4.1 Fisher information

**Exercise 4.1 (Additivity).** Consider $X_1, \ldots, X_n$ independent random variables. Show that the Fisher information is additive. That is, show that the Fisher information of the sample $(X_1, \ldots, X_n)$ is given by

$$I_\theta = I_\theta^{(1)} + \cdots + I_\theta^{(n)},$$

where $I_\theta^{(i)}$ is the Fisher information corresponding to the random variable $X_i$.

**Exercise 4.2 (Bernoulli random variables).** Let $X_1, \ldots, X_n$ be an i.i.d. sample of a Bernoulli random variable of parameter $\theta$. Show that the likelihood can be written $p_\theta(x) = \theta^x (1 - \theta)^{1-x}$ in that case. Compute the Fisher information. What happens when $\theta \to 0$ or $1$? Does it correspond to the intuition given in the lecture? *Hint:* Use the equivalent formulation $I_\theta = -E_\theta \left[ \frac{\partial^2}{\partial \theta^2} \log p_\theta \right]$ and use the previous exercise.

**Exercise 4.3 (Poisson random variables).** Consider i.i.d. Poisson random variables of parameter $\lambda > 0$. In that case, recall that the likelihood is $p_\lambda(k) = \frac{\lambda^k e^{-\lambda}}{k!}$. Show that the Fisher information is $I_\lambda = n/\lambda$.

**Exercise 4.4 (Gaussian random variables).** Consider i.i.d. Gaussian random variables of parameter $\theta = (\mu, \sigma^2)$. Show that the Fisher information in that case is

$$I_\theta = n \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^4} \end{pmatrix}.$$

*Hint:* look closely at our choice of parameters.

**Exercise 4.5 (Link with Kullback-Leibler).** Show that the Fisher information matrix is the Hessian of the Kullback-Leibler divergence, that is,

$$\forall i, j, \quad (I_{\theta_0})_{i,j} = \frac{\partial^2}{\partial \theta_i \partial \theta_j} D_{\text{KL}}(P_{\theta_0} \parallel P_\theta) \bigg|_{\theta = \theta_0}.$$

4.2 Cramér-Rao

**Exercise 4.6 (Gaussian distribution).** Let $X_1, \ldots, X_n$ be an i.i.d. sample from $N(\mu, \sigma^2)$. Find the Cramér-Rao lower bound for an unbiased estimator of

1. $\mu$ when $\sigma^2$ is known;
2. $\sigma^2$ when $\mu$ is known;
3. $\mu$ when $\sigma^2$ is unknown;
4. $\sigma^2$ when $\mu$ is unknown;
5. the coefficient of variation $\sigma/\mu$.

In cases 1. and 2., can you provide an estimator that achieves the bound?

**Exercise 4.7 (Trinomial distribution).** Let $X_1, \ldots, X_n$ be an i.i.d. sample from the trinomial distribution. The likelihood of observation $(x_1, x_2)$ is given by

$$p_\theta(x) = \frac{m!}{x_1!x_2!(m-x_1-x_2)!} \theta_1^{x_1} \theta_2^{x_2} (1 - \theta_1 - \theta_2)^{m-x_1-x_2},$$

with known parameter $m \in \mathbb{N}$ and unknown parameters $\theta = (\theta_1, \theta_2)$. Find lower bounds for the variance of unbiased estimators of $\theta_1$ and $\theta_2$. *Hint:* compute the Fisher information matrix and use the following formula to invert it in closed-form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. $$