hence $d(i_{\bar{X}} \sigma) = 0$ by (2). So the function $i_{\bar{X}} \sigma$ is constant on $\Gamma$; in other words, $\int_M D[u] L_{\bar{X}} \omega_\mu$ is independent of $u$, or else, integrating by parts, so is $\int_M X \cdot D[u] d\mu_u$ (where $X \cdot$ stands for $X$ acting as a derivation on real-valued functions on $M_n$).

To complete the proof of the first part of Theorem 2.1, we show that the integrand $X \cdot D(g_0)$ of the latter expression at $u = 0$ vanishes for a suitable choice of the metric
g₀ in the conformal class c. We recall the Ferrand–Obata theorem [Lelong-Ferrand 1969; Obata 1971/72], according to which either the conformal group G is compact or (Mₙ, e) is equal to Sⁿ equipped with its standard conformal class. In the former case, averaging on G, we may pick g₀ ∈ c invariant under the action of G: with this choice, D(g₀) is also G-invariant by naturality, hence X · D(g₀) ≡ 0 as needed. In the latter case, as observed in the proof of Proposition 4.2 below, D(g₀) is constant on /H₁¹ⁿ, and again the desired result follows.

Finally, the last assertion of the theorem (consistently with Proposition 4.2 below and the Fredholm theorem if L₀ is elliptic) follows from the first one, by taking for the vector field X the gradient of z with respect to the standard metric of /H₁¹ⁿ, which is known to be conformal Killing. □

3. Back to Q-curvatures on spheres: basic facts recalled

The special case n = 2m. Here we will consider the Q-curvature increment operator given by Q[u] = Q(gᵤ) − Q₀, with

$$Q(g_u) = e^{-2mu}(Q₀ + P₀[u])$$

where Q₀ = Q(g₀) is equal to Q₀ = (2m−1)! on (Sⁿ, g₀), and where

$$P₀ = \prod_{k=1}^{m} (\Delta₀ + (m-k)(m+k-1))$$

(see [Branson 1987; Beckner 1993]), Δ₀ denoting the positive laplacian relative to g₀. We call P₀ the Paneitz–Branson operator of the metric g₀.

**Remark 3.1.** Following [Branson 1995], one can define a Paneitz–Branson operator P₀ for any metric g₀ (given by a formula more general than (4) of course), and a Q-curvature Q(g₀) transforming like (3) under the conformal change of metrics gᵤ = e²μ₀ g₀. Importantly then, the map g → Q(g) ∈ C∞(Sⁿ) is natural, meaning (see [Stredder 1975, Definition 2.1], for instance) that any diffeomorphism ψ satisfies

$$ψ^*Q(g) = Q(ψ^*g).$$

**Remark 3.2.** From (3) and the formal self-adjointness of P₀ in L²(Sⁿ, dμ₀) [Graham and Zworski 2003, p. 91], one readily verifies that, for each u ∈ C∞(Sⁿ), the linear differential operator dQ[u] is formally self-adjoint in L²(Sⁿ, dμ_u).

The case n ≠ 2m. The expression of the Paneitz–Branson operator on (Sⁿ, g₀) becomes

$$P₀ = \prod_{k=1}^{m} (\Delta₀ + (1/₂n - k) (1/₂n + k - 1))$$
(see [Guillarmou and Naud 2006, Proposition 2.2]), while that for the metric $g_u = e^{2u} g_0$ is given by

$$P_u(\cdot) = e^{-(\frac{1}{2}n+m)u} P_0(e^{(\frac{1}{2}n-m)u} \cdot),$$

with the $Q$-curvature of $g_u$ given accordingly by $(\frac{1}{2}n-m) Q(g_u) = P_u(1)$. The analogue of Remark 3.1 still holds (now see [Graham et al. 1992; Graham and Zworski 2003]). We will consider the (renormalized) $Q$-curvature increment operator $Q[u] = (\frac{1}{2}n-m) (Q(g_u) - Q_0)$, now with

$$Q_0 = (\frac{1}{2}n-m) Q_0 = P_0(1) = \prod_{k=0}^{2m-1} (k + \frac{1}{2} n - m).$$

**Remark 3.3.** Finally, we note again that the linearized operator $d Q[u]$ is formally self-adjoint in $L^2(\mathbb{S}^n, d\mu_u)$. Indeed, a straightforward calculation yields

$$d Q[u](v) = (\frac{1}{2}n-m) P_u(v) - (\frac{1}{2}n+m) P_u(1) v,$$

and the Paneitz–Branson operator $P_u$ is known to be self-adjoint in $L^2(\mathbb{S}^n, d\mu_u)$ [Graham and Zworski 2003, p. 91].

For later use, and in all the cases for $(m,n)$, we will set $p_0$ for the degree $m$ polynomial such that $P_0 = p_0(\Delta_0)$.

### 4. Proof of Theorem 1.3

The case $m = 1$ was settled in [Delanoë 2003] with a proof robust enough to be followed again. For completeness, let us recall how it goes (see [Delanoë 2003] for details).

If $\mathcal{P}_1$ stands for the orthogonal projection of $L^2(\mathbb{S}^n, g_0)$ onto $\Lambda_1$, Lemma 1.1 and the self-adjointness of $L$ imply [Delanoë 2003, Theorem 7] that the modified operator

$$u \mapsto Q[u] + \mathcal{P}_1 u$$

is a local diffeomorphism of a neighborhood of 0 in $C^\infty(\mathbb{S}^n)$ onto another one: set $\mathcal{S}$ for its inverse and $\mathcal{D} = \mathcal{P}_1 \circ \mathcal{S}$ (defect map). Then $u = \mathcal{S} f$ satisfies the local nonlinear Fredholm-like equation

$$Q[u] = f - \mathcal{D}(f).$$

By [Delanoë 2003, Theorem 2], if a local constraint exists for $Q$ at 0, then $\mathcal{D} \circ Q = 0$ (recalling the symmetry fact above). Fixing $z \in \Lambda_1$, we will prove Theorem 1.3 by showing that $\mathcal{D} \circ Q[tz] \neq 0$ for small $t \in \mathbb{R}$; here is how. On the one hand, setting

$$u_t = \mathcal{S} \circ Q[tz] := tu_1 + t^2 u_2 + t^3 u_3 + O(t^4),$$
Lemma 1.1 yields $u_1 = 0$; also, as and easily verified general fact, we have

$$Q[u_t] + \mathcal{P}_1 u_t = t^2 (L + \mathcal{P}_1) u_2 + t^3 (L + \mathcal{P}_1) u_3 + O(t^4).$$

On the other hand, consider the expansion of $Q[tz]$:

$$Q[tz] = t^2 c_2[z] + t^3 c_3[z] + O(t^4),$$

and focus on its third order coefficient $c_3[z]$, for which we will prove:

**Lemma 4.1.** Let $(m, n)$ be positive integers such that $n > 1$ and $n \geq 2m$ in case $n$ is even. Then

$$\int_{\mathbb{S}^n} z c_3[z] d\mu_0 \neq 0.$$

Granted this lemma, we are done: indeed, the equality

$$Q[u_t] + \mathcal{P}_1 u_t = Q[tz],$$

combined with (10)–(11), yields

$$(L + \mathcal{P}_1) u_3 = c_3[z],$$

which, integrated against $z$, implies that

$$\int_{\mathbb{S}^n} z \mathcal{P}_1 u_3 d\mu_0 \neq 0$$

(recalling that $L$ is self-adjoint and $z \in \text{Ker} L$ by Lemma 1.1). Therefore $\mathcal{P}_1 u_3 \neq 0$, hence also $\mathcal{D} \circ Q[tz] \neq 0$.

Thus we have reduced the proof of Theorem 1.3 to that of Lemmas 1.1 and 4.1, which we now present.

**Proof of Lemma 1.1.** (1) *Proof of the inclusion $\Lambda_1 \subseteq \text{Ker} L$.* We need neither ellipticity nor conformal covariance for this inclusion to hold; the naturality property (5) suffices. We state a general result that implies at once what we need:

**Proposition 4.2.** Let $g \mapsto D(g)$ be any scalar natural differential operator on $\mathbb{S}^n$, defined on the open cone of Riemannian metrics, valued in $C^\infty(\mathbb{S}^n)$. For each $u \in C^\infty(\mathbb{S}^n)$, set $D[u] = D(g_u) - D(g_0)$ and $L = dD[0]$, where $g_u = e^{2u} g_0$. Then $\Lambda_1 \subseteq \text{Ker} L$.

**Proof:** Let us first observe that $D(g_0)$ must be constant. Indeed, for each isometry $\psi$ of $(\mathbb{S}^n, g_0)$, the naturality of $D$ implies $\psi^* D(g_0) \equiv D(g_0)$; so the result follows because the group of such isometries acts transitively on $\mathbb{S}^n$. Morally, since $g_0$ has constant curvature, this result is also expectable from the theory of Riemannian invariants (see [Stredder 1975] and references therein), here though, without any regularity (or polynomiality) assumption.
Given an arbitrary nonzero \( z \in \Lambda_1 \), let \( S = S(z) \in \mathbb{S}^n \) stand for its corresponding south pole (where \( z(S) = -M \) is minimum) and, for each small real \( t \), let \( \psi_t \) denote the conformal diffeomorphism of \( \mathbb{S}^n \) fixing \( S \) and composed elsewhere of: \( \text{Ster}_S \), the stereographic projection with pole \( S \), the dilation \( X \mapsto e^{Mt}X \in \mathbb{R}^n \), and the inverse of \( \text{Ster}_S \). As \( t \) varies, the family \( \psi_t \) satisfies

\[
\psi_0 = I, \quad \frac{d}{dt}(\psi_t)_{t=0} = -\nabla_0 z,
\]

where \( \nabla_0 \) denotes the gradient relative to \( g_0 \). If we set \( e^{2u}g_0 = \psi_t^*g_0 \), we get

\[
\frac{d}{dt}(u_t)_{t=0} = z.
\]

Recalling that \( D(g_0) \) is constant, the naturality of \( D \) implies

\[
D[u_t] = \psi_t^*D(g_0) - D(g_0) = 0;
\]

in particular, differentiating this equation at \( t = 0 \) yields \( Lz = 0 \) hence we conclude that \( \Lambda_1 \subset \text{Ker} \ L \).

(2) Proof of the reverse inclusion \( \ker L \subset \Lambda_1 \). For a contradiction, assume the existence of a nonzero \( v \in \Lambda_1^\perp \cap \text{Ker} \ L \). If \( \mathcal{B} \) is an orthonormal basis of eigenfunctions of \( \Delta_0 \) in \( L^2(\mathbb{S}^n, d\mu_0) \), there exists an integer \( i \neq 1 \) and a function \( \varphi_i \in \Lambda_i \cap \mathcal{B} \) (where \( \Lambda_i \) henceforth denotes the space of \( i \)-th spherical harmonics) such that

\[
\int_{\mathbb{S}^n} \varphi_i v d\mu_0 \neq 0
\]

(actually \( i \neq 0 \), due to \( \int_{\mathbb{S}^n} v d\mu_0 = 0 \), obtained just by averaging \( Lv = 0 \) on \( \mathbb{S}^n \)). By the self-adjointness of \( L \), we may write

\[
0 = \int_{\mathbb{S}^n} \varphi_i Lv d\mu_0 = \int_{\mathbb{S}^n} vL \varphi_i d\mu_0,
\]

then infer (see below) that

\[
0 = (p_0(\lambda_i) - p_0(\lambda_1)) \int_{\mathbb{S}^n} \varphi_i v d\mu_0,
\]

and finally get the desired contradiction, because \( p_0(\lambda_i) \neq p_0(\lambda_1) \) for \( i \neq 1 \) (see the Appendix). Here, we used the following auxiliary facts, obtained by differentiating (3) or (7) at \( u = 0 \) in the direction of \( w \in C^\infty(\mathbb{S}^n) \):

\[
\begin{align*}
 n = 2m & \Rightarrow Lw = P_0(w) - n! w, \\
 n \neq 2m & \Rightarrow Lw = \left(\frac{1}{2}n - m\right) P_0(w) - \left(\frac{1}{2}n + m\right) p_0(\lambda_0)w.
\end{align*}
\]
From $\Lambda_1 \subset \text{Ker } L$, we get, taking $w = z \in \Lambda_1$:

$$n = 2m \Rightarrow p_0(\lambda_1) - n! = 0,$$

(12)

$$n \neq 2m \Rightarrow \left( \frac{1}{2} n - m \right) p_0(\lambda_1) - \left( \frac{1}{2} n + m \right) p_0(\lambda_0) = 0.$$  

Moreover, taking $w = \varphi_i \in \Lambda_i$, we then have

$$n = 2m \Rightarrow L\varphi_i = \left( p_0(\lambda_i) - p_0(\lambda_1) \right) \varphi_i,$$

$$n \neq 2m \Rightarrow L\varphi_i = \left( \frac{1}{2} n - m \right) \left( p_0(\lambda_i) - p_0(\lambda_1) \right) \varphi_i.$$  

\[ \square \]

Proof of Lemma 4.1. (1) The case $m = 2n$. For fixed $z \in \Lambda_1$ and for $t \in \mathbb{R}$ close to 0, we compute the third order expansion of $Q[ tz ]$. By Lemma 1.1 it vanishes up to first order. Noting the identity $Q[v]/Q_0 \equiv e^{-nv}(1 + nv) - 1$, valid for all $v \in \Lambda_1$, we find at once

$$\frac{Q[ tz ]}{Q_0} = -2m^2 t^2 z^2 + \frac{8}{3} m^3 t^3 z^3 + O(t^4);$$

in particular (with the notation of Section 1), we have $c_3[z] = \frac{8}{3} m^3 Q_0 z^3$, and Lemma 4.1 holds trivially.

(2) The case $m \neq 2n$. In this case, calculations are drastically simplified by picking the nonlinear argument of $P_0$ in $P_\rho(1)$, namely $w := \exp((\frac{1}{2} n - m)u)$ (see (7)), as new parameter for the local image of the conformal curvature-increment operator. Since $w$ is close to 1, we further set $w = 1 + v$, so the conformal factor becomes

$$e^{2u} = (1 + v)^{4/(n-2m)}$$

and the renormalized $Q$-curvature increment operator accordingly becomes

(13)  $Q[u] \equiv \hat{Q}[v] := (1 + v)^{1-2^*} p_0(1 + v) - (\frac{1}{2} n - m) Q_0$

where $2^*$ stands for $2n/(n - 2m)$ in our context (admittedly a loose notation, customary for critical Sobolev exponents). Of course, Lemma 1.1 still holds for the operator $\hat{Q}$ (with $\hat{L} := d \hat{Q}[0] \equiv (2^*/n)L$) and proving Theorem 1.3 for $\hat{Q}$ is equivalent to proving it for $Q$. Altogether, we may thus focus on the proof of Lemma 4.1 for $Q$ instead of $\hat{Q}$. (The reader can instead prove Lemma 4.1 directly for $\hat{Q}$, but it takes a few pages.)

Picking $z$ and $t$ as above, plugging $v = tz$ in (13), and using the equality

$$P_0(z) = p_0(\lambda_1)z \equiv (2^* - 1) \left( \frac{1}{2} n - m \right) Q_0 z,$$

obtained from (12), we readily calculate the expansion

$$\frac{1}{(\frac{1}{2} n - m)} \hat{Q}[ tz ] = -\frac{1}{2}(2^* - 2)(2^* - 1) t^2 z^2 + \frac{1}{3}(2^* - 2)(2^* - 1) 2^* t^3 z^3 + O(t^4),$$

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and the renormalized $Q$-curvature increment operator accordingly becomes

(13)  $Q[u] \equiv \hat{Q}[v] := (1 + v)^{1-2^*} p_0(1 + v) - (\frac{1}{2} n - m) Q_0$

where $2^*$ stands for $2n/(n - 2m)$ in our context (admittedly a loose notation, customary for critical Sobolev exponents). Of course, Lemma 1.1 still holds for the operator $\hat{Q}$ (with $\hat{L} := d \hat{Q}[0] \equiv (2^*/n)L$) and proving Theorem 1.3 for $\hat{Q}$ is equivalent to proving it for $Q$. Altogether, we may thus focus on the proof of Lemma 4.1 for $Q$ instead of $\hat{Q}$. (The reader can instead prove Lemma 4.1 directly for $\hat{Q}$, but it takes a few pages.)

Picking $z$ and $t$ as above, plugging $v = tz$ in (13), and using the equality

$$P_0(z) = p_0(\lambda_1)z \equiv (2^* - 1) \left( \frac{1}{2} n - m \right) Q_0 z,$$

obtained from (12), we readily calculate the expansion

$$\frac{1}{(\frac{1}{2} n - m)} \hat{Q}[ tz ] = -\frac{1}{2}(2^* - 2)(2^* - 1) t^2 z^2 + \frac{1}{3}(2^* - 2)(2^* - 1) 2^* t^3 z^3 + O(t^4),$$

which in particular (with the notation of Section 1), we have $c_3[z] = \frac{1}{3} m^3 Q_0 z^3$, and Lemma 4.1 holds trivially.
thus finding for its third order coefficient
\[
\frac{1}{(\frac{1}{2}n - m)} \tilde{c}_3[z] = \frac{1}{3}(2^* - 2)(2^* - 1)2^* \ z^3.
\]

So Lemma 4.1 obviously holds, and with it Theorem 1.3. □

Appendix: Eigenvalue calculations

As well known (see [Berger et al. 1971], for instance), for each \( i \in \mathbb{N} \), the \( i \)-th eigenvalue of \( \Delta_0 \) on \( \mathbb{S}^n \) equals \( \lambda_i = i(i + n - 1) \). Recalling (6), we must calculate
\[
p_0(\lambda_i) = \prod_{k=1}^m \left( \lambda_i + \left( \frac{1}{2}n - k \right) \left( \frac{1}{2}n + k - 1 \right) \right).
\]

Setting provisionally
\[
r = \frac{1}{2}(n - 1), \quad s_k = k - \frac{1}{2},
\]
so that \( \frac{1}{2}n - k = r - s_k, \frac{1}{2}n + k - 1 = r + s_k \) and \( \lambda_i = i^2 + 2ir \), we can rewrite
\[
p_0(\lambda_i) = \prod_{k=1}^m \left( (i + r)^2 - s_k^2 \right)
\]
\[
= \prod_{k=1}^m \left( \frac{1}{2}i + r - k \right) \left( \frac{1}{2}i + r + k - 1 \right) = \prod_{k=0}^{2m-1} \left( \frac{1}{2}i + r - m + k \right),
\]

getting (back to \( m, n \) and \( k \) only)
\[
p_0(\lambda_i) = \prod_{k=0}^{2m-1} \left( i + \frac{1}{2}n - m + k \right).
\]

In particular,
\[
P_0(1) \equiv p_0(\lambda_0) = \left( \frac{1}{2}n - m \right) \prod_{k=1}^{2m-1} \left( \frac{1}{2}n - m + k \right)
\]
as asserted in (8) (and consistently there with the value of \( Q_0 \) in case \( n = 2m \)). An easy induction argument yields
\[
p_0(\lambda_{i+1}) = \frac{\left( \frac{1}{2}n + m + i \right)}{\left( \frac{1}{2}n - m + i \right)} p_0(\lambda_i) \quad \text{for all } i \in \mathbb{N}
\]

(consistently when \( i = 0 \) with (12)), which implies that \( |p_0(\lambda_{i+1})| > |p_0(\lambda_i)| \) for all \( i \in \mathbb{N} \), hence in particular \( p_0(\lambda_i) \neq p_0(\lambda_1) \) for \( i > 1 \), as required in the proof of Lemma 1.1. This also implies the final formula
\[
p_0(\lambda_i) = \frac{\left( \frac{1}{2}n + m \right) \ldots \left( \frac{1}{2}n + m + i - 1 \right)}{\left( \frac{1}{2}n - m \right) \ldots \left( \frac{1}{2}n - m + i - 1 \right)} p_0(\lambda_0) \quad \text{for all } i \geq 1.
ON THE LOCAL NIRENBERG PROBLEM FOR THE $Q$-CURVATURES

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References


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