6.1 Statistical hypothesis testing

Exercise 6.1 (Different point of view). Let $X$ be a pollution index measured close to a nuclear plant. We model it by a $\mathcal{N} (\mu, \sigma^2)$ distribution. The standard deviation is supposed to be known and equal to 4. The state regulations fix the maximal index at 30.

1. The nuclear plant head manager wants to show that his plant complies with the regulations. What hypotheses $H_0$ and $H_1$ should he test for? Propose an appropriate test. Establish the decision rule for that test at thresholds 5% and 1%.

2. The Green party wants to prove that the pollution is higher than prescribed. What hypotheses $H'_0$ and $H'_1$ should they test for? Establish the decision rule for that test at thresholds 5% and 1%.

Exercise 6.2 (Student’s test). Suppose that we want to compare the mean $\mu$ of a population to a reference value $\mu_0$. We have seen in the lecture how to proceed when the observations are Gaussian and the variance is known. When the variance is unknown, one can use the test statistic

$$Z = \sqrt{n} \frac{X_n - \mu_0}{\hat{\sigma}_n}, \quad \text{where} \quad \hat{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$ 

Let $N$ be a standard Gaussian random variable and $U$ an independent random variable such that $U \sim \chi^2_k$. Then by definition, the random variable $T = N/\sqrt{U/k}$ follows a Student law with $k$ degrees of freedom.

1. Use Cochran’s theorem to show that $Z$ follows a Student’s law. Specify the number of degrees of freedom.

2. Propose a test of level $\alpha$ based on the quantiles of the Student’s law to test for $H_0 : \mu = \mu_0$ v.s. $H_1 : \mu \neq \mu_0$.

3. Same question for $H_0 : \mu \geq \mu_0$ v.s. $H_1 : \mu < \mu_0$.

Exercise 6.3 ($\chi^2$ test). Consider $X_1, \ldots, X_n$ i.i.d. random variables with values in a discrete set $\mathcal{X} = \{x_1, \ldots, x_d\}$. We denote by $p = (p_1, \ldots, p_d)$ the law of the $X_i$s: for any $t \in \{1, \ldots, n\}$ and any $i \in \{1, \ldots, d\}$,

$$\mathbb{P}(X_t = x_i) = p_i.$$ 

Given a reference law $p^0$, with full support (that is, $p^0_1, \ldots, p^0_d$ positive real numbers such that $p^0_1 + \cdots + p^0_d = 1$), our goal is to test if the $X_t$ follow $p^0$.

1. Write the corresponding hypotheses.

2. Use the method of moments to propose an estimator of $p$. More precisely, for any $i \in \{1, \ldots, d\}$, propose an estimator $\hat{p}_{i,n}$ of $p_i$.

3. Show that these estimators are unbiased and consistent.
Thus, for large $n$, it is reasonable to assume that $\hat{p}_n = (\hat{p}_{1,n}, \ldots, \hat{p}_{d,n})$ is close to $p^0$ under the null. The idea of the $\chi^2$ test is to look at the chi-square distance between $\hat{p}$ and $p^0$, that is,

$$ D^2_n(\hat{p}, p) = n \sum_{i=1}^d \frac{(\hat{p}_{i,n} - p^0_i)^2}{p^0_i}. $$

We now study the asymptotic behavior of $D^2_n(\hat{p}, p)$ in order to build a test based on this statistic.

4. For any $t \in \{1, \ldots, n\}$, define

$$ Z_t = \left( \frac{1}{\sqrt{p_0^1}}(\mathbb{1}_{X_t = x_1} - p^0_1), \ldots, \frac{1}{\sqrt{p_0^d}}(\mathbb{1}_{X_t = x_d} - p^0_d) \right). $$

Prove that the $Z_t$s are i.i.d. and centered.

5. Compute the covariance matrix of $Z_t$.

6. Use the central limit theorem to show that, under the null,

$$ \frac{1}{\sqrt{n}} \sum_{t=1}^n Z_t \xrightarrow{d} V, $$

where $V \sim \mathcal{N}(0, I_d - \sqrt{p}^T \sqrt{p})$ (we denote by $\sqrt{p}$ the vector $(\sqrt{p_1}, \ldots, \sqrt{p_d})$).

7. Use the previous question, the continuity mapping, and Cochran’s theorem to show that, under the null, $D^2_n(\hat{p}, p) \xrightarrow{d} \chi^2_{d-1}$.

8. Show that $D^2_n \xrightarrow{a.s.} +\infty$ under $H_1$. Hint: notice that, under $H_1$, there is an integer $i$ such that $p_i \neq p^0_i$.

9. Let $c_{d-1,1-\alpha}$ be the quantile of order $1 - \alpha$ of the $\chi^2_{d-1}$ distribution. Use the previous questions to show that, for any $\alpha \in (0, 1)$, the test defined by

$$ \phi(X_1, \ldots, X_n) = 1_{D^2_n(\hat{p}, p) > c_{d-1,1-\alpha}} $$

is asymptotically of level $\alpha$. 