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Remarks on Bernoulli constants, gauge conditions and phase velocities in the context of water waves

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ABSTRACT

This short note is about the gauge condition for the velocity potential, the definitions of the Bernoulli constant and of the velocity speeds in the context of water waves. These definitions are often implicit and thus the source of confusion in the literature. This note aims at addressing this issue. The discussion is related to water waves because the confusion are frequent in this field, but it is relevant for more general problems in fluid mechanics.

1. Introduction

The Euler equations describe the momentum conservation of an inviscid fluid. For irrotational motions of incompressible fluids, the Euler equations can be integrated into a scalar equation called *Bernoulli equation* for steady flows and *Cauchy–Lagrange equation* for unsteady flows. The Bernoulli and Cauchy–Lagrange equations resulting of an integration procedure, they involve an arbitrary integration 'constant', the so-called *Bernoulli constant* (that is actually an arbitrary function of time for the Cauchy–Lagrange equation). This Bernoulli 'constant' and its physical meaning is a frequent source of confusion in the literature, especially in the study of water waves. Thus, there has been some recent works aiming at clarifying the situation [1].

The purpose of this short note is to address the issues related to the Bernoulli constants. This leads to clarify the definition of the velocity potential, its uniqueness being introduced by a gauge condition, and how this quantity is modified via Galilean transformations. Various frames of references are also discussed as they lead to the definition the phase velocity of a wave.

2. Cauchy–Lagrange equation

For the sake of simplicity, we consider the two-dimensional motion of an homogeneous incompressible fluid, but this is not a limitation for the purpose of the present note. For inviscid fluids, the equations of

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motion are [2]

$$u_x \;+\; u_y \;=\; 0, \tag{1}$$

$$u_t \;+\; u\,u_x \;+\; v\,u_y \;=\; -\,p_x, \tag{2}$$

$$v_t \;+\; u\,v_x \;+\; v\,v_y \;=\; -\,p_y \;-\; g, \tag{3}$$

where $\boldsymbol{x} = (x, y)$ are the Cartesian coordinates (y being directed upward), t is the time, $\boldsymbol{u} = (u, v)$ is the velocity field, p is the pressure divided by the (constant) density and g is the (constant) acceleration due to gravity directed toward the decreasing y-direction (downward).

In irrotational motion $v_x = u_y$, so there exists a velocity potential ϕ such that $u = \phi_x$ and $v = \phi_y$, i.e. $\boldsymbol{u} = \operatorname{grad}\phi$. The Euler equations (2)–(3) can then be rewritten [2]

$$\operatorname{grad}\left[\,\phi_t \;+\; \frac{1}{2}\,(\phi_x)^2 \;+\; \frac{1}{2}\,(\phi_y)^2 \;+\; g\,y \;+\; p\,\right] \;=\; 0, \tag{4}$$

that can be integrated into the Cauchy–Lagrange equation

$$\phi_t \;+\; \frac{1}{2}\,(\phi_x)^2 \;+\; \frac{1}{2}\,(\phi_y)^2 \;+\; g\,y \;+\; p \;=\; C(t), \tag{5}$$

where C is an integration 'constant' often called *Bernoulli constant* or *Bernoulli integral*.

3. Gauge condition

The velocity potential being defined via its gradient, ϕ is not an unique function: adding any arbitrary function of time to ϕ does not change the velocity field. Thus, if one makes the change of potential [2]

$$\phi(\boldsymbol{x}, t) \;=\; \phi^\star(\boldsymbol{x}, t) \;+\; \int C(t)\,\mathrm{d}t,$$

so that $\operatorname{grad}\phi = \operatorname{grad}\phi^\star$, the Cauchy–Lagrange equation (5) becomes

$$\phi_t^\star \;+\; \frac{1}{2}\,(\phi_x^\star)^2 \;+\; \frac{1}{2}\,(\phi_y^\star)^2 \;+\; g\,y \;+\; p \;=\; 0. \tag{6}$$

In other words, this shows that it is always possible, via a suitable definition of the velocity potential, to take

$$C(t) \;=\; 0, \tag{7}$$

without loss of generality and preserving the velocity field (i.e., $\operatorname{grad}\phi = \boldsymbol{u}$). Enforcing the unicity of the velocity potential ϕ (up to an additional constant) via (7) is a so-called *gauge condition*.

Hereafter, we always take the gauge condition (7) and the Cauchy–Lagrange equation is thus

$$\phi_t \;+\; \frac{1}{2}\,(\phi_x)^2 \;+\; \frac{1}{2}\,(\phi_y)^2 \;+\; g\,y \;+\; p \;=\; 0. \tag{8}$$

Of course, other gauge conditions could be introduced, as well as no gauge condition at all. In the latter case, the arbitrary function $C(t)$ should be carried along all the derivations.

Note that with the gauge (7), the Bernoulli 'constant' disappears from the Cauchy–Lagrange equation (8), but it has not completely been eliminated: it is now 'hidden' in the definition of the velocity potential ϕ and will reappear explicitly for some special flows, as shown below.

4. Galilean transformation

Let be a change of Galilean frames of reference $\mathcal{R}_0 \mapsto \mathcal{R}_1$, where the coordinate system attached to $\mathcal{R}_1$ appears to travel at a constant speed c in $\mathcal{R}_0$ along the x-direction. If (x, y, t) and (X, Y, T) denote the independent variables in $\mathcal{R}_0$ and $\mathcal{R}_1$, respectively, and if (u, v, p) and (U, V, P) are the corresponding velocity and pressure fields, the Galilean transformation from $\mathcal{R}_0$ to $\mathcal{R}_1$ is [3]

$$X = x - ct, \quad Y = y, \quad T = t, \quad U = u - c, \quad V = v, \quad P = p. \tag{9}$$

From the Galilean transformation (9), the transformation of the velocity potential $\phi \mapsto \Phi$ is necessarily (assuming $U = \Phi_X$ and $V = \Phi_Y$) of the general form

$$\phi = \Phi + cX + K(T), \tag{10}$$

where K is an arbitrary function of T to be determined. Substituting (10) into the Cauchy–Lagrange equation (8), one obtains at once

$$\Phi_T + \frac{1}{2}(\Phi_X)^2 + \frac{1}{2}(\Phi_Y)^2 + P + gY = \frac{c^2}{2} - \frac{\mathrm{d}K}{\mathrm{d}T}, \tag{11}$$

which is also a Cauchy–Lagrange equation. If we want to leave the Cauchy–Lagrange equation (8) invariant under Galilean transformations, the right-hand side of (11) must vanish, so one must take

$$K = \frac{1}{2}c^2 T, \tag{12}$$

thence the Galilean transformation for the velocity potential:

$$\Phi = \phi - cx + \frac{1}{2}c^2 t, \qquad \phi = \Phi + cX + \frac{1}{2}c^2 T. \tag{13}$$

The Galilean transformation (13) for the velocity potential is such that it preserves the gauge condition (7), i.e., the gauge condition (7) is the same in all Galilean frames of reference if the velocity potential is transformed according to (13).

Of course, it is not obligatory to choose the same gauge condition in every frame of reference. In that case, a suitable Galilean transformation should be introduced for the velocity potential, so that the gauge condition is transformed properly. However, it is simpler and less prone to confusion to take the same gauge condition for all frames of reference. Note that the Galilean transformation of a velocity potential is similar to the one for the action in classical mechanics (see the problem at the end of §8 in [3]).

The mishandling of the Galilean transformation for the velocity potential and the resulting (implicit) change of gauge condition is a source of apparent incompatibilities regarding the Bernoulli constants sometimes found in the literature. This confusion is also sometimes the source of physical misinterpretations as well. Discussion on this matter is given in the Section 8 below.

5. Steady flow and Bernoulli equation

An important class of physical problems involve steady flows where all measurable quantities (velocity, pressure, density) are independent of the time t in Eulerian description of motion.[1] A velocity potential is not directly measurable, only its gradient (i.e., the velocity field) can be measured. The question is thus to find out the general form of a velocity potential for steady flows.

[1] The definition in Lagrangian description of motion can be found in [4].

Since, for steady flows, grad ϕ is independent of time, it follows that the most general potential of such flows is of the form

$$\phi(x,y,t) \;=\; \Phi(x,y) \;-\; A(t), \tag{14}$$

where A is a function of time only to be determined. Substituting this relation into (8), one gets

$$\frac{1}{2}\,(\Phi_x)^2 \;+\; \frac{1}{2}\,(\Phi_y)^2 \;+\; p \;+\; g\,y \;=\; \frac{\mathrm{d}\,A}{\mathrm{d}t}. \tag{15}$$

The left-hand side of (15) being independent of the time t, so is the right-hand side. Therefore, A is necessarily a linear function of t:

$$A(t) \;=\; \frac{1}{2}\,B\,t \;+\; A_0, \tag{16}$$

where B is a Bernoulli constant (the factor $1/2$ is unessential and it is introduce only for later convenience). The Cauchy–Lagrange equation thus becomes the Bernoulli equation

$$\frac{1}{2}\,(\Phi_x)^2 \;+\; \frac{1}{2}\,(\Phi_y)^2 \;+\; p \;+\; g\,y \;=\; \frac{1}{2}\,B. \tag{17}$$

This shows that, for irrotational motions, the Bernoulli equation is just a special form of the Cauchy–Lagrange equation without inconsistencies or contractions with respect of the Bernoulli constants. The key point is that the velocity potential is not independent of the time for steady flows under the gauge condition (7).

6. Traveling waves

Consider now the more general case of a wave traveling at constant speed c along the x-direction and without change of form. This concept can be made precise as follow.

A traveling wave is such that it exists a frame of reference where the flow appears steady. Thus, in any other Galilean frame of reference, the velocity potential of a traveling wave is, according to the sections above, necessarily of the form

$$\phi(x,y,t) \;=\; \varphi(\xi,y) \;-\; \frac{1}{2}\,\mathscr{B}\,t, \qquad \xi \;=\; x \;-\; c\,t, \tag{18}$$

where $\mathscr{B}$ is a Bernoulli constant. The Cauchy–Lagrange equation thus yields the modified Bernoulli equation

$$-\,c\,\varphi_\xi \;+\; \frac{1}{2}\,(\varphi_\xi)^2 \;+\; \frac{1}{2}\,(\varphi_y)^2 \;+\; p \;+\; g\,y \;=\; \frac{1}{2}\,\mathscr{B}. \tag{19}$$

If $c = 0$ (frame of reference traveling with the wave) the Eq. (19) yields the Bernoulli equation (17). If $c \neq 0$ then $\mathscr{B} \neq B$, as it can be easily seen considering the Galilean transformation from (17) to (19).

This result shows that, for a traveling wave, the velocity potential is generally not of the form $\phi(x,y,t) = \varphi(x-ct,y)$ but of the more general form (18). The form of solution $\phi(x,y,t) = \varphi(x-ct,y)$ is, in most cases, incompatible with (7) as it implicitly implies a different gauge condition for the velocity potential. As for steady flows, the violation of the gauge condition is the source of confusion and apparent contradictions found in the literature.

7. Phase velocities

When looking for traveling waves propagating at constant speed c, one has to specify in which frame of reference the wave is observed. Otherwise confusion and inconsistencies may occur. Indeed, quite often in the literature, traveling wave solutions are sought as functions of $x - ct$, where c is called the phase velocity in the 'fixed' frame of reference without further precisions. This is not a definition of c because there are infinitely many 'fixed' Galilean frames. Below, we give the definitions of two 'fixed' frame often used in practice.

The fluid domain being $-d \leqslant y \leqslant \eta$ (d the constant depth, η the surface elevation from rest), there are two 'fixed' frame of references commonly used. The frame of reference where the average horizontal velocity is zero at the bottom is defines such that

$$\frac{1}{T\,L}\int_{-T/2}^{T/2}\int_{-L/2}^{L/2} u(x, y = -d, t)\,\mathrm{d}x\,\mathrm{d}t \;=\; 0, \tag{20}$$

where L and T are, respectively, the wavelength and the period. The condition (20) defines univocally the phase speed. This is *Stokes' first definition of wave celerity* [5], sometimes denoted c_e [6]. Since the flow is irrotational, this frame of reference is also the one where the horizontal velocity averages to zero along any horizontal line $y = \text{Constant}$ inside the fluid.

Another frame of reference of practical importance is the one where the mean flow is zero. The horizontal velocity is therefore such that

$$\frac{1}{T\,L\,d}\int_{-T/2}^{T/2}\int_{-L/2}^{L/2}\int_{-d}^{\eta} u(x, y, t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \;=\; 0. \tag{21}$$

The condition (21) also defines univocally the phase speed. This is *Stokes' second definition of wave celerity* [5], sometimes denoted c_s [6].

In general $c_\mathrm{e} \neq c_\mathrm{s}$, but these two velocities are equal in deep water ($d \to \infty$) and for solitary waves ($L \to \infty$). If B is the Bernoulli constant in the frame of reference moving with the wave (c.f. Eq. (17)), then $c_\mathrm{e}^2 \neq B \neq c_\mathrm{s}^2$ in general but $B = c_\mathrm{e}^2 = c_\mathrm{s}^2$ in deep water and for solitary waves [7].

Many other frames of reference can of course be defined, their interest depending on the problem at hand. Moreover, one can use different average operators than the ones considered above (arithmetic mean in Eulerian description of motion). These remarks are also valid, of course, for unsteady flows. In any case, precise definitions are necessary to avoid confusion.

For practical determination of traveling waves, it is simpler to first determine the solutions for $\varPhi$ and B in the frame of reference moving with the wave where the flow is steady. In particular, since η is the surface elevation from rest where the pressure is zero, the Bernoulli constant can be obtained averaging the Bernoulli equation at the free surface, i.e.

$$B \;=\; \frac{1}{L}\int_{-L/2}^{L/2}\left[\,\mathrm{grad}\,\varPhi\,\right]_{y=\eta}^{2}\mathrm{d}x.$$

At this stage, no phase speed needs to be defined. The phase speed c is subsequently obtained for whatever frame of reference of interest, such as (20) or (21). Thus, in the frame moving with the wave, the celerities c_e and c_s are computed with the formulae

$$\frac{1}{L}\int_{-L/2}^{L/2}\varPhi_x(x, y = -d)\,\mathrm{d}x \;=\; -\,c_\mathrm{e}, \qquad \frac{1}{L\,d}\int_{-L/2}^{L/2}\int_{-d}^{\eta}\varPhi_x(x, y)\,\mathrm{d}y\,\mathrm{d}x \;=\; -\,c_\mathrm{s}.$$

Finally, the velocity potential φ and the Bernoulli constant $\mathscr{B}$ are obtained in the 'fixed' frame of reference using the Galilean transformation (13).

It should be noted that for experiments in closed flumes, the fixed (laboratory) frame of reference is the one without mean flow. But since viscous fluids (generally water) are used in practice, the fluid sticks to rigid walls and the velocity is zero at the bottom. Thus, the fixed frame is also the one without mean velocity at the bed, in contraction with the potential flow theory. This example shows that precise comparisons with experiments are not an easy matter.

8. Discussion

Apparent incompatible formulations of the Bernoulli equations found in the literature (e.g. in [8,9]) has been investigated by Vasan and Deconinck [1]. They have shown that under the gauge condition (7), considering $\phi(x-ct,y)$ for traveling waves yields (19) with $\mathscr{B}=0$, that is generally incorrect. They solve this problem reintroducing a Bernoulli constant and they show that the resulting equation corresponds to another velocity potential. This velocity potential corresponds to a different gauge condition. Vasan and Deconinck [1] also discuss the relation between Bernoulli constants and uniform currents. Such considerations are not necessary if traveling waves are defined by (18) and if Galilean transformations between velocity potentials are defined by (13) in order to preserve the gauge condition.

In many applications, these apparent incompatibilities are of little consequence since they can be circumvented via redefinition of some parameters and variables, as shown in [1]. In some applications, however, the consequences can be dramatic and no simple redefinitions can be introduced to solve the issue.

An example is the case of a varying dissipation used to model sponge layers described in section §4.2.2 of [10]. Indeed, in order to introduce a sponge layer, the Cauchy–Lagrange equation(5) is modified as

$$\phi_t \;+\; \frac{1}{2}\,(\phi_x)^2 \;+\; \frac{1}{2}\,(\phi_y)^2 \;+\; g\,y \;+\; p \;+\; \gamma(x,y)\,\phi \;=\; C(t), \tag{22}$$

where $\gamma>0$ in the regions where damping is required and $\gamma=0$ elsewhere. If γ is constant, the Bernoulli integral $C(t)$ can be set to zero without loss of generality via the change of potential

$$\phi(\boldsymbol{x},t) \;=\; \phi^\star(\boldsymbol{x},t) \;+\; \int_{t_0}^{t} \mathrm{e}^{\gamma(t'-t)}\,C(t')\,\mathrm{d}t'.$$

When $\gamma=\gamma(x,y)$, the gauge condition $C(t)=0$ cannot be applied because

$$\operatorname{grad}\phi \;=\; \operatorname{grad}\phi^\star \;+\; (\operatorname{grad}\gamma)\int_{t_0}^{t}(t'-t)\,\mathrm{e}^{\gamma(t'-t)}\,C(t')\,\mathrm{d}t',$$

so Eq. (22) is modified. One can easily verify that no transformation $\phi\to\phi^\star$ with $\operatorname{grad}\phi=\operatorname{grad}\phi^\star$ leads to (22) with $C=0$ if γ is not constant. Therefore, if one uses (22) together with $C=0$ then $\operatorname{grad}\phi$ is not the velocity field and spurious unphysical phenomena may appear as discussed in [10].

9. Conclusion

We have seen that the confusion regarding the Bernoulli constants can be the result of incorrect Galilean transformations violating the Gauge condition used for the velocity potential. This can also leads to incorrect mathematical definitions of the velocity potential for steady flows and traveling waves.

Although the focus was on irrotational water waves, such considerations also apply for rotational waves [11] and for ideal fluid motions in general [12]. Remarks on the possible gauge conditions for variants of the Cauchy–Lagrange equation involving varying dissipative terms are also discussed.

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